Lógica Quântica
Lecture notes and exercise sheet 5
Curry–Howard–Lambek correspondence

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Natural deduction system for intuitionistic logic

Formulae are built out of basic propositions with binary connectives $\wedge$ (conjunction) and $\supset$ (implication). A *sequent* is a judgement of the form

$$\Gamma \vdash A$$

where A is a formula and $\Gamma = \{A_1, \ldots, A_n\}$ a finite set of formulae. We write Γ, A for $\Gamma \cup \{A\}$. The sequent above is meant to assert that A can be proved from assertions in Γ. Proofs are constructed using the following rules:

$$\frac{}{\Gamma, A \vdash A}\ \text{Id}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\text{-I} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-E}_1 \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-E}_2$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \supset\text{-I} \qquad \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}\ \supset\text{-E}$$

Exercise 1. Give proofs of:

(a) $\vdash (A \wedge B) \supset A$

(b) $\vdash ((A \supset B) \wedge (B \supset C)) \supset (A \supset C)$

(c) $\vdash (A \supset (B \supset C)) \supset (B \supset (A \supset C))$

(d) $\vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

Definition 1. A proof rule

$$\frac{\Gamma_1 \vdash A_1 \quad \cdots \quad \Gamma_k \vdash A_k}{\Delta \vdash B}$$

is said to be admissible if whenever there are proofs of each $\Gamma_i \vdash A_i$ $(i = 1, \ldots, n)$, there is a proof of $\Delta \vdash B$.

If a proof is admissible then it can be added to a proof system without altering whether a sequent is provable.

Exercise 2. Show that the following rules are admissible:

(a) $\dfrac{\Gamma \vdash A}{\Gamma, C \vdash A}$ Weakening

(b) $\dfrac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$ Cut

Simply-typed lambda-calculus

Let BaseTypes be a set of base types. Let Var be a countable set of variables. Types and terms in the simply-typed λ-calculus are given by the following syntax

$$\begin{aligned} \mathsf{Var} \ni x, y, \dots & & \\ \mathsf{Types} \ni A, B, \dots & ::= & b \in \mathsf{BaseTypes} \mid A \longrightarrow B \mid A \times B \\ \mathsf{Terms} \ni t, u, \dots & ::= & x \mid tu \mid \lambda x.t \mid \langle t, u \rangle \mid \pi_1 t \mid \pi_2 t \end{aligned}$$

A *typing judgement* has the form

$$\Gamma \vdash t : A$$

where $\Gamma = \{x_1 : A_1, \dots, x_k : A_k\}$ is a type assignment to a finite set of variables, called a typing context. The judgement $\Gamma \vdash t : A$ asserts that the term t is given type A in the context Γ.

Typing judgements are derived by the following typing rules:

$$\frac{}{\Gamma, x : A \vdash x : A}\ \text{Id}$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \langle t, u \rangle : A \times B}\ \wedge\text{-I} \qquad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \pi_1 t : A}\ \wedge\text{-E}_1 \qquad \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \pi_2 t : B}\ \wedge\text{-E}_2$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \longrightarrow B}\ \wedge\text{-I} \qquad \frac{\Gamma \vdash t : A \longrightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash tu : B}\ \wedge\text{-E}$$

Exercise 3. Give (if possible) the most general type for the following lambda terms:

(a) $\lambda x.\lambda y.xy$

(b) $\lambda x.\lambda y.\lambda z.x(yz)$

(c) $\lambda x.\lambda y.x$

(d) $\lambda x.\lambda y.\langle x, \lambda z.y \rangle$

(e) $\lambda x.x$

(f) $\lambda x.\lambda y.xyy$

(g) $\lambda x.xx$

Could you automate this exercise?

Exercise 4. Compute β-reductions of the following terms:

(a) $(\lambda x.\lambda y.yx)w(\lambda z.z)$

(b) $(\lambda x.\lambda y.xyy)(\lambda z.\lambda w.\langle z, w \rangle)(\pi_1 \langle \lambda w.w, v \rangle)\ (\lambda x.)(\lambda y.yy)$

Exercise 5. Show that in the untyped λ-calculus, the term

$$\Omega = (\lambda x.x)(\lambda x.x)$$

has an infinite β-reduction sequence. Is it typable?

Exercise 6. Consider the formula $((A \supset B) \supset A) \supset A$

(a) Check its validity by truth tables.

(b) Is there a natural deduction proof of this formula? Equivalently, is there lambda-term of this type (for some types A, B)?

Exercise 7. What are the corresponding λ-terms to the proofs in 1?

Cartesian-closed categories

Definition 2. Let **C** be a category with binary products, and let A, B be objects of **C**. An exponential or hom-object is an object $A \Rightarrow B$ of **C** together with an arrow $\mathsf{ev}_{A,B} \colon (A \Rightarrow B) \times A \longrightarrow B$ satisfying the following universal property: for any arrow $g \colon C \times A \longrightarrow B$ there is a unique arrow $\Lambda(g) \colon C \longrightarrow A \Rightarrow B$ satisfying

$$ev_{A,B} \circ (\Lambda(g) \times \mathsf{id}_A) = g.$$

Exercise 8. If $A \Rightarrow B$ exists, what is $\Lambda(ev_{A,B})$ equal to?

Exercise 9. Show that the uniqueness requirement in the definition of exponential can be replaced by an equational requirement:

$$\text{for all } h \colon C \longrightarrow (A \Rightarrow B), \quad \Lambda(ev_{A,B} \circ (h \times \mathsf{id}_A)) = h \ .$$

Exercise 10. Recall that a poset $(P, \leq)$ can be seen as a category.

(a) Show that a Boolean algebra (seen as a posetal category) is cartesian closed.

(b) A subset $S \subseteq P$ is downwards-closed if $x \leq y$ and $y \in S$ implies $x \in P$. Let $D(P)$ be the poset of downwards-closed subsets of P, ordered by inclusion. Show that $D(P)$ is indeed a poset, and that (as a category) it has products.

(c) Show that $D(P)$ is cartesian closed with the following definition of exponential object: given $S, T \in D(P)$,

$$(S \Rightarrow T) := \{x \in P \mid y \leq x \text{ and } y \in S \text{ implies } y \in T\}.$$

Definition 3. A category **C** is *cartesian closed* if:

- it has a terminal object $\mathbf{1}$;
- any pair of objects A, B have a product $A \times B$;
- any pair of objects A, B have an exponential $A \Rightarrow B$.

Exercise 11. Let **C** be cartesian closed. Assume you are given interpretations $[\![A]\!]$, $[\![B]\!]$, … of A, B, … as objects of **C**.

(a) Give the interpretations of the (typable) terms from exercise 3.

(b) Give interpretations of the terms in exercise 4 and of their reducts, and verify these yield the same arrow.

Exercise 12. Use the lambda calculus to show that the objects $A \Rightarrow (B \Rightarrow C)$ and $(A \times B) \Rightarrow C$ are isomorphic in any cartesian closed category.