Lógica Quântica
Lecture notes and exercise sheet 7
Categorical quantum mechanics

Rui Soares Barbosa

2022–2023

Daggers, scalars, and linearity

Let $\mathbf{C}$ be a monoidal category. Recall that a scalar is an arrow $s\colon I \longrightarrow I$.

Exercise 1. Given a scalar $s\colon I \longrightarrow I$ and an arrow $f\colon A \longrightarrow B$, the scalar multiplication is the arrow $s \bullet f\colon A \longrightarrow B$ given by

$$A \;\cong\; I \otimes A \xrightarrow{s \otimes f} I \otimes B \cong\; B.$$

Show the following properties, where s, t are scalars and f, g arbitrary morphisms (of the appropriate types):

(a) $\mathsf{id}_I \bullet f = f$

(b) $s \bullet (t \bullet f) = (s \circ t) \bullet f$

(c) $(t \bullet g) \circ (s \bullet f) = (t \circ s) \bullet (g \circ f)$

(d) $(t \bullet f) \otimes (s \bullet g) = (t \circ s) \bullet (f \otimes g)$

Definition 1. A dagger on a category $\mathbf{C}$ is a functor $\dagger\colon \mathbf{C}^{\mathsf{op}} \longrightarrow \mathbf{C}$ that is involutive and identity on objects. A dagger category is a category equipped with a dagger.

Exercise 2. Show that relational converse yields a dagger in **Rel**.

Definition 2. A superposition rule is a binary operation $(f, g) \longmapsto f + g$ defined over arrows $f, g\colon A \longrightarrow B$ between any object A, B such that:

- $f + g = g + f$
- $(f + g) + h = f + (g + h)$
- for all A, B there is $u_{A,B}\colon A \longrightarrow B$ such that $f + u = f$ for all f
- $(g + g') \circ f = (g \circ f) + (g' \circ f)$
- $g \circ (f + f') = (g \circ f) + (g \circ f')$
- $u_{C,B} = f \circ u_{C,A}$
- $u_{A,D} = u_{B,D} \circ f$

Exercise 3. Let $\mathbf{C}$ be a category with a zero object 0 and a superposition rule. Show that, for all objects A, B, $u_{A,B} = 0_{A,B}$, where $0_{A,B}$ is the unique arrow $A \longrightarrow 0 \longrightarrow B$.

Exercise 4. Let $\mathbf{C}$ be a category with zero object and a superposition rule.

(a) Show that the notion of biproduct from sheet 2 agrees with the following (more traditional) definition: a biproduct of objects A_1 and SA_2 is an object $A_1 \oplus A_2$ equipped with arrows $\iota_j \colon A_j \longrightarrow A_1 \oplus A_2$ and $\pi_j \colon A_1 \oplus A_2 \longrightarrow A_j$ satisfying

$$\begin{aligned} id_{A_j} &= \pi_j \circ \iota_j \\ 0_{A_k, A_j} &= \pi_j \circ \iota_k \quad (j \neq k) \\ id_{A_1 \oplus A_2} &= i_1 \circ p_1 + i_2 \circ p_2 \end{aligned}$$

(b) Show that if **C** has biproducts then it has a unique superposition rules.

Dagger compact closed categories

Exercise 5. Show that:

- Trace is cyclic.
- If $s \colon I \longrightarrow I$ is a scalar, then $\mathrm{Tr}(s) = s$
- $\mathrm{Tr}(f \otimes g) = \mathrm{Tr}(f) \otimes \mathrm{Tr}(g)$
- $\mathrm{Tr}(f^\dagger) = \mathrm{Tr}(f)^\dagger$

Exercise 6. Recall the definition of transpose. Show that the transpose $f \longmapsto f^T$ is involutive and contravariantly functorial.

Exercise 7. An arrow $f \colon A \longrightarrow A$ is positive if there is an arrow $h \colon A \longrightarrow B$ such that $f = h^\dagger \circ h$. Show that:

(a) If f is positive, then $\mathrm{Tr}(f)$ is positive.

(b) If f, g are positive then $\mathrm{Tr}(g \circ f)$ is positive.

Exercise 8. Interpret the teleportation protocol in **Rel** using the sets $2 = \{0, 1\}$, cup $\eta \colon \{*\} \longrightarrow 2 \times 2$ defined by choosing the pairs $(0, 0)$ and $(1, 1)$ to constitute the image of $*$. Note that this can be interpreted as classical encrypted communication using a one-time pad.

Monoids, comonoids, dagger Frobenius structures

Exercise 9. Let $(A, m_1 \colon A \otimes A \longrightarrow A, u_1 \colon I \longrightarrow A)$ and $(A, m_2 \colon A \otimes A \longrightarrow A, u_2 \colon I \longrightarrow A)$ be monoids on the same object in a symmetric monoidal category and suppose that

$$m_1 \circ (m_2 \otimes m_2) = m_2 \circ (m_1 \otimes m_1) \circ (\mathsf{id}_A \otimes \sigma_{A,A} \otimes \mathsf{id}_A)$$

Show that

(a) $u_1 = u_2$

(b) $m_1 = m_2$

(c) m_1 is commutative

Exercise 10. A state $a \colon I \longrightarrow A$ is copyable by a comonoid (A, m, u) if $m \circ a = a \otimes a$ (up to the unique iso $I \cong I \otimes I$). Show that the states copyable by a commutative dagger Frobenius structure in **FHilb** form an orthogonal basis, and moreover that speciality of the structure implies orthonormality of the basis.

Exercise 11. Let A have a dagger Frobenius structure. Show how one can use it to define cups and caps for the object A satisfying the snake equations.

Exercise 12. Show that two commutative dagger Frobenius structures are complementary if and only if the corresponding bases $\{e_i\}$, $\{f_j\}$ are complementary with $\langle e_i | f_j \rangle \langle f_j | a e_i \rangle = 1$.

Exercise 13. Show that two commutative dagger Frobenius structures on the same object are complementary if and only if $(c_2 \otimes \mathsf{id}) \circ (id \otimes c_1)$ is unitary.