Lógica Quântica
Assessment 1

2022–2023

Part 1

Definition 1. Given a linear map $f\colon H \longrightarrow K$ between Hilbert spaces H and K, its adjoint is the unique linear map $f^\dagger\colon K \longrightarrow H$ such that, for all $v \in H$ and $w \in K$,

$$\langle f(v), w\rangle = \langle v, f^\dagger(w)\rangle\,.$$

Exercise 1. Show that this construction is functorial, i.e. it defines a functor $(-)^\dagger\colon \mathbf{Hilb}^{\mathsf{op}} \longrightarrow \mathbf{Hilb}$. Concretely, this means showing that $\mathsf{id}_H^\dagger = \mathsf{id}_H$ and $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$ whenever f and g are composable. Moreover, show that this functor is involutive, i.e. for any $f\colon H \longrightarrow K$, $(f^\dagger)^\dagger = f$. ◀

Part 2

Definition 2. A *monoidal category* is a category $\mathbf{C}$ equipped with:

- a functor $\otimes\colon \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ (called *tensor*);
- an object I of $\mathbf{C}$ (called the *unit*);
- natural isomorphisms α, λ, ρ with components

$$\alpha_{A,B,C}\colon A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$$
$$\lambda_A\colon I \otimes A \xrightarrow{\cong} A \qquad \rho_A\colon A \otimes I \xrightarrow{\cong} A$$

  (called the *associator*, the *left unitor*, and the *right unitor*, respectively);

satisfying the equations

$(\mathsf{id}_A \otimes \lambda_B) \circ \alpha_{A,I,B} = \rho_A \otimes \mathsf{id}_B$ and $(\mathsf{id}_A \times \alpha_{B,C,D}) \circ \alpha_{A,B\otimes C,D} \circ (\alpha_{A,B,C} \otimes \mathsf{id}_D) = \alpha_{A,B,C\otimes D} \circ \alpha_{A\otimes B,C,D}$

which are expressed by the commutativity of the following *triangle* a *pentagon* diagrams:

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathsf{id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathsf{id}_A \otimes \lambda_B"] \\
& (A \otimes B) &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A\otimes(B\otimes C))\otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A\otimes((B\otimes C)\otimes D) \arrow[dr, "\mathrm{id}_A\times\alpha_{B,C,D}"] & \\
((A\otimes B)\otimes C)\otimes D \arrow[ur, "\alpha_{A,B,C}\otimes\mathrm{id}_D"] \arrow[drr, "\alpha_{A\otimes B,C,D}"'] & & & & A\otimes(B\otimes(C\otimes D)) \\
& & (A\otimes B)\otimes(C\otimes D) \arrow[urr, "\alpha_{A,B,C\otimes D}"'] & &
\end{tikzcd}$$

These equations guarantee what is called *coherence*: that all diagrams involving only α, λ, and ρ commute.

In any monoidal category, one can reason using string diagrams (and you may use those in this exercise if you prefer).

We have seen that any category $\mathbf{C}$ with binary products $\times$ and a terminal object $\mathbf{1}$ is a monoidal category: exercise 3.13, exercise 4.7. Moreover, we have seen in exercises 4.2 and 4.3 that, in the case of products there are natural transformations Δ, p, q with components

$$\delta_A \colon A \longrightarrow A \times A\ , \quad p_{A,B} \colon A \times B \longrightarrow A\ , \quad q_{A,B} \colon A \times B \longrightarrow B,$$

which one can interpret as *copying* and *projections*. The goal of this second part is to show a converse to this, therefore characterising products as precisely monoidal structures (tensors) that admit copying and projections in some sense.

Exercise 2. Let $\mathbf{C}$ be a monoidal category and suppose there are natural transformations with components of type

$$d_A \colon A \longrightarrow A \otimes A\ , \quad p_{A,B} \colon A \otimes B \longrightarrow A\ , \quad q_{A,B} \colon A \otimes B \longrightarrow B,$$

such that the following diagrams commute:

$$\begin{tikzcd}
A & A\otimes A \arrow[l, "p_{A,A}"'] \arrow[r, "q_{A,A}"] & A \\
& A \arrow[ul, "\mathrm{id}_A"] \arrow[u, dashed, "d_A"] \arrow[ur, "\mathrm{id}_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (A\otimes B)\otimes(A\otimes B) \arrow[dr, "p_{A,B}\otimes q_{A,B}"] & \\
A\otimes B \arrow[ur, "d_{A\otimes B}"] \arrow[rr, "\mathrm{id}_{A\otimes B}"'] & & A\otimes B
\end{tikzcd}$$

Show that $\otimes$ gives a product structure, i.e. that $A \otimes B$ is the categorical product of A and B. ◀