Structural reason for monogamy (and locality of average macroscopic behaviour)

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Introduction

- Monogamy of violation of Bell inequalities from the no-signalling condition.
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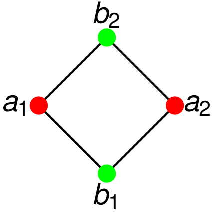
- General framework of Abramsky & Brandenburger (2011) and provide a structural explanation using Vorob'ev's theorem (1962).
- Illustrate with a very simple example.

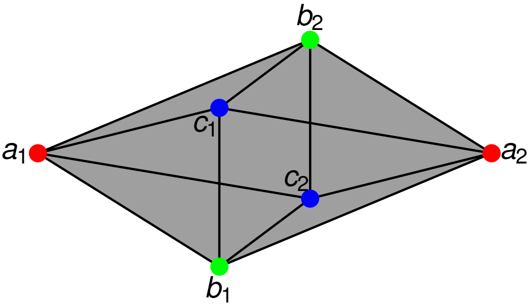
The setting

Measurement Scenarios

Abramsky-Brandenburger framework

- a finite set of measurements X;
- a cover $\mathcal{U}$ of X (or an abstract simplicial complex Σ on X), indicating the **compatibility** of measurements.





Examples: Bell-type scenarios, KS configurations, and more.

Non-locality and contextuality

"No-signalling" **empirical model**:

- a family $(p_C)_{C \in \mathcal{U}}$ of probability distributions on the outcomes of the measurements in each context C;
- compatibility condition: p_C and $p_{C'}$ marginalise to the same distribution on the outcomes of measurements in $C \cap C'$.
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Is there a **global distribution** p_X that marginalises to all the p_C?

Obstructions to such extensions are witnessed by violations of **general Bell inequalities.** E.g. in bipartite scenario:

$$\sum_{i,j,x,y} \alpha(i,j,x,y)\, p(x,y \mid a_i, b_j) \leq R$$

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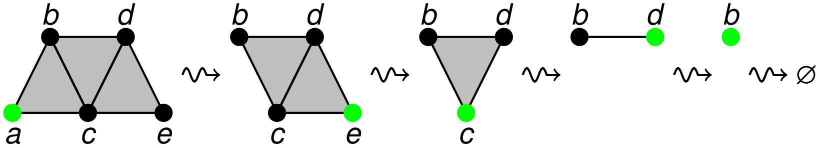
Vorob'ev (1962) derived a **necessary and sufficient** combinatorial condition on $\mathcal{U}/\Sigma$ for this to be the case.

- Turns out to be equivalent to the notion of acyclicity of a database schema.
- Graham reduction step: delete a measurement that belongs to only one maximal context.
- A cover is acyclic when it is Graham reducible to $\varnothing$.

Vorob'ev's theorem: "Proof" by example

- acyclic cover (Graham reductible to $\varnothing$).



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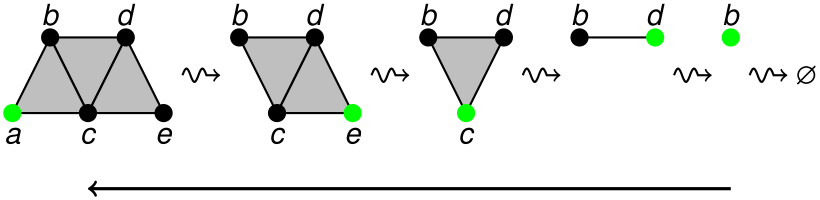

construct the global distribution by glueing
(recall Cecilia Flori & Tobias Fritz talk yesterday)

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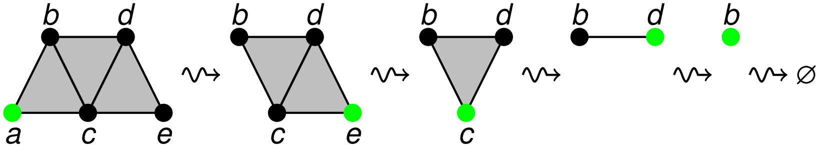
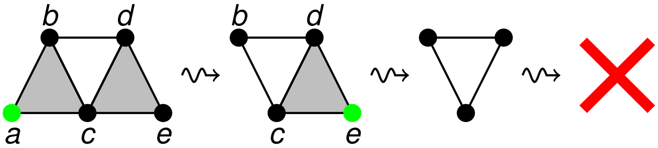


- non-acyclic cover (Graham reduction fails).



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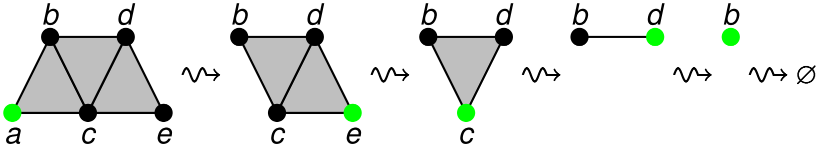

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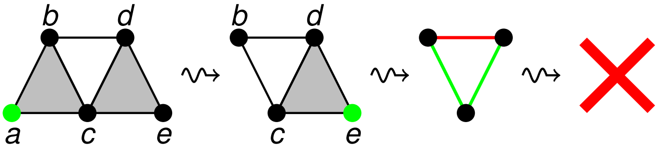


There is a "cycle"!

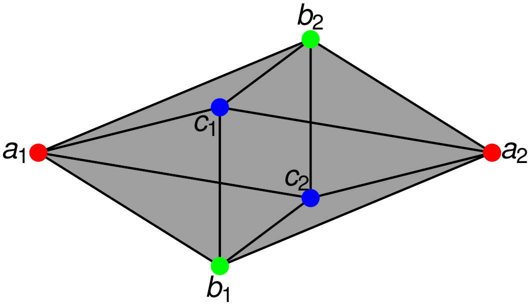
Monogamy of non-locality (and locality of average behaviour)

Tripartite example

Consider a tripartite scenario:

$$X = \{a_1, a_2, b_1, b_2, c_1, c_2\}$$

$$\mathcal{U} = \{\{a_i, b_j, c_k\} \mid i, j, k \in \{1, 2\}\}$$



Tripartite example

- Empirical model: no signalling probabilities

$$p(x, y, z \mid a_i, b_j, c_k)$$

where x, y, z are possible outcomes.

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- Consider the subsystem composed of A and B only, given by marginalisation (in QM, partial trace):

$$p(x, y \mid a_i, b_j) = \sum_z p(x, y, z \mid a_i, b_j, c_k)$$

(this is independent of c_k due to no-signalling).

Similarly define $p(x, z \mid a_i, c_k)$. (A and C)

Tripartite example: monogamy of non-locality

Take a **(general) Bell inequality** for a bipartite scenario: a set of coefficients $\alpha(i, j, x, y)$ and a bound R.

- Applied to the partial system A, B:

$$\sum_{i,j,x,y} \alpha(i, j, x, y) p(x, y \mid a_i, b_j) \leq R$$

- Applied to the partial system A, C:

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$$\sum_{i,j,x,y} \alpha(i,j,x,y) p(x,y \mid a_i, c_j) \leq R$$

- Monogamy relation:

$$\sum_{i,j,x,y} \alpha(i,j,x,y) p(x,y \mid a_i, b_j) + \sum_{i,j,x,y} \alpha(i,j,x,y) p(x,y \mid a_i, c_j) \leq 2R$$

Tripartite example: average macroscopic scenario

- Ramanathan et al.: a **macroscopic scenario** is obtained from an underlying microscopic scenario by **lumping together** certain measurements
  (e.g. spins in a given direction of several particles give rise to a magnetisation measurement in that direction).
  Such merged measurements must be 'symmetric' wrt the compatibility structure.
- Consider B and C to be in the same 'macroscopic' site: the symmetry identifies the measurements $b_1 \sim c_1$ and $b_2 \sim c_2$, giving rise to "macroscopic" measurements m_1 and m_2.

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- They consider the emergent 'macroscopic' average behaviour; probabilities given by an **average**:

$$p(a_i, m_j = x, y) = \frac{1}{2}\Big(p(x, y \mid a_i, b_j) + p(x, y \mid a_i, c_j) \Big)$$

Monogamy and locality of quotient model

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$$\Leftrightarrow$$

$$\sum_{i,j,x,y} \frac{1}{2} \alpha(i,j,x,y) \Big(p(x,y \mid a_i, b_j) + p(x,y \mid a_i, c_j) \Big) \leq R$$

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The quotient model $p(a_i, m_j = \cdots)$ **satisfies the inequality** if and only if Alice in the microscopic model is **monogamous** with respect to violating it with Bob and Charlie.

Example: W state

Z and X measurements on the W state:

	000	001	010	011	100	101	110	111
$a_1 b_1 c_1$	9	1	1	1	1	1	1	9
$a_1 b_1 c_2$	8	2	0	2	0	2	8	2
$a_1 b_2 c_1$	8	0	2	2	0	8	2	2
$a_1 b_2 c_2$	4	4	4	0	4	4	4	0
$a_2 b_1 c_1$	8	0	0	8	2	2	2	2
$a_2 b_1 c_2$	4	4	4	4	4	0	4	0
$a_2 b_2 c_1$	4	4	4	4	4	4	0	0
$a_2 b_2 c_2$	0	8	8	0	8	0	0	0

(every entry should be divided by 24)

Example: W state

	00	01	10	11
$a_1 m_1$	10	2	2	10
$a_1 m_2$	8	4	8	4
$a_2 m_1$	8	8	4	4
$a_2 m_2$	8	8	8	0

(every entry should be divided by 24)

This is **local**!

Another example model

	000	001	010	011	100	101	110	111
$a_1 b_1 c_1$	1	1	0	0	0	0	1	1
$a_1 b_1 c_2$	1	1	0	0	0	0	1	1
$a_1 b_2 c_1$	1	1	0	0	0	0	1	1
$a_1 b_2 c_2$	1	1	0	0	0	0	1	1
$a_2 b_1 c_1$	1	1	0	0	0	0	1	1
$a_2 b_1 c_2$	1	1	0	0	0	0	1	1
$a_2 b_2 c_1$	0	0	1	1	1	1	0	0
$a_2 b_2 c_2$	0	0	1	1	1	1	0	0

(every entry should be divided by 4)

Another example model

	00	01	10	11
$a_1 b_1$	2	0	0	2
$a_1 b_2$	2	0	0	2
$a_2 b_1$	2	0	0	2
$a_2 b_2$	0	2	2	0

(divided by 4)

maximally non-local

	00	01	10	11
$a_1 c_1$	1	1	1	1
$a_1 c_2$	1	1	1	1
$a_2 c_1$	1	1	1	1
$a_2 c_2$	1	1	1	1

(divided by 4)

local

	00	01	10	11
$a_1 m_1$	3	1	1	3
$a_1 m_1$	3	1	1	3
$a_1 m_1$	3	1	1	3
$a_1 m_1$	1	3	3	1

(every entry should be divided by 8)

Again, this is **local**!

Structural Reason

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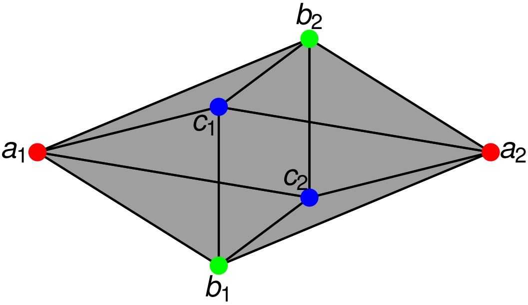

- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.

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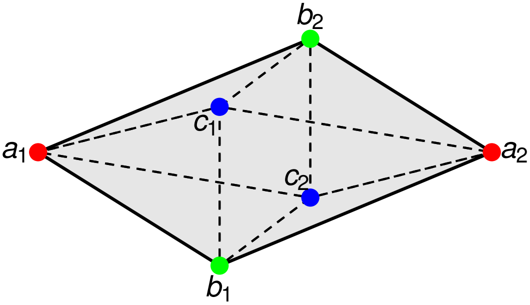

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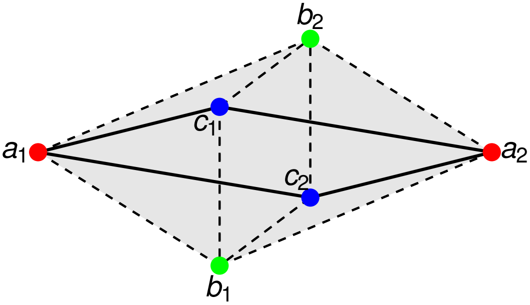

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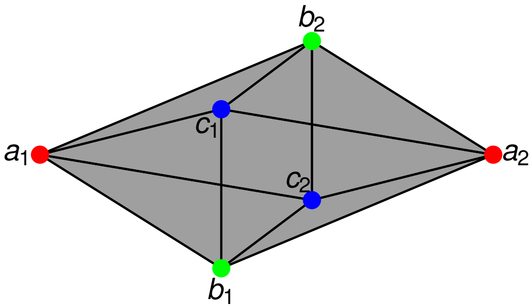

- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.
- We identify B and C: $b_1 \sim c_1$, $b_2 \sim c_2$.
- The macro scenario arises as a quotient.

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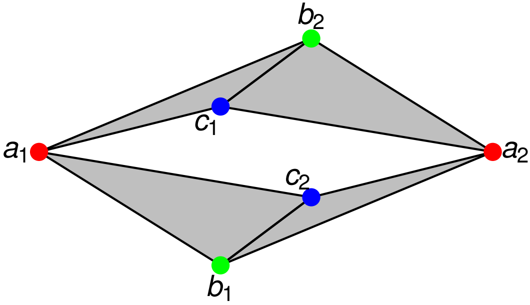

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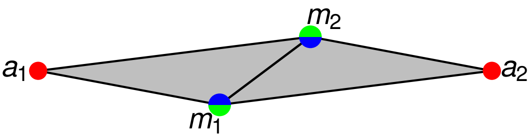

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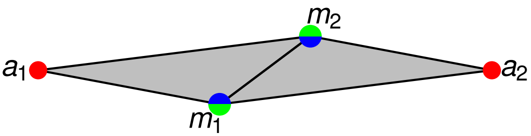

- This quotient complex is **acyclic.**
- Therefore, no matter which micro model $p(a_i, b_j, c_k = \cdots)$ we start from, the averaged macro correlations $p(a_i, m_j = \cdots)$ are local.
- In particular, they satisfy any Bell inequality. Hence, the original tripartite model also satisfies a **monogamy relation** for any Bell inequality.

General multipartite scenarios

- *n*-partite Bell inequality;
- k_i measurement settings in site i.

$B(a, b, c, \ldots) =$

$$\sum_{i_1=1}^{k_i} \cdots \sum_{i_n=1}^{k_n} \sum_{o_1,\ldots,o_n} \alpha(i_1, \ldots, i_n, o_1, \ldots, o_n) p(o_1, \ldots, o_n | a_{i_1}, b_{i_2}, c_{i_3}, \ldots) \leq R$$

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- take r_i 'micro' sites on each 'macro' site i. $a^{(1)}, \ldots, a^{(r_1)}, b^{(1)}, \ldots, b^{(r_2)}, \ldots$.
- the quotient scenario is acyclic when either of these holds:
  - $r_1 = 1$ and $\forall_{i=2,\ldots,n}.\ r_i \geq k_i$;
  - $\forall_{i=1,\ldots,n}.\ r_i \geq k_i$.
- We get monogamy relations

$$\sum_{j_1=1}^{r_1} \cdots \sum_{j_n=1}^{r_n} B(a^{(j_1)}, b^{(j_2)}, c^{(j_3)}, \ldots) \leq R \prod_i r_i$$

Summary/Conclusions

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- A model satisfies a **monogamy** relation for a Bell inequality iff the macro average correlations (quotient model) satisfy the Bell inequality.
- So, if the quotient scenario is acyclic, then **any no-signalling empirical model** is monogamous wrt to all Bell inequalities (since the average correlations, being defined in this quotient scenario, must be local/non-contextual).

Summary/Conclusions

- In particular, we proved that this is the case for multipartite Bell-type scenarios provided the number of parties being identified as belonging to each 'macro' site is larger than the number of measurement settings available to each of them.
- Our approach highlights the **reason why** monogamy relations for general multipartite Bell inequalities follow from no-signalling alone, generalising the result of Pawłowski and Brukner (2009) from bipartite to multipartite. It also shows that what Ramanathan et al. proved holds not only for QM but for any no-signalling theory.
- The approach is not restricted to multipartite Bell-type scenarios. More generally, we can apply the same ideas to derive monogamy relations for contextuality inequalities.

Questions...

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