# A structural reason for monogamy and locality of average macroscopic behaviour

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# Overview

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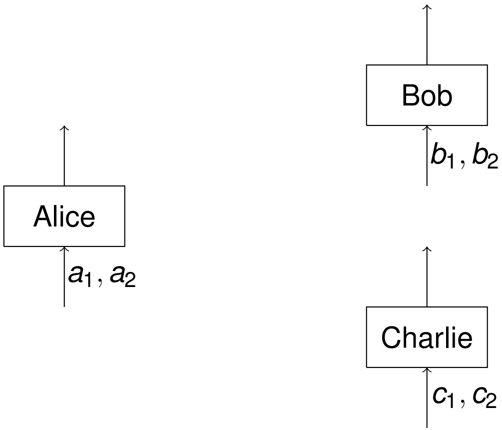

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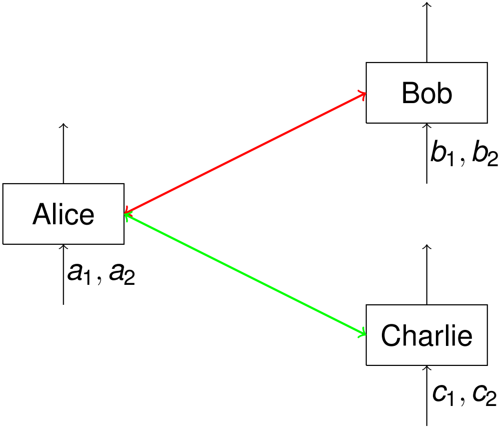

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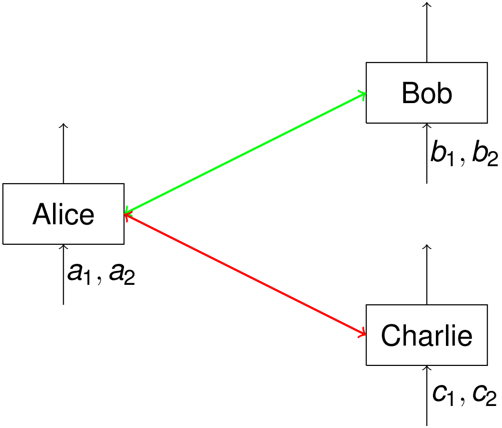

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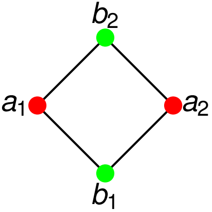
- General framework of Abramsky & Brandenburger (2011):
- Generalise the results above;
- Provide a structural explanation related to Vorob'ev's theorem (1962).
- In this talk, we'll mainly consider a simple illustrative example.

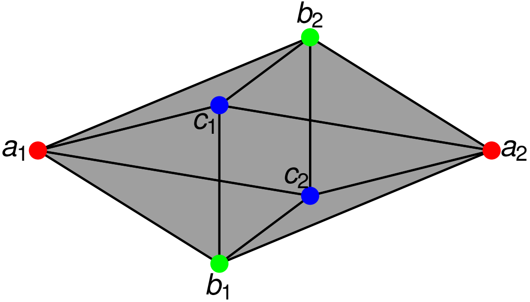
# The setting
(recap of Samson's and Shane's talks)

# Measurement Scenarios

Abramsky-Brandenburger framework

- a finite set of measurements $X$;
- a cover $\mathcal{U}$ of $X$ (or an abstract simplicial complex $\Sigma$ on $X$), indicating the **compatibility** of measurements.




Examples: Bell-type scenarios, KS configurations, and more.

# Empirical models

a family $(p_C)_{C \in \mathcal{U}}$, where $p_C$ is a probability distribution on the outcomes of measurements in context $C$.

E.g. $Z$ and $X$ measurements on the $W$ state:

| | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| $a_1 b_1 c_1$ | 9 | 1 | 1 | 1 | 1 | 1 | 1 | 9 |
| $a_1 b_1 c_2$ | 8 | 2 | 0 | 2 | 0 | 2 | 8 | 2 |
| $a_1 b_2 c_1$ | 8 | 0 | 2 | 2 | 0 | 8 | 2 | 2 |
| $a_1 b_2 c_2$ | 4 | 4 | 4 | 0 | 4 | 4 | 4 | 0 |
| $a_2 b_1 c_1$ | 8 | 0 | 0 | 8 | 2 | 2 | 2 | 2 |
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(every entry should be divided by 24)

# The no-signalling condition

- Suppose Alice and Bob are space-like separated;
- Alice chooses to measure $a_1$; Bob can choose $b_1$ or $b_2$.
- What is $p_{a_1}(x)$ (prob of Alice obtaining the outcome $x$ when measuring $a_1$)?

$$p_{a_1,b_1}(x) := \sum_y p_{a_1,b_1}(x,y)$$

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- Relativity implies that her measurement statistics cannot depend on Bob's choice of measurement:

$$p_{a_1,b_1}(x) = p_{a_1,b_2}(x)$$

I.e. it makes sense to speak of $p_{a_1}(x)$.

# The no-signalling condition

In general, we require that our empirical models $(p_C)_{C \in \mathcal{U}}$ satisfy a compatibility condition:

| |
|---|
| $p_C$ and $p_{C'}$ marginalise to the same distribution on the outcomes of measurements in $C \cap C'$. |

For Bell-type multipartite scenarios, this condition corresponds to the usual **no-signalling**.

# Non-locality and Contextuality

We are interested in whether a given empirical model admits a **local/non-contextual hidden variable** explanation (in the sense of Bell's theorem).

This is equivalent to the existence of a **global distribution** $p_X$ (i.e. for all measurements at the same time) that marginalises to all $p_C$. (Abramsky, Brandenburger 2011).

Obstructions to such extensions are witnessed by **general Bell inequalities**. E.g. in bipartite scenario:

$$\sum_{i,j,x,y} \alpha(i,j,x,y) p_{a_i,b_j}(x,y) \leq R$$

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For which measurement compatibility structures $\mathcal{U}$ (or $\Sigma$) is it so that **any** no-signalling empirical model admits a global extension, i.e. is local/non-contextual?

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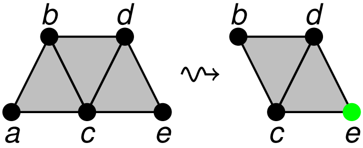
For which measurement compatibility structures $\mathcal{U}$ (or $\Sigma$) is it so that **any** no-signalling empirical model admits a global extension, i.e. is local/non-contextual?

Vorob'ev (1962) derived a **necessary and sufficient** combinatorial condition on $\Sigma$ (or $\mathcal{U}$) for this to be the case.
It turns out to be equivalent to the notion of acyclicity of a database schema.

# Acyclicity

- Graham reduction step: delete a measurement that belongs to only one maximal context.



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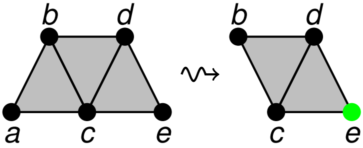
- A cover is **acyclic** when it is Graham reducible to $\varnothing$.



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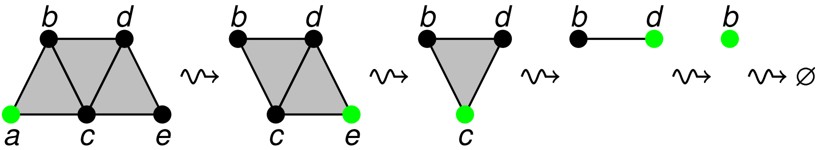

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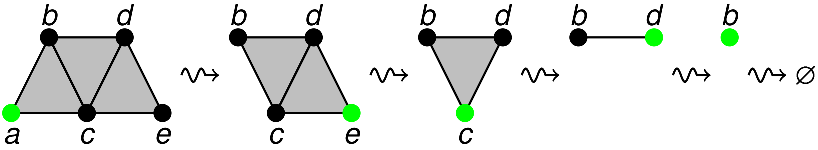


Theorem (Vorob'ev 1962, adapted)

*All empirical models on $\Sigma$ are extendable **iff** $\Sigma$ is acyclic*

# Glueing (natural join)

Consider three random variables (measurements) $a$, $b$, and $c$. Suppose we are given distributions $P_{ab}$ over $\{a, b\}$ and $P_{bc}$ over $\{b, c\}$ which are compatible on $b$:

$$\sum_{x \in O} P_{ab}(x, y) = \sum_{z \in O} P_{bc}(y, z)$$

so it makes sense to refer to $P_b(y)$.

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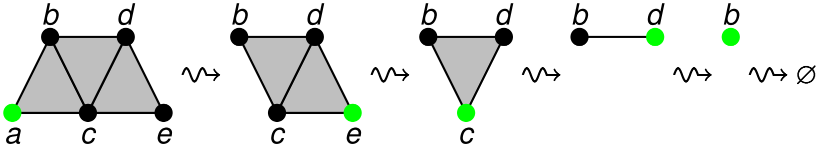
Note: in the possibilistic case, this is the natural join from database theory!

# Sketch of proof of Vorob'ev's theorem

- If $\Sigma$ is acyclic,



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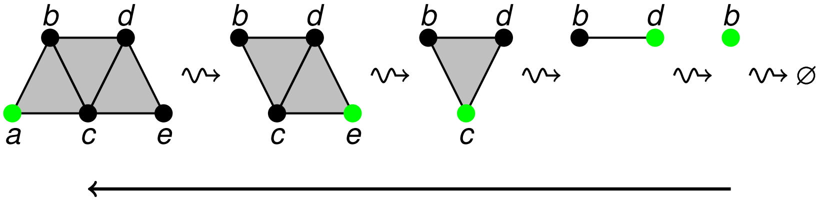


then construct a global distribution by glueing

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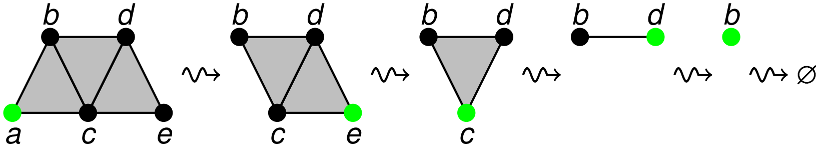
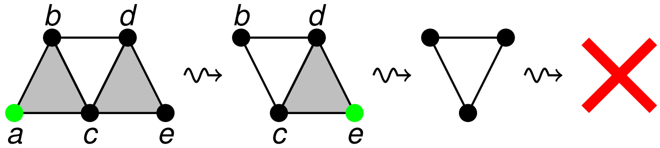



- If Σ is not acyclic (Graham reduction fails).



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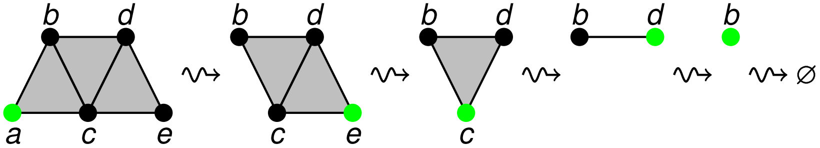


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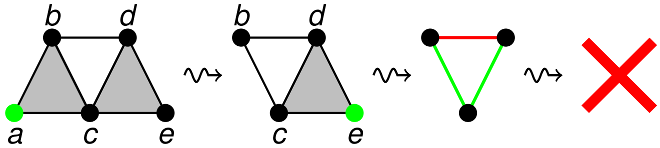



There is a "cycle"!

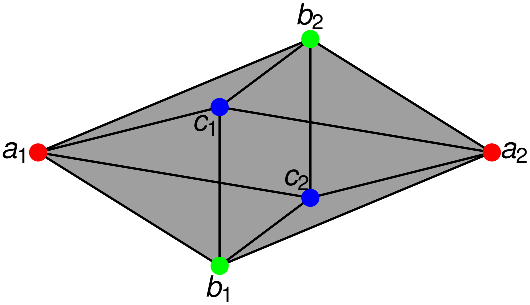
# Monogamy of non-locality (and locality of average behaviour)

# Tripartite example

Consider a tripartite scenario:

$$X = \{a_1, a_2, b_1, b_2, c_1, c_2\}$$

$$\mathcal{U} = \{\{a_i, b_j, c_k\} \mid i, j, k \in \{1, 2\}\}$$



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- Empirical model: no signalling probabilities

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- Consider the subsystem composed of $A$ and $B$ only, given by marginalisation (in QM, partial trace):

$$p_{a_i,b_j}(x,y) = \sum_z p_{a_i,b_j,c_k}(x,y,z)$$

(this is independent of $c_k$ due to no-signalling).

Similarly define $p_{a_i,c_k}(x,z)$. (A and C)

# Tripartite example: monogamy of non-locality

Take a **(general) Bell inequality** for a bipartite scenario:
a set of coefficients $\alpha(i, j, x, y)$ and a bound $R$.

- Applied to the partial system $A, B$:

$$\sum_{i,j,x,y} \alpha(i, j, x, y) p_{a_i, b_j}(x, y) \leq R$$

- Applied to the partial system $A, C$:

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- Monogamy relation:

$$\sum_{i,j,x,y} \alpha(i, j, x, y) p_{a_i, b_j}(x, y) + \sum_{i,j,x,y} \alpha(i, j, x, y) p_{a_i, c_j}(x, y) \leq 2R$$

# Tripartite example: average macroscopic scenario

- Ramanathan et al.: a **macroscopic scenario** is obtained from an underlying microscopic scenario by **lumping together** certain measurements
  (e.g. spins in a given direction of several particles give rise to a magnetisation measurement in that direction).
  Such merged measurements must be 'symmetric' wrt the compatibility structure.
- Consider $B$ and $C$ to be in the same 'macroscopic' site: the symmetry identifies the measurements $b_1 \sim c_1$ and $b_2 \sim c_2$, giving rise to "macroscopic" measurements $m_1$ and $m_2$.

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- They consider the 'macroscopic' average behaviour; probabilities given by an **average**:

$$p_{a_i,m_j}(x,y) = \frac{1}{2}\left( p_{a_i,b_j}(x,y) + p_{a_i,c_j}(x,y) \right)$$

# Monogamy and locality of quotient model

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$$\Leftrightarrow$$

$$\sum_{i,j,x,y} \frac{1}{2}\alpha(i,j,x,y) \left( p_{a_i,b_j}(x,y) + p_{a_i,c_j}(x,y) \right) \leq R$$

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$$\Leftrightarrow$$

$$\sum_{i,j,x,y} \alpha(i,j,x,y) p_{a_i,b_j}(x,y) + \sum_{i,j,x,y} \alpha(i,j,x,y) p_{a_i,c_j}(x,y) \leq 2R$$

The quotient model $p_{a_i,m_j}$ **satisfies the inequality** if and only if Alice in the microscopic model is **monogamous** with respect to violating it with Bob and Charlie.

# Example: $W$ state

$Z$ and $X$ measurements on the $W$ state:

| | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| $a_1 b_1 c_1$ | 9 | 1 | 1 | 1 | 1 | 1 | 1 | 9 |
| $a_1 b_1 c_2$ | 8 | 2 | 0 | 2 | 0 | 2 | 8 | 2 |
| $a_1 b_2 c_1$ | 8 | 0 | 2 | 2 | 0 | 8 | 2 | 2 |
| $a_1 b_2 c_2$ | 4 | 4 | 4 | 0 | 4 | 4 | 4 | 0 |
| $a_2 b_1 c_1$ | 8 | 0 | 0 | 8 | 2 | 2 | 2 | 2 |
| $a_2 b_1 c_2$ | 4 | 4 | 4 | 4 | 4 | 0 | 4 | 0 |
| $a_2 b_2 c_1$ | 4 | 4 | 4 | 4 | 4 | 4 | 0 | 0 |
| $a_2 b_2 c_2$ | 0 | 8 | 8 | 0 | 8 | 0 | 0 | 0 |

(every entry should be divided by 24)

# Example: $W$ state

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $a_1 m_1$ | 10 | 2 | 2 | 10 |
| $a_1 m_2$ | 8 | 4 | 8 | 4 |
| $a_2 m_1$ | 8 | 8 | 4 | 4 |
| $a_2 m_2$ | 8 | 8 | 8 | 0 |

(every entry should be divided by 24)

This is **local**!

# Another example model

| | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| $a_1 b_1 c_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_1 b_1 c_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_1 b_2 c_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_1 b_2 c_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_2 b_1 c_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_2 b_1 c_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_2 b_2 c_1$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| $a_2 b_2 c_2$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |

(every entry should be divided by 4)

# Another example model

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $a_1 b_1$ | 2 | 0 | 0 | 2 |
| $a_1 b_2$ | 2 | 0 | 0 | 2 |
| $a_2 b_1$ | 2 | 0 | 0 | 2 |
| $a_2 b_2$ | 0 | 2 | 2 | 0 |

(divided by 4)

maximally non-local

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $a_1 c_1$ | 1 | 1 | 1 | 1 |
| $a_1 c_2$ | 1 | 1 | 1 | 1 |
| $a_2 c_1$ | 1 | 1 | 1 | 1 |
| $a_2 c_2$ | 1 | 1 | 1 | 1 |

(divided by 4)

local

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $a_1 m_1$ | 3 | 1 | 1 | 3 |
| $a_1 m_1$ | 3 | 1 | 1 | 3 |
| $a_1 m_1$ | 3 | 1 | 1 | 3 |
| $a_1 m_1$ | 1 | 3 | 3 | 1 |

(every entry should be divided by 8)

Again, this is **local**!

# Structural Reason

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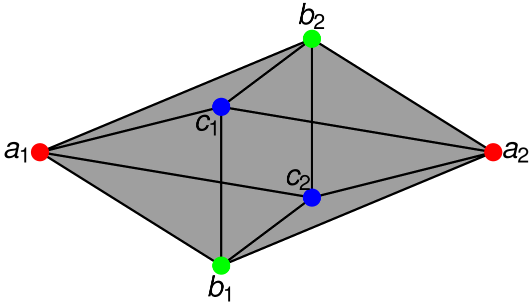


- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.

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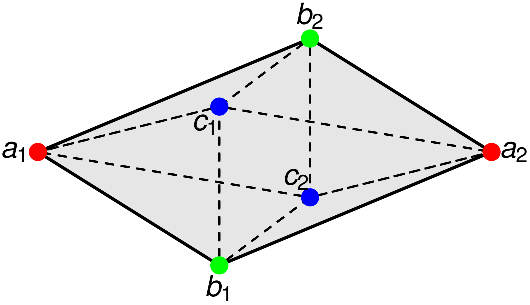


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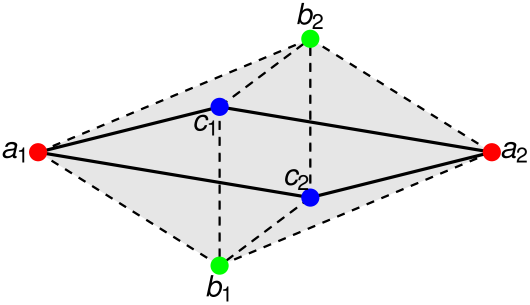


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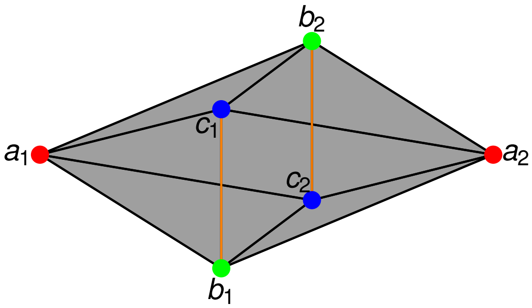


- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.
- We identify $B$ and $C$: $b_1 \sim c_1$, $b_2 \sim c_2$.
- The macro scenario arises as a quotient.

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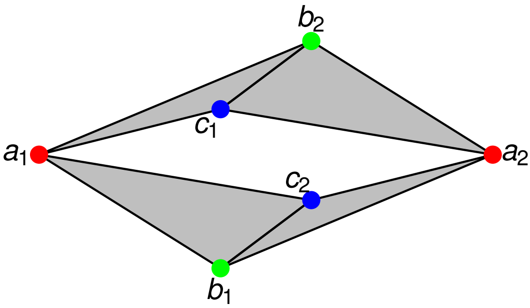


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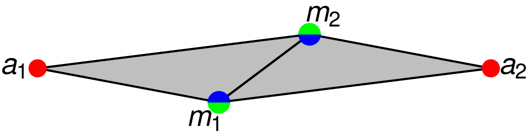


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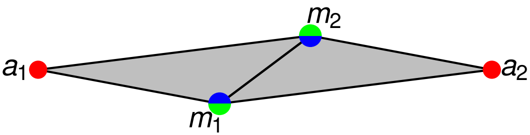


- This quotient complex is **acyclic.**
- Therefore, no matter which micro model $p_{a_i,b_j,c_k}$ we start from, the averaged macro correlations $p_{a_i,m_j}$ are local.
- In particular, they satisfy any Bell inequality. Hence, the original tripartite model also satisfies a **monogamy relation** for any Bell inequality.

# General multipartite scenarios

- $n$-partite Bell inequality;
- $k_i$ measurement settings in site $i$. $\mathfrak{D}_{k_1} * \cdots * \mathfrak{D}_{k_n}$

$$B(a, b, c, \ldots) =$$

$$\sum_{i_1=1}^{k_i} \cdots \sum_{i_n=1}^{k_n} \sum_{o_1, \ldots, o_n} \alpha(i_1, \ldots, i_n, o_1, \ldots, o_n) p_{a_{i_1}, b_{i_2}, c_{i_3}, \ldots}(o_1, \ldots, o_n) \leq R$$

# General multipartite scenarios

- $n$-partite Bell inequality;
- $k_i$ measurement settings in site $i$. $\mathfrak{D}_{k_1} * \cdots * \mathfrak{D}_{k_n}$

$B(a, b, c, \ldots) =$

$$\sum_{i_1=1}^{k_i} \cdots \sum_{i_n=1}^{k_n} \sum_{o_1,\ldots,o_n} \alpha(i_1, \ldots, i_n, o_1, \ldots, o_n) p_{a_{i_1}, b_{i_2}, c_{i_3}, \ldots}(o_1, \ldots, o_n) \leq R$$

- take $r_i$ copies at each site $i$:

$$a^{(1)}, \ldots, a^{(r_1)}, b^{(1)}, \ldots, b^{(r_2)}, \ldots$$

  scenario: $\mathfrak{D}_{k_1}^{(*r_1)} * \cdots * \mathfrak{D}_{k_n}^{(*r_n)}$.
- symmetry by $S_{r_1} \times \cdots \times S_{r_n}$ identifies the copies at each site.

# General multipartite scenarios

- the quotient scenario is acyclic when either of these holds:
  - $r_1 = 1$ and $\forall_{i=2,\ldots,n}.\ r_i \geq k_i$;
  - $\forall_{i=1,\ldots,n}.\ r_i \geq k_i$.
- We get monogamy relations

$$\sum_{j_1=1}^{r_1} \cdots \sum_{j_n=1}^{r_n} B(a^{(j_1)}, b^{(j_2)}, c^{(j_3)}, \ldots) \leq R \prod_i r_i$$

# Summary/Conclusions

# Summary/Conclusions

- A symmmetry ($G$-action) on $\Sigma$ identifies measurements.
- A model satisfies a $G$-**monogamy** relation for a Bell inequality iff the macro average correlations (quotient model by $G$) satisfy the Bell inequality.

- So, if the quotient scenario is acyclic, then **any no-signalling empirical model** is $G$-monogamous wrt to all Bell inequalities (since the average correlations, being defined in this quotient scenario, must be local/non-contextual).

# Summary/Conclusions

- In particular, we proved that this is the case for multipartite Bell-type scenarios provided the number of parties being identified as belonging to each ’macro’ site is larger than the number of measurement settings available to each of them.
- Our approach highlights the **reason why** monogamy relations for general multipartite Bell inequalities follow from no-signalling alone, generalising the result of Pawłowski and Brukner (2009) from bipartite to multipartite. It also shows that what Ramanathan et al. proved holds not only for QM but for any no-signalling theory.
- The approach is not restricted to multipartite Bell-type scenarios. More generally, we can apply the same ideas to derive monogamy relations for contextuality inequalities.

# Questions...

# ?

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