On monogamy of non-locality and macroscopic averages

(examples and preliminary results)

Rui Soares Barbosa

Quantum Group
Department of Computer Science
University of Oxford

rui.soares.barbosa@cs.ox.ac.uk

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Overview

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  (Pawłowski & Brukner 2009: bipartite Bell inequalities)

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- General framework of Abramsky & Brandenburger (2011):
  - generalise the results above
  - provide a structural explanation related to Vorob'ev's theorem (1962)

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- Average macro correlations arising from micro models
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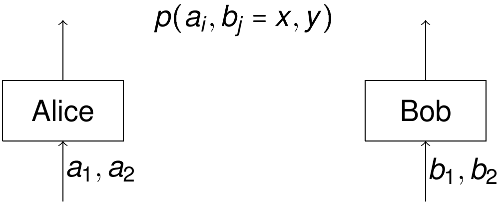
- General framework of Abramsky & Brandenburger (2011):
  - generalise the results above
  - provide a structural explanation related to Vorob'ev's theorem (1962)
- This talk: we mainly consider a simple illustrative example.

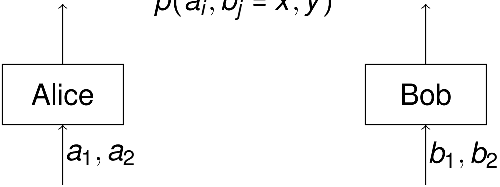
Monogamy of non-locality

Non-locality



Non-locality

$$p(a_i, b_j = x, y)$$



	00	01	10	11
$a_1 b_1$	$1/2$	0	0	$1/2$
$a_1 b_2$	$3/8$	$1/8$	$1/8$	$3/8$
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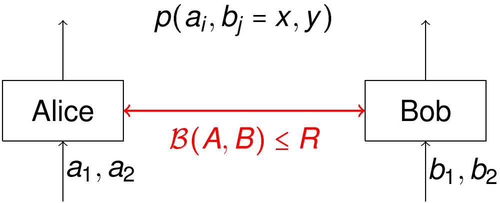

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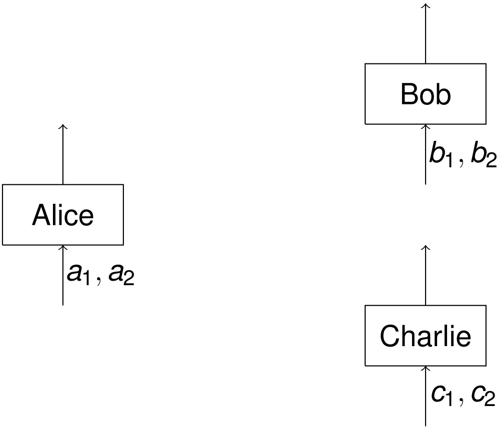
Monogamy of non-locality



Monogamy of non-locality

- Empirical model: no signalling probabilities

$$p(a_i, b_j, c_k = x, y, z)$$

where x, y, z are possible outcomes.

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where x, y, z are possible outcomes.

- Consider the subsystem composed of A and B only, given by marginalisation (in QM, partial trace):

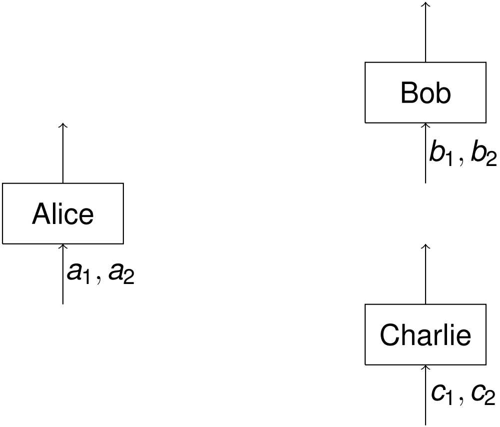
$$p(a_i, b_j = x, y) = \sum_z p(a_i, b_j, c_k = x, y, z)$$

(this is independent of c_k due to no-signalling).

Similarly define $p(a_i, c_k = x, z)$. (A and C)

Monogamy of non-locality

Given a Bell inequality $\mathcal{B}(-,-,) \leq R$,



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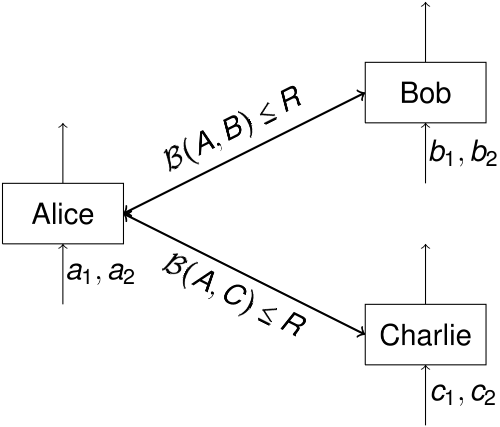

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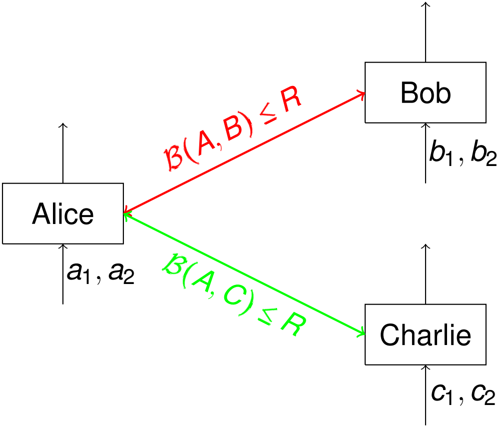

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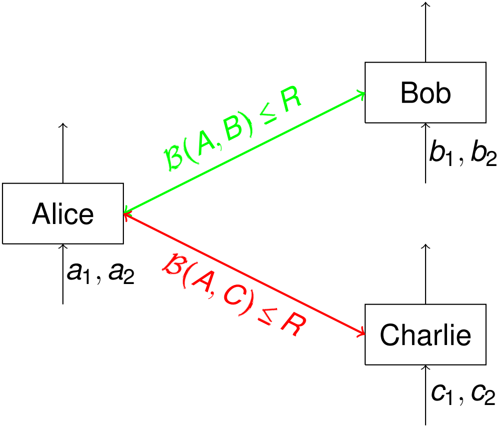

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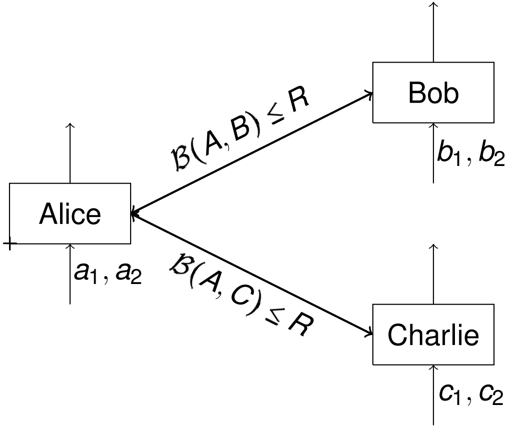

Monogamy relation: $\mathcal{B}(A, B) + \mathcal{B}(A, C) \leq 2R$

Macroscopic average behaviour

Macroscopic measurements

- (Micro) dichotomic measurement: a single particle is subject to an interaction *a* and collides with one of two detectors: outcomes 0 and 1.
- The interaction is probabilistic: $p(a = x)$, $x = 0, 1$.

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- Consider beam (or region) of N particles, differently prepared.
- Subject each particle to the interaction a: the beam gets divided into 2 smaller beams hitting each of the detectors.
- Outcome represented by the intensity of resulting beams: $I_a \in [0, 1]$ proportion of particles hitting the detector 1.
- We are concerned with the mean, or expected, value of such intensities.

Macroscopic average behaviour

- This mean intensity can be interpreted as the average behaviour among the particles in the beam or region: if we would randomly select one of the N particles and subject it to the microscopic measurement a, we would get outcome 1 with probability I_a:

$$I_a = \sum_{i=1}^{N} p_i(a = 1) \; .$$

- The situation is analogous to statistical mechanics, where a macrostate arises as an averaging over an extremely large number of microstates, and hence several different microstates can correspond to the same macrostate.

Macroscopic average behaviour: multipartite

- Multipartite macroscopic measurements:
  - several 'macroscopic' sites consisting of a large number of microscopic sites/particles;
  - several (macro) measurement settings at each site.
- Average macroscopic Bell experiment: the (mean) values of the macroscopic intensities indicate the behaviour of a randomly chosen tuple of particles: one from each of the beams, or sites.

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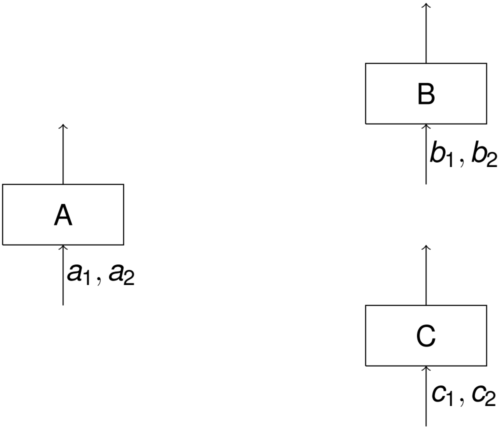
- Average macroscopic Bell experiment: the (mean) values of the macroscopic intensities indicate the behaviour of a randomly chosen tuple of particles: one from each of the beams, or sites.
- We shall show that, as long as there are enough particles (microscopic sites) in each macroscopic site, such average macroscopic behaviour is always local no matter which no-signalling model accounts for the underlying microscopic correlations.

Macroscopic average behaviour: tripartite example

- Consider again the tripartite scenario:



Macroscopic average behaviour: tripartite example

- Consider again the tripartite scenario.
- We regard sites B and C as forming one ‘macroscopic’ site, M, and site A as forming another.
- In order to be ‘lumped together’, B and C must be symmetric/of the same type: the symmetry identifies the measurements $b_1 \sim c_1$ and $b_2 \sim c_2$, giving rise to ‘macroscopic’ measurements m_1 and m_2.

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- Given an empirical model $p(a_i, b_j, c_k = x, y, z)$, the 'macroscopic' average behaviour is a bipartite model (with two macro sites A and M) given by the following average of probabilities of the partial models:

$$p_{a_i,m_j}(x,y) = \frac{p_{a_i,b_j}(x,y) + p_{a_i,c_j}(x,y)}{2}$$

Example: W state

Z and X measurements on the W state:

	000	001	010	011	100	101	110	111
$a_1 b_1 c_1$	9	1	1	1	1	1	1	9
$a_1 b_1 c_2$	8	2	0	2	0	2	8	2
$a_1 b_2 c_1$	8	0	2	2	0	8	2	2
$a_1 b_2 c_2$	4	4	4	0	4	4	4	0
$a_2 b_1 c_1$	8	0	0	8	2	2	2	2
$a_2 b_1 c_2$	4	4	4	4	4	0	4	0
$a_2 b_2 c_1$	4	4	4	4	4	4	0	0
$a_2 b_2 c_2$	0	8	8	0	8	0	0	0

(every entry should be divided by 24)

Example: W state

	00	01	10	11
$a_1 m_1$	10	2	2	10
$a_1 m_2$	8	4	8	4
$a_2 m_1$	8	8	4	4
$a_2 m_2$	8	8	8	0

(every entry should be divided by 24)

This is **local**!

Another example model

	000	001	010	011	100	101	110	111
$a_1 b_1 c_1$	1	1	0	0	0	0	1	1
$a_1 b_1 c_2$	1	1	0	0	0	0	1	1
$a_1 b_2 c_1$	1	1	0	0	0	0	1	1
$a_1 b_2 c_2$	1	1	0	0	0	0	1	1
$a_2 b_1 c_1$	1	1	0	0	0	0	1	1
$a_2 b_1 c_2$	1	1	0	0	0	0	1	1
$a_2 b_2 c_1$	0	0	1	1	1	1	0	0
$a_2 b_2 c_2$	0	0	1	1	1	1	0	0

(every entry should be divided by 4)

Another example model

	00	01	10	11
$a_1 b_1$	2	0	0	2
$a_1 b_2$	2	0	0	2
$a_2 b_1$	2	0	0	2
$a_2 b_2$	0	2	2	0

(divided by 4)

	00	01	10	11
$a_1 c_1$	1	1	1	1
$a_1 c_2$	1	1	1	1
$a_2 c_1$	1	1	1	1
$a_2 c_2$	1	1	1	1

(divided by 4)

maximally non-local

local

	00	01	10	11
$a_1 m_1$	3	1	1	3
$a_1 m_1$	3	1	1	3
$a_1 m_1$	3	1	1	3
$a_1 m_1$	1	3	3	1

(every entry should be divided by 8)

Again, this is **local**!

Monogamy and macroscopic averages

A simple observation

Consider any bipartite Bell inequality $\mathcal{B}(-,-) \leq R$, given by a set of coefficients $\alpha(i, j, x, y)$ and a bound R.

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$$\mathcal{B}(A,M) \leq R$$

$$\Leftrightarrow$$

$$\sum_{i,j,x,y} \alpha(i,j,x,y) p(a_i, m_j = x, y) \leq R$$

$$\Leftrightarrow$$

$$\sum_{i,j,x,y} \alpha(i,j,x,y) \frac{p_(a_i, b_j = x, y) + p(a_i, c_j = x, y)}{2} \leq R$$

$$\Leftrightarrow$$

$$\sum_{i,j,x,y} \alpha(i,j,x,y) p(a_i, b_j = x, y) + \sum_{i,j,x,y} \alpha(i,j,x,y) p(a_i, c_j = x, y) \leq 2R$$

$$\Leftrightarrow$$

$$\mathcal{B}(A,B) + \mathcal{B}(A,C) \leq R$$

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$$\Leftrightarrow$$

$$\mathcal{B}(A,B) + \mathcal{B}(A,C) \leq R$$

The average model p_{a_i,m_j} **satisfies the inequality** if and only if in the microscopic model Alice is **monogamous** with respect to violating it with Bob and Charlie.

A simple observation

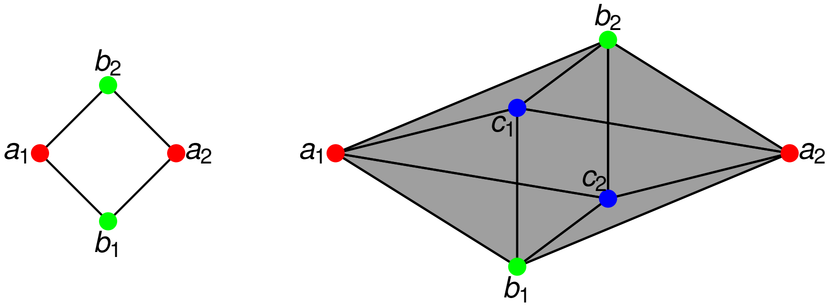
- In the two examples above, the average models were local. Equivalently, the examples satisfied the monogamy relation for **any** Bell inequality.
- This is true for all no-signalling empirical models on the tripartite scenario under consideration, with two measurement settings per site.
- We give a structural explanation for this...

Vorob’ev’s theorem

Abramsky-Brandenburger framework

Measurement scenarios:

- a finite set of measurements X;
- a cover $\mathcal{U}$ of X (or an abstract simplicial complex Σ on X), indicating the **compatibility** of measurements.



Examples: Bell-type scenarios, KS configurations, and more.

Abramsky-Brandenburger framework

No-signalling **empirical model**:

- a family $(p_C)_{C \in \mathcal{U}}$, where p_C is a probability distribution on the outcomes of measurements in context C.
- compatibility condition: p_C and $p_{C'}$ marginalise to the same distribution on the outcomes of measurements in $C \cap C'$. (on multipartite scenarios: no-signalling)

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An empirical model admits a **local/non-contextual hidden variable** explanation (in the sense of Bell's theorem) iff there exists a **global distribution** p_X (i.e. for all measurements at the same time) that marginalises to all the p_C.

Obstructions to such extensions are witnessed by violations of general Bell inequalities.

Vorob'ev's theorem

For which measurement compatibility structures $\mathcal{U}$ (or Σ) is it so that **any** no-signalling empirical model admits a global extension, i.e. is local/non-contextual?

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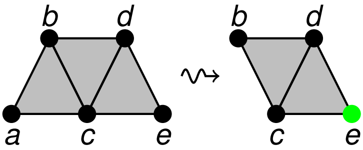
Vorob'ev (1962) derived a **necessary and sufficient** combinatorial condition on Σ (or $\mathcal{U}$) for this to be the case.

- Turns out to be equivalent to the notion of acyclicity of a database schema.

Acyclicity

- Graham reduction step: delete a measurement that belongs to only one maximal context.



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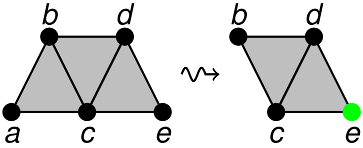

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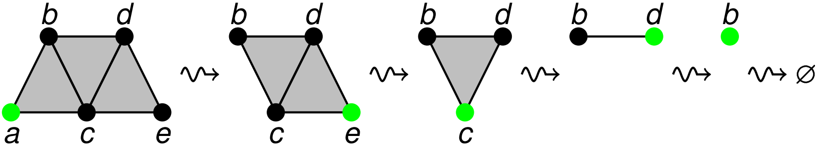

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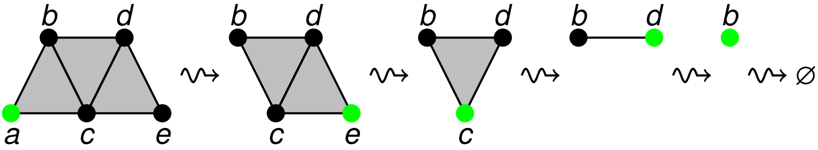
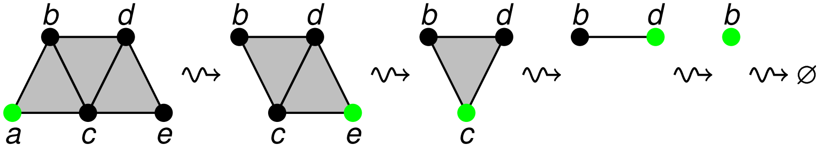


Theorem (Vorob'ev 1962, adapted)

All empirical models on Σ are extendable ***iff*** *Σ is acyclic*

Sketch of proof of Vorob'ev's theorem

- If Σ is acyclic,



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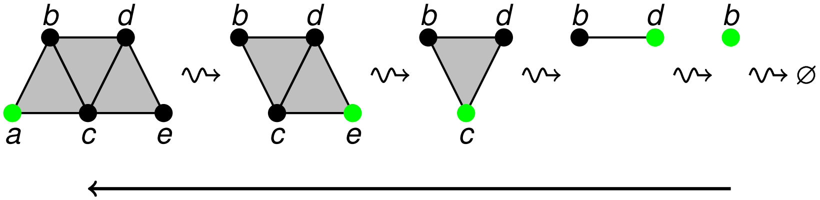

then construct a global distribution by glueing

Given distributions P_{ab} over $\{a, b\}$ and P_{bc} over $\{b, c\}$ compatible on b,

$$\sum_{x \in O} P(a, b = x, y) = \sum_{z \in O} P(b, c = y, z) \ ,$$

we can define an extension:

$$P(a, b, c = x, y, z) = \frac{P(a, b = x, y) P(b, c = y, z)}{P(b = y)} \ .$$

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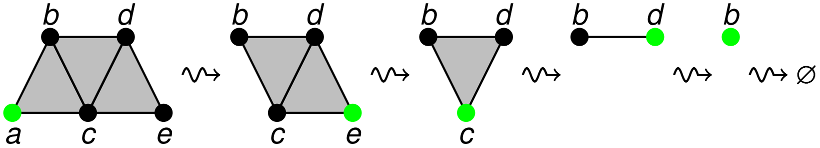
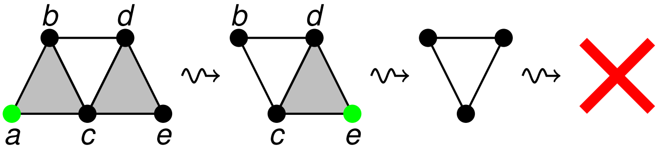


- If Σ is not acyclic (Graham reduction fails).



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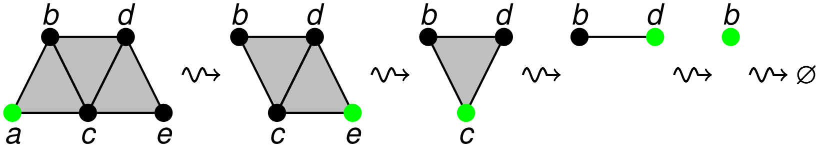

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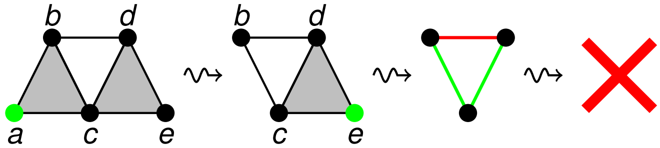
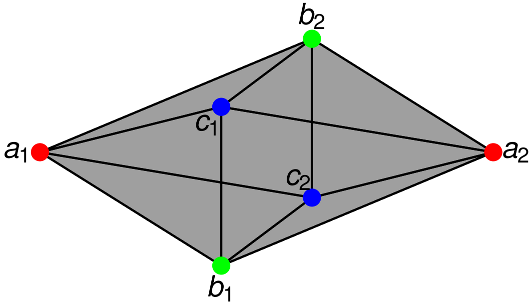


There is a “cycle”!

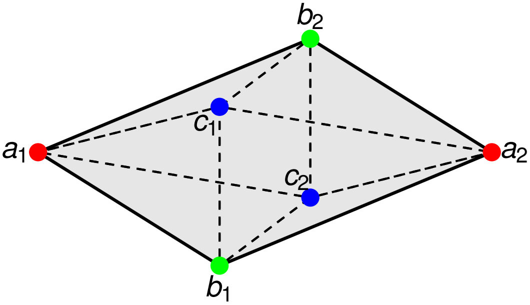
A structural explanation

Structural Reason



- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.

Structural Reason



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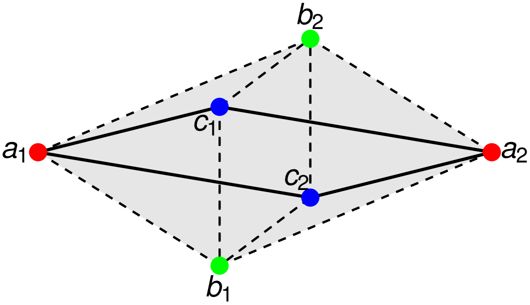

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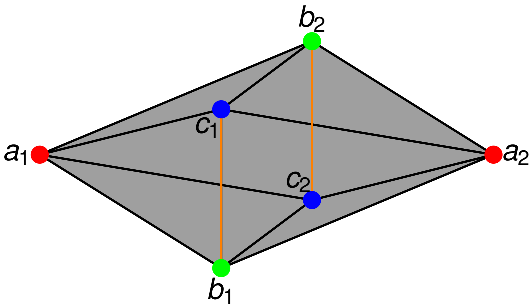

- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.
- We identify B and C: $b_1 \sim c_1$, $b_2 \sim c_2$.
- The macro scenario arises as a quotient.

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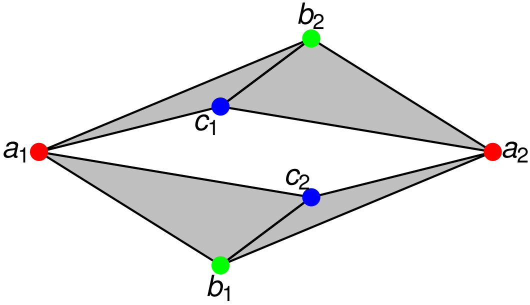

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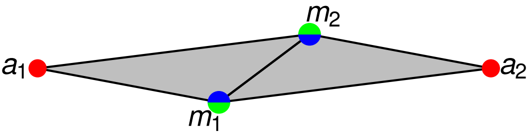

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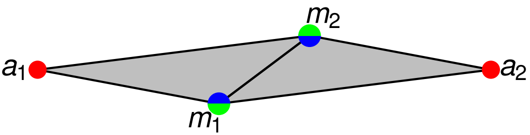

- This quotient complex is **acyclic.**
- Therefore, no matter which micro model p_{a_i,b_j,c_k} we start from, the averaged macro correlations p_{a_i,m_j} are local.
- In particular, they satisfy any Bell inequality. Hence, the original tripartite model also satisfies a **monogamy relation** for any Bell inequality.

General multipartite scenarios

General multipartite scenarios

- n macroscopic sites $A, B, C, \ldots$
- k_i measurement settings at site i
- take r_i copies of each site i, or r_i micro sites constituting i. For a macro site A:
  - copies / micro sites: $A^{(1)}, \ldots, A^{(r_1)}$
  - measurement settings art $A^{(m)}$: $a_1^{(m)}, \ldots, a_{k_A}^{(m)}$

Scenario: $\Sigma_{n,\vec{k},\vec{r}} := \mathfrak{D}_{k_1}^{(*r_1)} * \cdots * \mathfrak{D}_{k_n}^{(*r_n)}$.

General multipartite scenarios

- n macroscopic sites $A, B, C, \ldots$
- k_i measurement settings at site i
- take r_i copies of each site i, or r_i micro sites constituting i. For a macro site A:
  - copies / micro sites: $A^{(1)}, \ldots, A^{(r_1)}$
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  Scenario: $\Sigma_{n,\vec{k},\vec{r}} := \mathfrak{D}_{k_1}^{(*r_1)} * \cdots * \mathfrak{D}_{k_n}^{(*r_n)}$.

- Symmetry by $S_{r_1} \times \cdots \times S_{r_n}$ identifies the copies at each macro site.

$$a_j^{(1)} \sim \cdots \sim a_j^{(r_A)} \quad (\forall j \in \{1, \ldots, k_A\}),$$
$$b_j^{(1)} \sim \cdots \sim a_j^{(r_A)} \quad (\forall j \in \{1, \ldots, k_A\}),$$

etc.

General multipartite scenarios

Proposition

The quotient of the measurement scenario $\Sigma_{n,\vec{k},\vec{r}}$ by the symmetry above is acyclic iff one of the following holds:

- *each site has at least as many microscopic sites or copies as it has measurement settings, i.e.* $\forall_{i\in\{1,\ldots,n\}}.\ k_i \leq r_i$;
- *one of the sites has a single copy and the condition above is satisfied by all the other sites, i.e.* $\exists_{i_0}.\ \left(r_{i_0} = 1\ \wedge\ \forall_{i\in\{1,\ldots\widehat{i_0}\ldots,n\}}.\ k_i \leq r_i\right)$.

General multipartite scenarios

Proposition

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We get monogamy relations

$$\sum_{m_B=1}^{r_B} \sum_{m_C=1}^{r_C} \cdots \mathcal{B}(A, B^{(m_B)}, c^{(m_C)}, \ldots) \leq \ r_B r_C \cdots R$$

Conclusions

Summary

- A symmmetry (G-action) on Σ identifies measurements.
- A model satisfies a G-**monogamy** relation for a Bell inequality iff the macro average correlations (quotient model by G) satisfy the Bell inequality.

- So, if the quotient scenario is acyclic, then **any no-signalling empirical model** is G-monogamous wrt to all Bell inequalities (since the average correlations, being defined in this quotient scenario, must be local/non-contextual).

Summary

- In particular, we proved that this is the case for multipartite Bell-type scenarios provided the number of parties being identified as belonging to each 'macro' site is larger than the number of measurement settings available to each of them.
- Our approach highlights the **reason why** monogamy relations for general multipartite Bell inequalities follow from no-signalling alone, generalising the result of Pawłowski and Brukner (2009) from bipartite to multipartite. It also shows that what Ramanathan et al. proved holds not only for QM but for any no-signalling theory.
- The approach is not restricted to multipartite Bell-type scenarios. More generally, we can apply the same ideas to derive monogamy relations for contextuality inequalities.

Questions...

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