Possibilities determine the structure of the no-signalling polytope

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Introduction

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- general measurement scenarios

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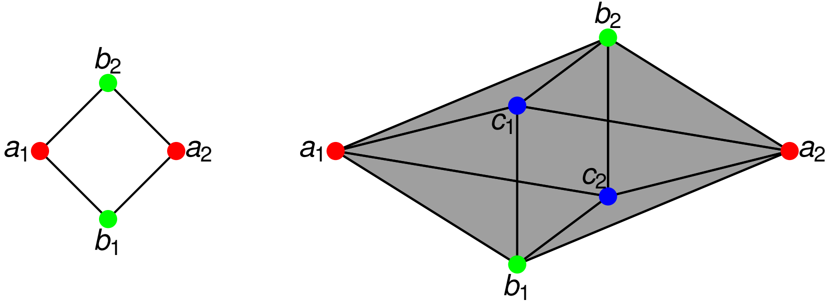
  (proofs *without inequalities*)

- supports determine the **combinatorial structure** of the **no-signalling polytope.**
- . . . but not quite possibilities alone!

Abramsky–Brandenburger framework

Measurement scenarios:

- a finite set of measurements X;
- a finite set of outcomes O;
- a cover $\mathcal{M}$ of X, indicating **joint measurability.**



Examples: Bell-type scenarios, KS configurations, and more.

Abramsky-Brandenburger framework

No-signalling **empirical model**:

- a family $(e_C)_{C \in \mathcal{M}}$, where e_C is a probability distribution on the outcomes of measurements in context C.
- compatibility condition:

$$e_C|_{C \cap C'} = e_{C'}|_{C \cap C'}$$

(on multipartite scenarios: no-signalling)

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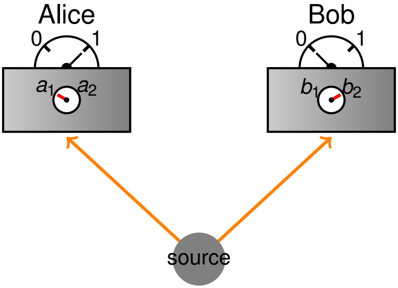
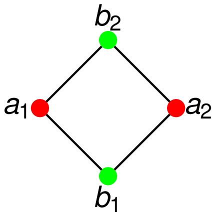
An empirical model is **local/non-contextual** when there exists a **global distribution** p_X (i.e. for all measurements at the same time) that marginalises to all the e_C.

Bell scenario





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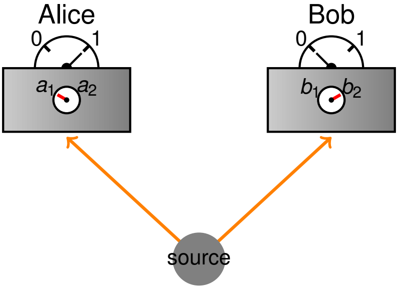

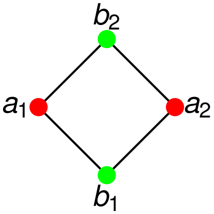

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- Given an empirical model e, define possibilistic model $\mathrm{poss}(e)$ by taking the support of each distributions.
- Contains the possibilistic, or logical, information of that model.

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Logical contextuality: Hardy model

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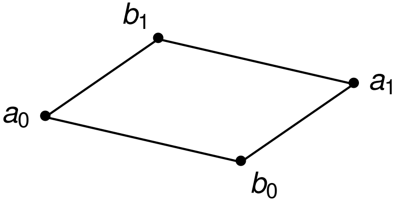

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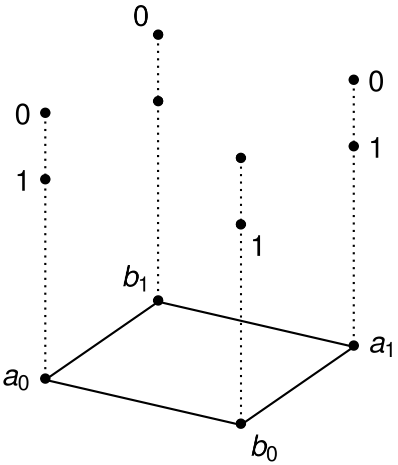

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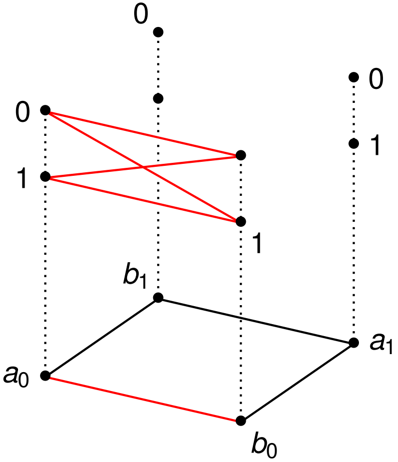

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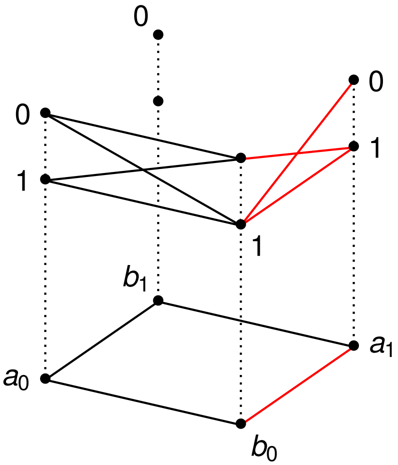

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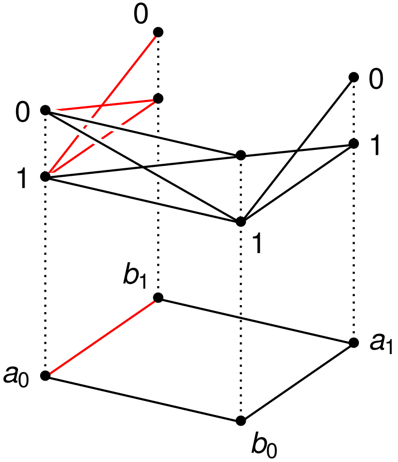

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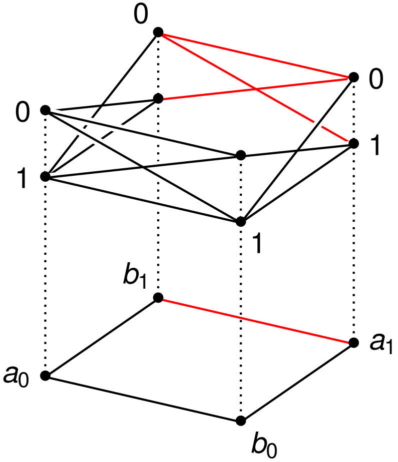

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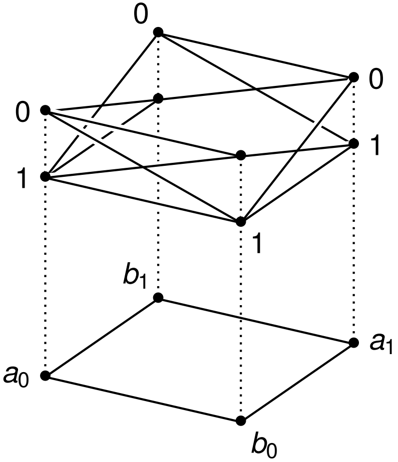

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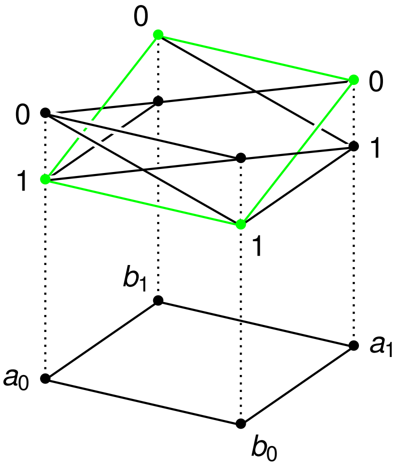

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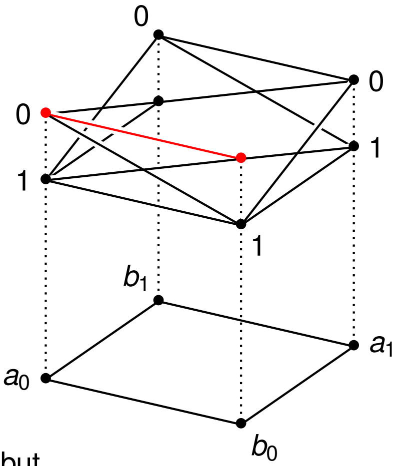

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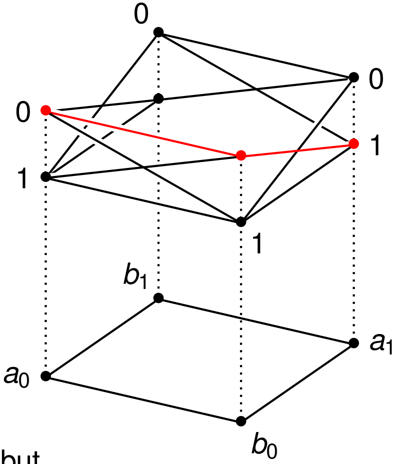

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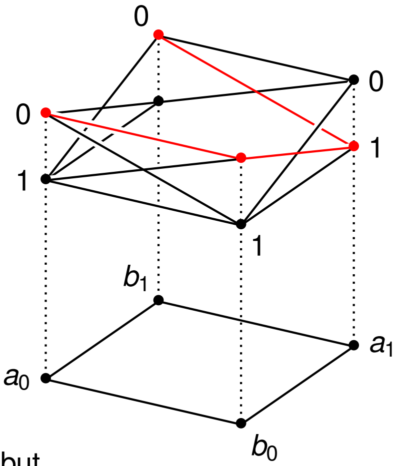

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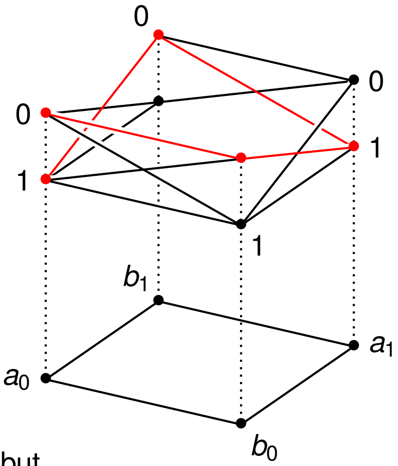

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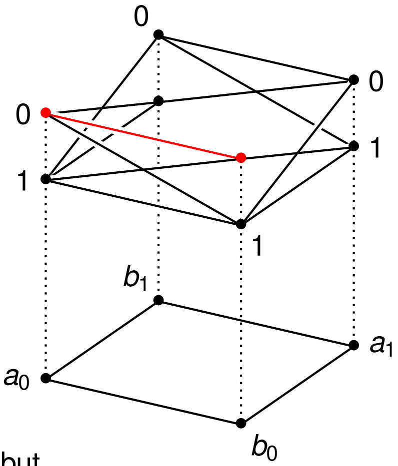

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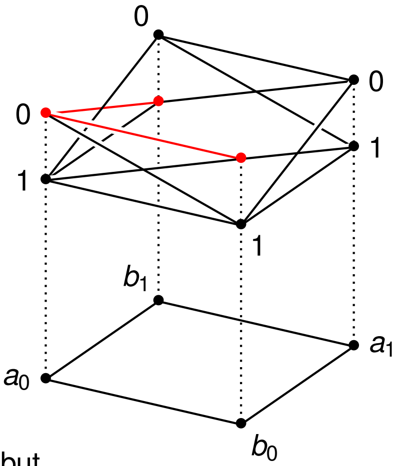

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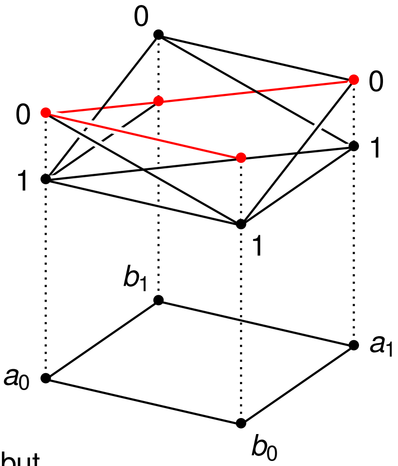

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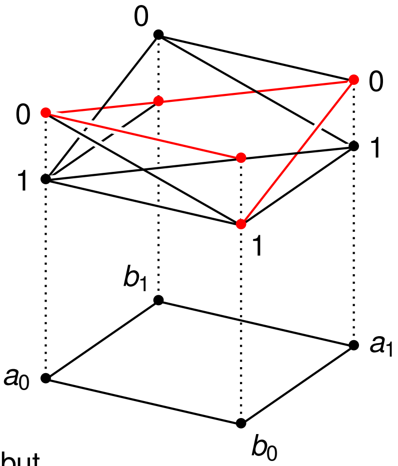

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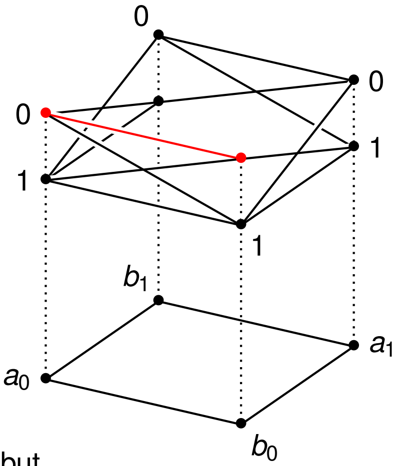

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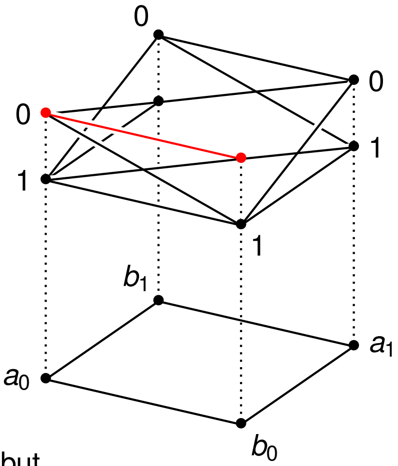

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Logical contextuality: Not all sections extend to global ones.

Also: strong contextuality, cohomological, All-versus-nothing.

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- explicitly represent models as points in $\mathbb{R}^N$, with $N = \sum_{C \in \mathcal{M}} |C|$.
- $\mathcal{N}$ is a polytope: defined by a finite number of linear constraints.

The structure of the no-signalling polytope

- $\mathcal{N}$: set of probabilistic empirical models
- $\mathcal{F}$: the face lattice of this polytope (vertices, edges, ...)

- $\mathcal{S}$: possibilistic models of the form $\mathsf{poss}(e)$ for some $e \in \mathcal{N}$
- ordered contextwise by support

  Then

$$\mathcal{F} \cong \mathcal{S}_{\perp}$$

In fact, the result applies to a much wider class of polytopes.

$\mathcal{N}$ is defined by constraints:

- Non-negativity;
- Linear equations: viz. normalisation and no-signalling.

In geometric terms: $\mathcal{N} = \mathcal{H}_{\geq \mathbf{0}} \cap \mathrm{Aff}(\mathcal{N})$
where $\mathrm{Aff}(\mathcal{N})$ is the affine subspace generated by $\mathcal{N}$,
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For any P is **standard form**, there is an order-isomorphism between:

- $\mathcal{F}(P)$, the face lattice of P.
- $\mathcal{S}(P)$, set of “supports” of points in P, ordered by inclusion.

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Fundamental Theorem of Polytopes: the two notions coincide.

Face lattice

- $\mathbf{a} \cdot \mathbf{x} \geq b$ is **valid** for P if it is satisfied by every $\mathbf{x} \in P$.
- A valid inequality defines a **face** F of P:

$$F := \{\mathbf{x} \in P \mid \mathbf{a} \cdot \mathbf{x} = b\}.$$

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- It is atomistic, coatomistic, and graded.
- Meets in $\mathcal{F}(P)$ are given by intersection of faces, joins defined indirectly.

Called the **face lattice** of P, aka the **combinatorial type** of P.

Relative interior

Relative interior of a set S:

$$\text{relint}\,(S) = \{\mathbf{x} \in S \mid \exists \epsilon > 0.\ \text{Aff}(S) \cap B_\epsilon(x) \subseteq S\}$$

For a convex set:

$$\text{relint}\,(S) = \{\mathbf{x} \in S \mid \forall \mathbf{y} \in S.\ \exists \epsilon > 0.\ (1+\epsilon)\mathbf{x} - \epsilon\mathbf{y} \in S\}$$

Intuitively: a point $\mathbf{x}$ is in the relative interior if the line segment $[\mathbf{y}, \mathbf{x}]$ from any point $\mathbf{y}$ of S in to $\mathbf{x}$ can be extended beyond $\mathbf{x}$ while remaining in S.

Carrier face

Every polytope P can be written as the disjoint union of the relative interiors of its non-empty faces:

$$P = \bigsqcup_{F \in \mathcal{F}^+(P)} \operatorname{relint} F.$$

This means that for any polytope P we can define a map

$$\operatorname{carr} : P \longrightarrow \mathcal{F}^+(P)$$

which assigns to each point $\mathbf{x}$ of P its **carrier face** — the unique face F such that $\mathbf{x} \in \operatorname{relint} F$.

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- $\mathcal{S}(P) := \{\mathrm{supp}\,\mathbf{x} \mid \mathbf{x} \in P\}$, ordered componentwise.

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- Join of $\mathbf{u}, \mathbf{v}$ is componentwise boolean disjunction: $(\mathbf{u} \vee \mathbf{v})_i \;:=\; \mathbf{u}_i \vee \mathbf{v}_i$.
- For $\mathbf{x}, \mathbf{y} \in P$ and $0 < \lambda < 1$, $\mathrm{supp}\,(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \;=\; \mathrm{supp}\, x \vee \mathrm{supp}\, y$.

Supports

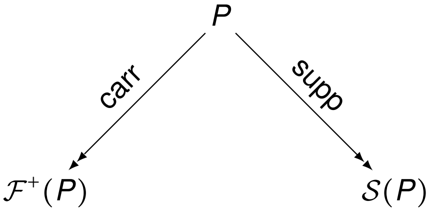
Polytope P in **standard form**: $P = \mathcal{H}_{\geq \mathbf{0}} \cap \mathrm{Aff}(P)$.

- Define a map supp $: \mathcal{H}_{\geq \mathbf{0}} \longrightarrow \{0,1\}^n$:

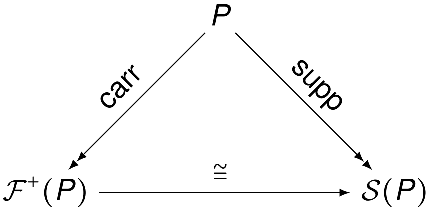
$$(\operatorname{supp} \mathbf{x})_i = \begin{cases} 0, & \mathbf{x}_i = 0 \\ 1, & \mathbf{x}_i > 0 \end{cases}$$

- $\mathcal{S}(P) := \{\operatorname{supp} \mathbf{x} \mid \mathbf{x} \in P\}$, ordered componentwise.

- Join of $\mathbf{u}, \mathbf{v}$ is componentwise boolean disjunction: $(\mathbf{u} \vee \mathbf{v})_i \; := \; \mathbf{u}_i \vee \mathbf{v}_i$.
- For $\mathbf{x}, \mathbf{y} \in P$ and $0 < \lambda < 1$, $\operatorname{supp}(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \; = \; \operatorname{supp} x \vee \operatorname{supp} y$.
- So $\mathcal{S}(P)_\perp$ is a finite lattice.



WTS: $\operatorname{carr} x \subseteq \operatorname{carr} y \quad \Leftrightarrow \quad \operatorname{supp} x \leq \operatorname{supp} y$



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For $\mathbf{x}$ in P, define a vector $\mathbf{x}^\sigma$ in $\mathbb{R}^n$: $\mathbf{x}_i^\sigma = \begin{cases} 0, & \mathbf{x}_i > 0 \\ 1, & \mathbf{x}_i = 0 \end{cases}$.

Clearly, $\mathbf{x}^\sigma \cdot \mathbf{z} \geq 0$ is valid for all $\mathbf{z} \in P$, and defines a face

$$\begin{aligned} F_\mathbf{x} &= \{\mathbf{z} \in P \mid \mathbf{x}^\sigma \cdot \mathbf{z} = 0\} \\ &= \{\mathbf{z} \in P \mid \operatorname{supp} \mathbf{z} \leq \operatorname{supp} \mathbf{x}\}. \end{aligned}$$

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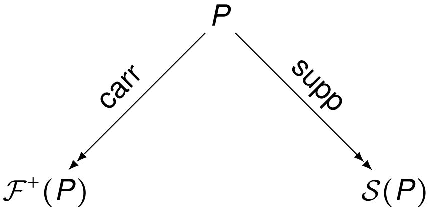
- Choose ϵ such that $\epsilon\mathbf{z} \leq \mathbf{x}$.
- $\mathbf{v} := (1 + \epsilon)\mathbf{x} - \epsilon\mathbf{z} \geq \mathbf{0}$.
- Hence, $\mathbf{v} \in F_{\mathbf{x}}$.



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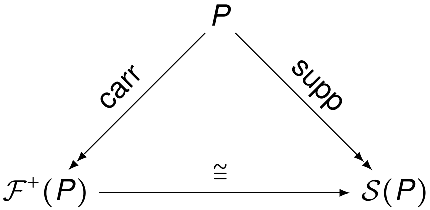

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Lattice iso: $\mathcal{F}(P) \cong \mathcal{S}(P)_\perp$

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- The vertices of the no-signalling polytope are exactly the probability models with minimal support. Moreover, there is only one probability model for each such minimal support.
- Therefore, the extremal empirical models are exactly those models which are completely and uniquely determined by their supports.
- These vertices of the polytope can be written as the disjoint union of the non-contextual, deterministic models – the vertices of the polytope of classical models – and the strongly contextual models with minimal support.

But . . .

- Note the mention of support!
- We still start from probabilistic models and take their supports.

Can we characterise the combinatorial type of $\mathcal{N}$ using **only** possibilistic notions?

- Recall that empirical models are families of **consistent distributions**.
- These can be defined over any commutative semiring R.
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The support lattice $\mathcal{S}(\mathcal{N}_{\mathbb{R}_{\geq 0}})$ is the image of this map.

Can we give an **intrinsic characterisation** of the image of the possibilistic collapse map, using only possibilistic notions?

$$\mathcal{S}(\mathcal{N}_{\mathbb{R}_{\geq 0}}) \neq \mathcal{N}_{\mathbb{B}}$$

i.e. there exist possibilistic empirical models that are not the support of any (probabilistic) empirical model (Abramsky, 2012).

A	B	00	01	10	11
a_1	b_1	1	0	0	1
a_1	b_2	1	1	0	1
a_2	b_1	1	0	0	1
a_2	b_2	1	0	0	1

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A	B	00	01	10	11
a_1	b_1	c	0	0	c'
a_1	b_2	d	g	0	d'
a_2	b_1	e	0	0	e'
a_2	b_2	f	0	0	f'

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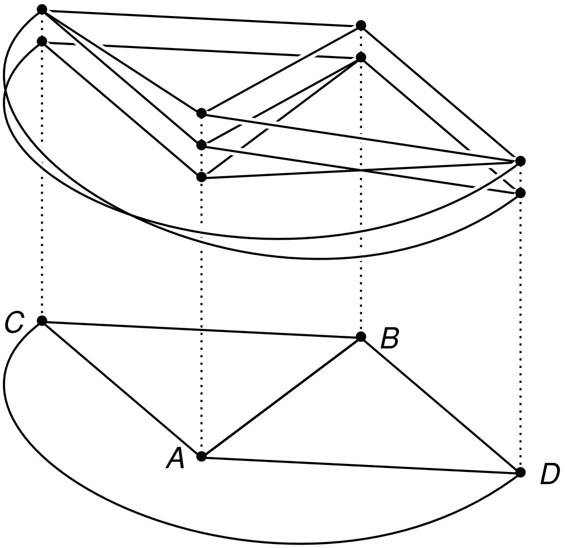
- A sensible question would be: given a possibilistic empirical model, is there always a (probabilistic) empirical model whose support is at most the original one?
- That is, are minimal possibilistic models always realisable as supports?

- Also, NO!

$$X = \{A, B, C, D\}$$
$$\mathcal{M} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\}$$
$$O = \{0, 1, 2\}$$

Possible assignments:

$$\begin{array}{lllll}
AB & \mapsto & 00, & 10, & 21 \\
 & & a & b & c \\
AC & \mapsto & 00, & 11, & 21 \\
 & & d & e & f \\
AD & \mapsto & 01, & 10, & 21 \\
 & & k & l & m \\
BC & \mapsto & 00, & 11 & \\
 & & g & h & \\
BD & \mapsto & 00, & 11 & \\
 & & i & j & \\
CD & \mapsto & 01, & 10 & \\
 & & n & o &
\end{array}$$


C
B
A
D

$$
\begin{array}{llccc}
AB & \mapsto & \underset{a}{00}, & \underset{b}{10}, & \underset{c}{21} \\
AC & \mapsto & \underset{d}{00}, & \underset{e}{11}, & \underset{f}{21} \\
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BD & \mapsto & \underset{i}{00}, & \underset{j}{11} & \\
CD & \mapsto & \underset{n}{01}, & \underset{o}{10} &
\end{array}
$$

$$
\begin{array}{llllllll}
a = k, & b = l, & g = i, & h = j, & c = n, & d = k, & e = l, & f = m \\
c = h, & h = o, & g = n, & i = o, & j = n, & c = j, & l = o, & d = n.
\end{array}
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\end{array}
$$

- All variables must be equated.
- Minimality: set any variable to zero, then all must be zero.
- Only remaining non-trivial equation is $a = a + a$.
- No non-zero, real solution!

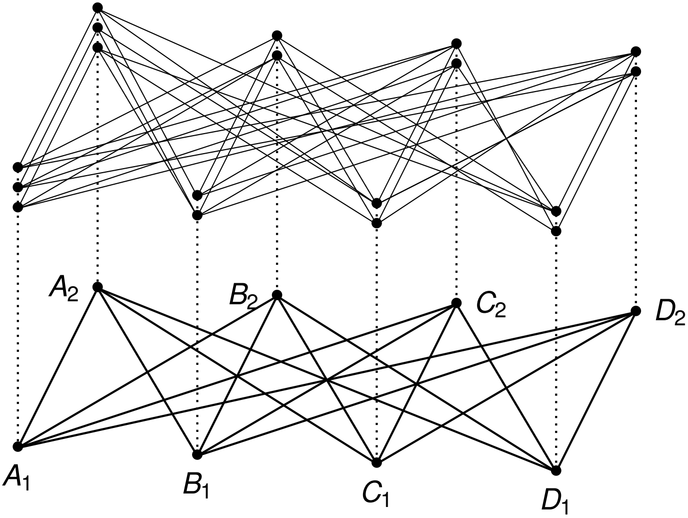
A Bell-type example

$$
\begin{aligned}
X_{\text{Bell}} &= \{A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2\} \\
\mathcal{M}_{\text{Bell}} &= \{A_1, B_1, C_1, D_1\} \times \{A_2, B_2, C_2, D_2\} \\
O &= \{0, 1, 2\}
\end{aligned}
$$

Possible sections:

$$
\begin{array}{llllllll}
A_1 A_2 & & & \mapsto & 00, & 11, & 22 \\
B_1 B_2, & C_1 C_2, & D_1 D_2 & \mapsto & 00, & 11 & \\
A_1 B_2, & A_2 B_1 & & \mapsto & 00, & 10, & 21 \\
A_1 C_2, & A_2 C_1 & & \mapsto & 00, & 11, & 21 \\
A_1 D_2, & A_2 D_1 & & \mapsto & 01, & 10, & 21 \\
B_1 C_2, & B_2 C_1 & & \mapsto & 00, & 11 & \\
B_1 D_2, & B_2 D_1 & & \mapsto & 00, & 11 & \\
C_1 D_2, & C_2 D_1 & & \mapsto & 01, & 10 &
\end{array}
$$

A Bell-type example



Still an open question

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