# The contextual fraction and contextuality as a resource

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Workshop on Quantum Contextuality
in Quantum Mechanics and Beyond
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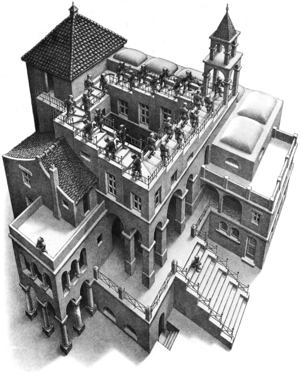
# Introduction

- **Contextuality**: a fundamental non-classical phenomenon of QM



# Introduction

- **Contextuality**: a fundamental non-classical phenomenon of QM

- Contextuality as a **resource** for QI and QC:
  - **Non-local games**
    quantum homomorphisms, constraint satisfaction, etc.
  - **MBQC** – Raussendorf (2013)
    "Contextuality in measurement-based quantum computation"
  - **MSD** – Howard, Wallman, Veith, & Emerson (2014)
    "Contextuality supplies the 'magic' for quantum computation"

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- Abramsky–Brandenburger: unified framework for non-locality and contextuality in general measurement scenarios
- qualitative hierarchy of contextuality for empirical models
- quantitative grading – **measure of contextuality**
  (NB: there may be more than one useful measure)

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  $\rightsquigarrow$ **resource theory**
- Relates to quantifiable **advantages** in QC and QIP tasks

# Contextuality

# Empirical data

| A | B | (0, 0) | (0, 1) | (1, 0) | (1, 1) |
|---|---|---|---|---|---|
| $a_1$ | $b_1$ | $1/2$ | 0 | 0 | $1/2$ |
| $a_1$ | $b_2$ | $3/8$ | $1/8$ | $1/8$ | $3/8$ |
| $a_2$ | $b_1$ | $3/8$ | $1/8$ | $1/8$ | $3/8$ |
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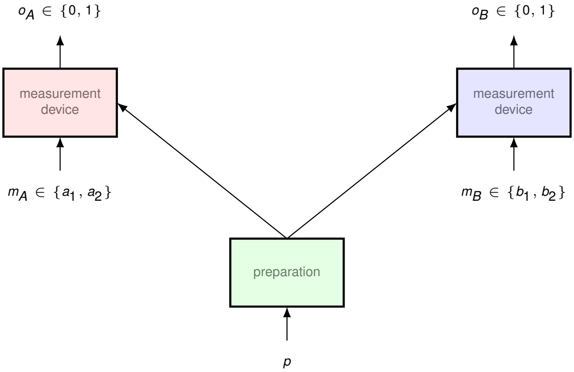

# Abramsky–Brandenburger framework

Measurement scenario $\langle X, \mathcal{M}, O \rangle$:

- $X$ is a finite set of measurements or variables
- $O$ is a finite set of outcomes or values
- $\mathcal{M}$ is a cover of $X$, indicating **joint measurability** (contexts)

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**Example**: (2,2,2) Bell scenario

- The set of variables is $X = \{a_1, a_2, b_1, b_2\}$.
- The outcomes are $O = \{0, 1\}$.
- The measurement contexts are:

$$\{\ \{a_1, b_1\},\quad \{a_1, b_2\},\quad \{a_2, b_1\},\quad \{a_2, b_2\}\ \}.$$

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A joint outcome or **event** in a context $C$ is $s \in O^C$, e.g.

$$s = [a_1 \mapsto 0, b_1 \mapsto 1] \ .$$

(These correspond to the cells of our probability tables.)

# Another example: 18-vector Kochen–Specker

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- A set of 18 variables, $X = \{A, \ldots, O\}$
- A set of outcomes $O = \{0, 1\}$
- A measurement cover $\mathcal{M} = \{C_1, \ldots, C_9\}$, whose contexts $C_i$ correspond to the columns in the following table:

| $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ | $U_6$ | $U_7$ | $U_8$ | $U_9$ |
|---|---|---|---|---|---|---|---|---|
| $A$ | $A$ | $H$ | $H$ | $B$ | $I$ | $P$ | $P$ | $Q$ |
| $B$ | $E$ | $I$ | $K$ | $E$ | $K$ | $Q$ | $R$ | $R$ |
| $C$ | $F$ | $C$ | $G$ | $M$ | $N$ | $D$ | $F$ | $M$ |
| $D$ | $G$ | $J$ | $L$ | $N$ | $O$ | $J$ | $L$ | $O$ |

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**Compatibility** condition: these distributions "agree on overlaps", i.e.

$$\forall_{C, C' \in \mathcal{M}}.\; e_C|_{C \cap C'} \;=\; e_{C'}|_{C \cap C'} \;.$$

where marginalisation of distributions: if $D \subseteq C$, $d \in \mathrm{Prob}(O^C)$,

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For multipartite scenarios, compatibility = the **no-signalling** principle.

# Contextuality

A (compatible) empirical model is **non-contextual** if there exists a **global distribution** $d \in \mathrm{Prob}(O^X)$ (on the joint assignments of outcomes to all measurements) that marginalises to all the $e_C$:

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The import of results such as Bell's and Bell–Kochen–Specker's theorems is that there are empirical models arising from quantum mechanics that are contextual.

# Strong contextuality

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E.g. K–S, GHZ, the PR box:

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| $a_1$ | $b_1$ | ✓ | × | × | ✓ |
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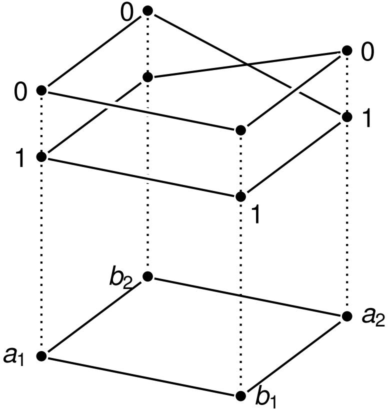
| $a_2$ | $b_2$ | × | ✓ | ✓ | × |



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Equivalently, maximum weight $\lambda$ over all convex decompositions

$$e = \lambda e^{NC} + (1 - \lambda) e'$$

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$$\mathrm{NCF}(e) = \lambda \qquad \mathrm{CF}(e) = 1 - \lambda$$

# (Non-)contextual fraction via linear programming

Checking contextuality of $e$ corresponds to solving

$$\begin{array}{ll} \texttt{Find} & \mathbf{d} \in \mathbb{R}^n \\ \texttt{such that} & \mathbf{M}\,\mathbf{d} = \mathbf{v}^e \\ \texttt{and} & \mathbf{d} \geq \mathbf{0} \end{array} .$$

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Computing the non-contextual fraction corresponds to solving the following linear program:

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# E.g. Equatorial measurements on GHZ($n$)

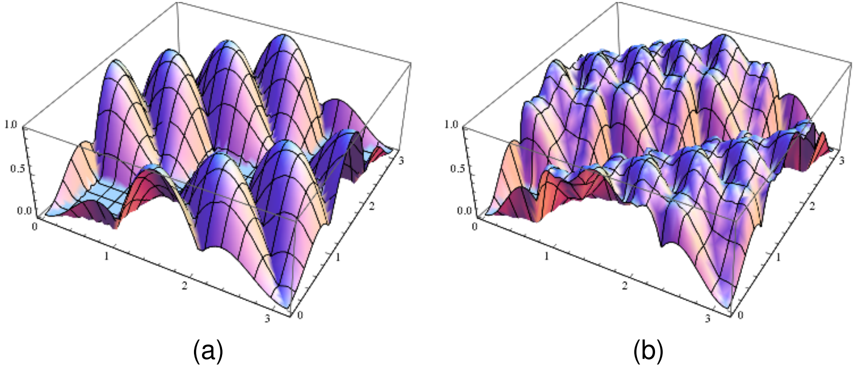


Figure: Contextual fraction of empirical models obtained with equatorial measurements at $\phi_1$ and $\phi_2$ on each qubit of $|\psi_{\mathrm{GHZ}(n)}\rangle$ with: (a) $n = 3$; (b) $n = 4$.

# Violations of Bell inequalities

# Generalised Bell inequalities

An **inequality** for a scenario $\langle X, \mathcal{M}, O \rangle$ is given by:

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For a model $e$, the inequality reads as

$$\mathcal{B}_\alpha(e) \; \leq \; R \, ,$$

where

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It is called a **Bell inequality** if it is satisfied by every NC model. If it is saturated by some NC model, the Bell inequality is said to be **tight**.

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The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by an empirical model $e$ is the value

$$\frac{\max\{0, \mathcal{B}_\alpha(e) - R\}}{\|\alpha\| - R} \ .$$

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### Proposition

Let $e$ be an empirical model.

- The normalised violation by $e$ of any Bell inequality is at most $\mathsf{CF}(e)$.
- This bound is attained: there exists a Bell inequality whose normalised violation by $e$ is exactly $\mathsf{CF}(e)$.
- Moreover, this Bell inequality is tight at "the" non-contextual model $e^{NC}$ and maximally violated by "the" strongly contextual model $e^{SC}$ for any decomposition:

$$e = \mathsf{NCF}(e)e^{NC} + \mathsf{CF}(e)e^{SC} \ .$$

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Quantifying Contextuality LP:

$$\begin{array}{ll} \texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\ \texttt{maximising} & \mathbf{1} \cdot \mathbf{c} \\ \texttt{subject to} & \mathbf{M}\,\mathbf{c} \leq \mathbf{v}^e \\ \texttt{and} & \mathbf{c} \geq \mathbf{0} \quad . \end{array}$$

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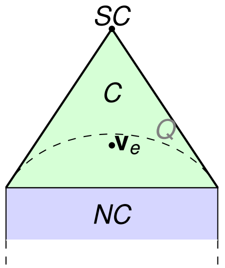

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Dual LP:

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|---|---|
| Find | $\mathbf{y} \in \mathbb{R}^m$ |
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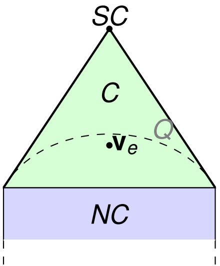

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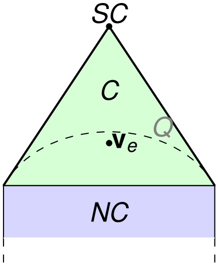

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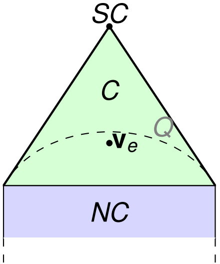


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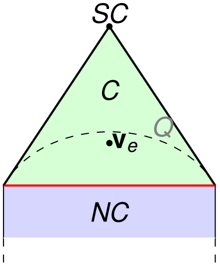
| and | $\mathbf{c} \geq \mathbf{0}$ . |

$e = \lambda e^{NC} + (1-\lambda) e^{SC}$ with $\lambda = \mathbf{1} \cdot \mathbf{x}^*$.




Dual LP:

| | |
|---|---|
| Find | $\mathbf{y} \in \mathbb{R}^m$ |
| minimising | $\mathbf{y} \cdot \mathbf{v}^e$ |
| subject to | $\mathbf{M}^T \mathbf{y} \geq \mathbf{1}$ |
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| | |
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computes tight Bell inequality (separating hyperplane)

# Operations on empirical models

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  to mean that $e$ is a (compatible) emprical model on $\langle X, \mathcal{M}, O \rangle$.
- The operations remind one of process algebras.

# Operations

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Relabelling

$$\begin{array}{l} e : \langle X, \mathcal{M}, O \rangle \\ \alpha : (X, \mathcal{M}) \cong (X', M') \end{array} \quad \rightsquigarrow \quad e[\alpha] : \langle X', \mathcal{M}', O \rangle$$

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# Operations

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For $C \in \mathcal{M}$, $D \in \mathcal{M}'$, $s = \langle s_1, s_2\rangle : C \sqcup D \longrightarrow O$,

$$(e \otimes e')_{C \sqcup D}\langle s_1, s_2\rangle := e_C(s_1)\, e'_D(s_2)$$

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| | $\mathsf{NCF}(e_1 \otimes e_2) = \mathsf{NCF}(e_1)\mathsf{NCF}(e_2)$ |

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- Measure of contextuality $\rightsquigarrow$ to quantify such advantages.

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- Probabilistic version: non-linear function computed with sufficently large probability of success implies contextuality.

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- Then, $1 - \bar{p}_S \;\geq\; \mathrm{NCF}(e)\nu(f)$.

# Contextual fraction and cooperative games

Constraint system $\langle V, D, \Gamma \rangle$

- $V$ finite set of variables
- $D$ finite domain of values
- $\Gamma$ finite set of formulae on the variables in $V$

Write $V(\phi)$ for variables that occur in $\phi$.

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  - Cf. “Noncontextual wirings”
    Amaral, Cabello, Terra Cunha, & Aolita (2017)
  - What (else) is this resource useful for?

# Questions...

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