The contextual fraction and contextuality as a resource



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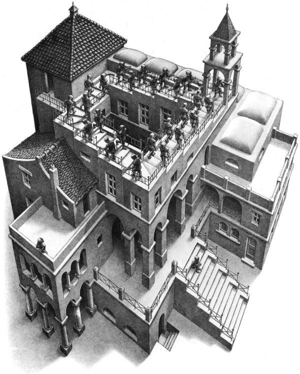
Motivation

- **Contextuality**: a fundamental non-classical phenomenon of QM



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    constraint satisfaction (Cleve & Mittal)
    etc. (Abramsky, B, de Silva, & Zapata)

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  - **MSD**
    Howard, Wallman, Veith, & Emerson (2014)
    "Contextuality supplies the 'magic' for quantum computation"

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- Abramsky & Brandenburger:
  unified framework for non-locality and contextuality
- qualitative hierarchy of contextuality for empirical models
- quantitative grading – **measure of contextuality**
  (NB: there may be more than one useful measure)

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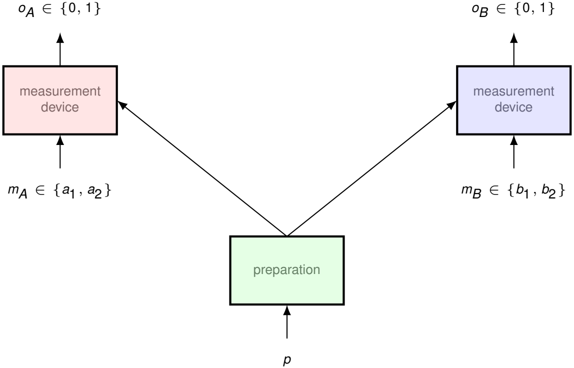
- **Monotone** wrt operations that don't introduce contextuality
  $\rightsquigarrow$ **resource theory**
- Relates to quantifiable **advantages** in QC and QIP tasks

Contextuality

Empirical data

A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_2	$1/8$	$3/8$	$3/8$	$1/8$



Abramsky–Brandenburger framework

Measurement scenario $\langle X, \mathcal{M}, O \rangle$:

- X is a finite set of measurements or variables
- O is a finite set of outcomes or values
- $\mathcal{M}$ is a cover of X, indicating **joint measurability** (contexts)

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Example: (2,2,2) Bell scenario

- The set of variables is $X = \{a_1, a_2, b_1, b_2\}$.
- The outcomes are $O = \{0, 1\}$.
- The measurement contexts are:

$$\{ \, \{a_1, b_1\}, \quad \{a_1, b_2\}, \quad \{a_2, b_1\}, \quad \{a_2, b_2\} \, \}.$$

Empirical Models

Joint outcome or **event** in a context C is $s \in O^C$, e.g.

$$s = [a_1 \mapsto 0, b_1 \mapsto 1] \, .$$

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It specifies a probability distribution over the events in each context.
Each distribution is a row of the probability table.

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$$\forall \, C, C' \in \mathcal{M}. \quad e_C|_{C \cap C'} \; = \; e_{C'}|_{C \cap C'} \, .$$

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In multipartite scenarios, compatibility = the **no-signalling** principle.

Contextuality

A (compatible) empirical model is **non-contextual** if there exists a **global distribution** $d \in \text{Prob}(O^X)$ on the joint assignments of outcomes to all measurements that marginalises to all the e_C:

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The import of results such as Bell's and Bell–Kochen–Specker's theorems is that there are empirical models arising from quantum mechanics that are contextual.

Strong contextuality

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no event can be extended to a global assignment.

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E.g. K–S, GHZ, the PR box:

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a_1	b_1	✓	×	×	✓
a_1	b_2	✓	×	×	✓
a_2	b_1	✓	×	×	✓
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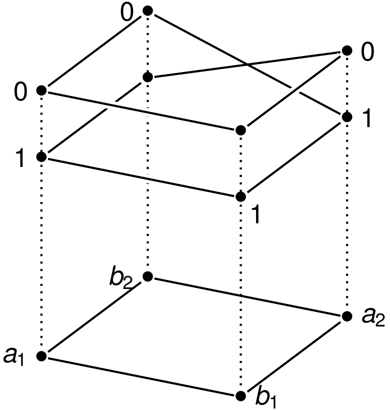

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Non-contextuality: global distribution $d \in \mathrm{Prob}(O^X)$ such that:

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Consider **subdistributions** $c \in \text{SubProb}(O^X)$ such that:

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Non-contextual fraction: maximum weight of such a subdistribution.

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Equivalently, maximum weight λ over all convex decompositions

$$e = \lambda e^{NC} + (1 - \lambda) e'$$

where e^{NC} is a non-contextual model.

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$$e = \lambda e^{NC} + (1 - \lambda) e^{SC}$$

where e^{NC} is a non-contextual model. e^{SC} is strongly contextual!

$$\text{NCF}(e) = \lambda \qquad \text{CF}(e) = 1 - \lambda$$

(Non-)contextual fraction via linear programming

Checking contextuality of e corresponds to solving

$$\begin{array}{ll} \texttt{Find} & \mathbf{d} \in \mathbb{R}^n \\ \texttt{such that} & \mathbf{M}\,\mathbf{d} = \mathbf{v}^e \\ \texttt{and} & \mathbf{d} \geq \mathbf{0} \end{array} \quad .$$

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Computing the non-contextual fraction corresponds to solving the following linear program:

$$\begin{array}{ll} \texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\ \texttt{maximising} & \mathbf{1} \cdot \mathbf{c} \\ \texttt{subject to} & \mathbf{M}\,\mathbf{c} \leq \mathbf{v}^e \\ \texttt{and} & \mathbf{c} \geq \mathbf{0} \end{array} \quad .$$

E.g. Equatorial measurements on GHZ(n)

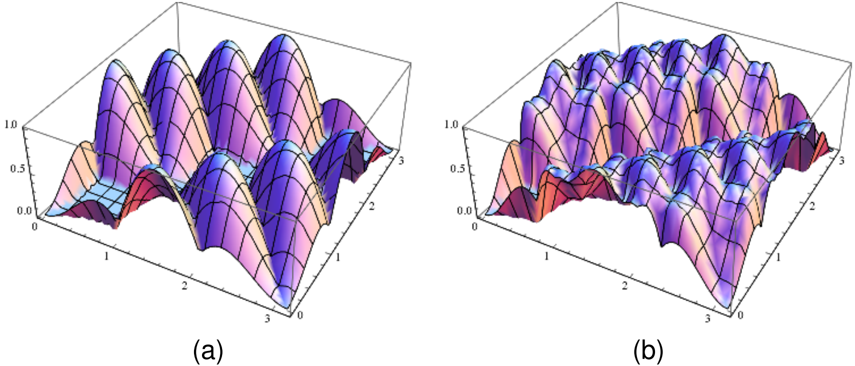

Figure: Contextual fraction of empirical models obtained with equatorial measurements at ϕ_1 and ϕ_2 on each qubit of $|\psi_{\text{GHZ}(n)}\rangle$ with: (a) $n = 3$; (b) $n = 4$.

Violations of Bell inequalities

Generalised Bell inequalities

An **inequality** for a scenario $\langle X, \mathcal{M}, O \rangle$ is given by:

- a set of coefficients $\alpha = \{\alpha(C, s)\}_{C \in \mathcal{M}, s \in O^C}$
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For a model e, the inequality reads as

$$\mathcal{B}_\alpha(e) \leq R \, ,$$

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It is called a **Bell inequality** if it is satisfied by every NC model. If it is saturated by some NC model, the Bell inequality is said to be **tight**.

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A Bell inequality establishes a bound for the value of $\mathcal{B}_\alpha(e)$ amongst NC models.

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The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by an empirical model e is the value

$$\frac{\max\{0, \mathcal{B}_\alpha(e) - R\}}{\|\alpha\| - R} .$$

Bell inequality violation and the contextual fraction

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Let e be an empirical model.

- The normalised violation by e of any Bell inequality is at most $\mathsf{CF}(e)$.
- This bound is attained: there exists a Bell inequality whose normalised violation by e is exactly $\mathsf{CF}(e)$.
- Moreover, this Bell inequality is tight at “the” non-contextual model e^{NC} and maximally violated by “the” strongly contextual model e^{SC} for any decomposition:

$$e = \mathsf{NCF}(e)e^{NC} + \mathsf{CF}(e)e^{SC} .$$

Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

Find	$\mathbf{c} \in \mathbb{R}^n$
maximising	$\mathbf{1} \cdot \mathbf{c}$
subject to	$\mathbf{M}\,\mathbf{c} \leq \mathbf{v}^e$
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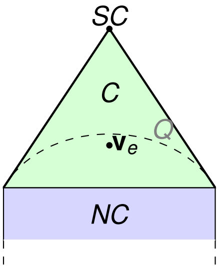

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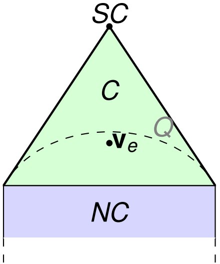

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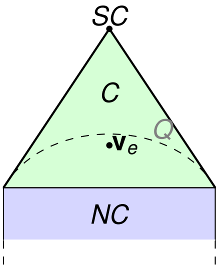

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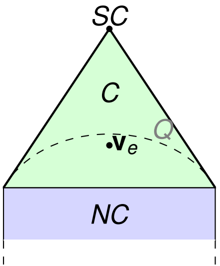

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and	$\mathbf{a} \leq \mathbf{1}$.

Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

```
Find           c ∈ ℝ^n
maximising     1 · c
subject to     M c ≤ v^e
and            c ≥ 0 .
```

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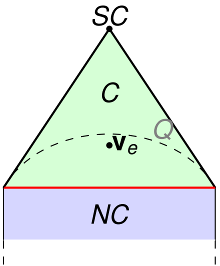

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computes tight Bell inequality (separating hyperplane)

Operations on empirical models

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- What properties should a good measure satisfy?

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- More than one possible measure of contextuality.
- What properties should a good measure satisfy?
- Monotonicity wrt operations that do not introduce contextuality
- Towards a resource theory
  as for entanglement (e.g. LOCC), non-locality, ...

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Coarse-graining e/f

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Relabelling	$e[\alpha]$
Restriction	$e \restriction \mathcal{M}'$
Coarse-graining	e/f
Mixing	$\lambda e + (1 - \lambda)e'$

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Contextual fraction and quantum advantages

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- Measure of contextuality $\rightsquigarrow$ quantify such advantages.

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- Probabilistic version: non-linear function computed with sufficently large probability of success implies contextuality.

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  - Connections with Contextuality-by-Default (Dzhafarov et al.)

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  - What (else) is this resource useful for?

Questions...

?

“The contextual fraction as a measure of contextuality”
Samson Abramsky, RSB, & Shane Mansfield
`arXiv:1705.07918[quant-ph]`