Minimum quantum resources for strong non-locality

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Quantum Physics & Logic (QPL 2017)
Radboud Universiteit, Nijmegen, 7th July 2017

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- . . . but in this talk we are interested in quantum realisations

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Empirical model: $p(o_1, \ldots, o_n \mid m_1, \ldots, m_n)$

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Local hidden variable model for $p(\mathbf{o} \mid \mathbf{m})$:

- space of hidden variables Λ
- $\mu \in D(\Lambda)$
- $\mathcal{P} : \Lambda \times X_1 \times \cdots \times X_n \longrightarrow D(\{0, 1\}^n)$
- explain the empirical data:
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- Bell locality:
  $\mathcal{P}(o_1, \ldots, o_n \mid m_1, \ldots, m_n, \lambda) = \mathcal{P}(o_1 \mid m_1, \lambda) \cdots \mathcal{P}(o_n \mid m_n, \lambda)$

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- strict relationship of strengths of non-locality:

$$\text{Bell} < \text{Hardy} < \text{GHZ}\ ,$$

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- **Strong non-locality**: (e.g. GHZ–Mermin, PR box)
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- local fraction $=$ maximal normalised violation of a Bell inequality
  Hence, SNL means violation of a Bell inequality up to the algebraic bound

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- What are the minimum resources necessary to witness quantum strong contextuality?
- SNL can be realised in bipartute two-qutrit system
(Heywood & Redhead)
We are focusing on qubits.

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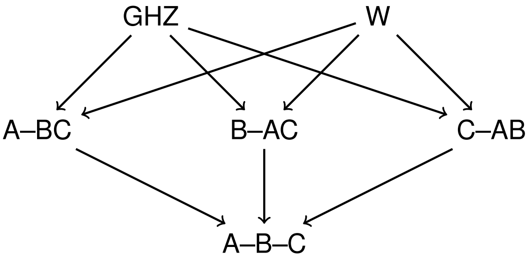
- Subtle counterpoint (Barrett, Kent, & Pironio):
  - maximally-entangled two-qubit state
  - SNL is achieved “in the limit” of infinitely many measurements
  - increasing number of measurements $\rightsquigarrow$ squeezes local fraction

  We’ll revisit this later.

Three-qubit states: SLOCC classes

Stochastic Local Operations and Classical Communication

(Dür, Vidal, & Cirac)



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- A new infinite family of SNL models
  - states not LU-equiv to GHZ

    $n = 0$: GHZ

    $n \to \infty$: $|\Phi^+\rangle \otimes |+\rangle$ in AB–C class

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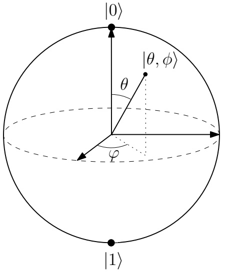
  - trade-off: measurements in A, B (upper bound for local fraction)
    entanglement necessary between C and AB

Local measuremets

A local projective measurement is represented by a vector

$$|\theta, \varphi\rangle := \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$$

on the Bloch sphere, corresponding to the $+1$ eigenvalue or outcome.



Set of local measurements for each qubit: $\mathsf{LM} := [0, \pi] \times [0, 2\pi)$.

Proof strategy

Find global assignment:

$$g = \bigsqcup_{i=1}^{n} g_i : \bigsqcup_{i=1}^{n} \mathsf{LM} \longrightarrow \{0, 1\}$$

such that for all contexts $(\boldsymbol{\theta}, \boldsymbol{\varphi})$,

$$\langle \boldsymbol{\theta}, \boldsymbol{\varphi} \mapsto g(\boldsymbol{\theta}, \boldsymbol{\varphi}) | \psi \rangle \neq 0$$

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If g satisfies

$$g_i(\theta, \varphi) = -g_i(\pi - \theta, \varphi + \pi)$$

it suffices to verify:

$$\langle (\boldsymbol{\theta}, \boldsymbol{\varphi}) | \psi \rangle \neq 0$$

for all contexts $(\boldsymbol{\theta}, \boldsymbol{\varphi})$ whose measurements are all assigned $+1$ by g.

Two-qubit states

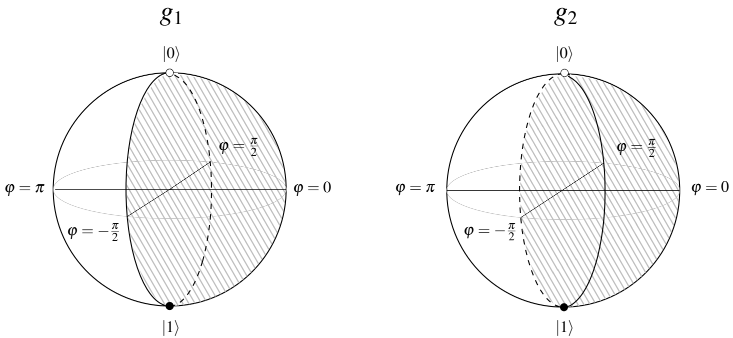
2-qubit states

Every two-qubit state can be written, up to LU, uniquely as

$$|\psi\rangle = \cos\delta|00\rangle + \sin\delta|11\rangle$$

where $\delta \in [0, \frac{\pi}{4}]$. Assume $\delta > 0$ (SLOCC of Bell).
Measuring $(\boldsymbol{\theta}, \boldsymbol{\varphi}) = \langle(\theta_1, \varphi_1), (\theta_2, \varphi_2)\rangle$, outcome $\langle +1, +1\rangle$:

$$\langle\boldsymbol{\theta}, \boldsymbol{\varphi}|\psi\rangle = \cos\delta \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2} + \sin\delta \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} e^{-i(\varphi_1+\varphi_2)}$$



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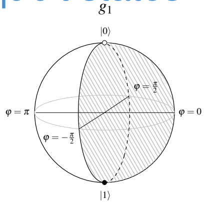

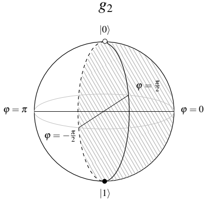

Let $(\boldsymbol{\theta}, \boldsymbol{\varphi})$ mapped to $+1$ by g. Then $\theta_1, \theta_2 \neq 0$. Hence,

$$s := \sin\delta \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} > 0 \text{ and } c := \cos\delta \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \geq 0.$$

- If $\theta_1 = \pi$ or $\theta_2 = \pi$, then $c = 0$
- Otherwise, $\langle \boldsymbol{\theta}, \boldsymbol{\varphi} | \psi \rangle = c + se^{-i(\varphi_1 + \varphi_2)}$ is positive real number plus non-zero complex number.
- To be zero, the latter must be real and negative:

$$\varphi_1 + \varphi_2 = \pi \mod 2\pi,$$

not satisfiable in the domain $\varphi_1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\varphi_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

States in the W SLOCC class

General state in W SLOCC class, up to LU:

$$|\psi_{\mathrm{W}}\rangle = \sqrt{a}|001\rangle + \sqrt{b}|010\rangle + \sqrt{c}|100\rangle + \sqrt{d}|000\rangle$$

with $a, b, c \in \mathbb{R}_{>0}$, and $d = 1 - (a + b + c)$.

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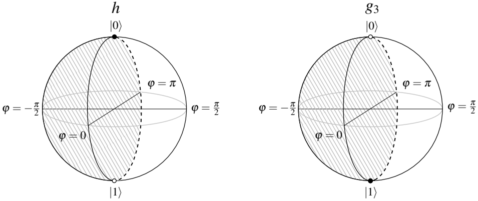
$\langle \boldsymbol{\theta}, \boldsymbol{\varphi} | \psi_{\mathrm{W}} \rangle = \sqrt{d}\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2}\right) + \sum_{k=1}^{3} z_k$,
with $z_k := \cos\frac{\theta_i}{2}\cos\frac{\theta_j}{2}\sin\frac{\theta_k}{2}e^{-i\phi_k}$



$\langle \boldsymbol{\theta}, \boldsymbol{\varphi} \mapsto g(\boldsymbol{\theta}, \boldsymbol{\varphi}) | \psi \rangle \neq 0$ for all contexts with measurements in shaded

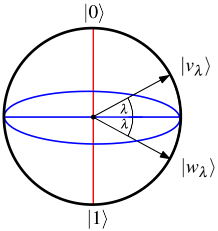
States in W SLOCC class do not realise SNL

States in the GHZ SLOCC class

General state in GHZ SLOCC class, up to LU:

$$|\psi_{\text{GHZ}}\rangle = \cos\delta |v_{\lambda_1}\rangle|v_{\lambda_2}\rangle|v_{\lambda_3}\rangle + \sin\delta e^{i\Phi}|w_{\lambda_1}\rangle|w_{\lambda_2}\rangle|w_{\lambda_3}\rangle,$$

with $\delta \in (0, \pi/4]$, $\Phi \in [0, 2\pi)$, and $\lambda_i \in [0, \pi/2)$,



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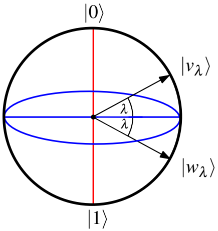

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A state in the GHZ SLOCC class realises SNL must be balanced. Moreover, any such SNL behaviour can be witnessed using only *equatorial* measurements.

A family of SNL 3-qubit models

Scope of our search for SNL: equatorial measurements on

$$|\mathrm{B}_{\boldsymbol{\lambda},\Phi}\rangle := \sqrt{\frac{K}{2}}(|v_{\lambda_1}\rangle|v_{\lambda_2}\rangle|v_{\lambda_3}\rangle + e^{i\Phi}|w_{\lambda_1}\rangle|w_{\lambda_2}\rangle|w_{\lambda_3}\rangle),$$

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- $N > 0$ even
- Third party can perform $\{X, Y\} = \left\{|\frac{\pi}{2}, 0\rangle, |\frac{\pi}{2}, \frac{\pi}{2}\rangle\right\}$
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- The state is $|\mathrm{B}_{\langle 0,0,\lambda_N\rangle,0}\rangle$, where $\lambda_N := \frac{\pi}{2} - \frac{\pi}{N}$

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These models are SNL.

A conditional AvN argument

A global assignment picks outcomes for all the measurements:

$$a_0, \ldots, a_{N-1}, b_0, \ldots, b_{N-1}, c_0, c_m \in \mathbb{Z}_2.$$

From algebraic structure of $\mathbb{Z}_{2N}$, derive $\mathbb{Z}_2$-system:

$$\begin{cases} a_0 \oplus b_0 \oplus c_0 = 0 & & \\ a_i \oplus b_{N-i} \oplus c_0 = 1 & \forall i \text{ s.t. } 1 \leq i \leq N-1 & \\ & & \\ a_i \oplus b_{N-i-1} = 1 & \forall i \text{ s.t. } 0 \leq i \leq N-1 & \text{if } c_m = 0 \\ & & \\ a_0 \oplus b_1 = 0 & & \\ a_1 \oplus b_0 = 0 & & \text{if } c_m = 1 \\ a_i \oplus b_{N+1-i} = 1 & \forall i \text{ s.t. } 2 \leq i \leq N-1 & \end{cases}$$

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A global assignment picks outcomes for all the measurements:

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From algebraic structure of $\mathbb{Z}_{2N}$, derive $\mathbb{Z}_2$-system:

$$\begin{cases} a_0 \oplus b_0 \oplus c_0 = 0 & & \\ a_i \oplus b_{N-i} \oplus c_0 = 1 & \forall i \text{ s.t. } 1 \leq i \leq N-1 & \\ & & \\ a_i \oplus b_{N-i-1} = 1 & \forall i \text{ s.t. } 0 \leq i \leq N-1 & \text{if } c_m = 0 \\ & & \\ a_0 \oplus b_1 = 0 & & \\ a_1 \oplus b_0 = 0 & & \text{if } c_m = 1 \\ a_i \oplus b_{N+1-i} = 1 & \forall i \text{ s.t. } 2 \leq i \leq N-1 & \end{cases}$$

$$\begin{cases} \bigoplus_i a_i \oplus \bigoplus_j b_j = 1. \\ \bigoplus_i a_i \oplus \bigoplus_j b_j = 0, \end{cases}$$

A family of SNL 3-qubit models

$$|0\rangle|0\rangle|v_{\frac{\pi}{2}-\frac{\pi}{N}}\rangle + |1\rangle|1\rangle|w_{\frac{\pi}{2}-\frac{\pi}{N}}\rangle$$

$n = 0$: GHZstate $\cdots$ $n \to \infty$: $|\Phi^+\rangle \otimes |+\rangle$ in AB–C class

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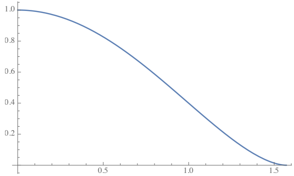

(von Neumann entanglement entropy as a function of λ)

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Experimentally bound local fraction:

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- Also: other states with less tripartite entanglement than GHZ

Questions...

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