The quantum monad on relational structures

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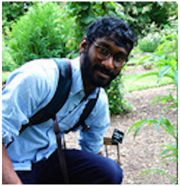
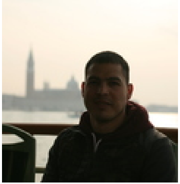
Rui Soares Barbosa[1]



Nadish de Silva[2]



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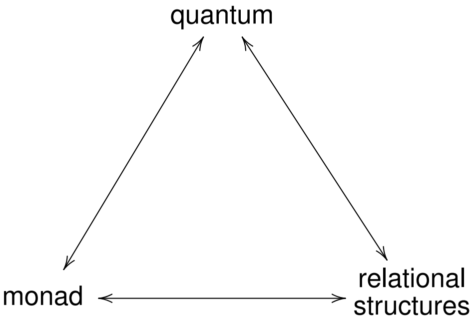
MFCS 2017

42nd International Symposium on
Mathematical Foundations of Computer Science

Aalborg Universitet, Aalborg, 22nd August 2017

Keywords



Keywords

quantum

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$\left(\begin{array}{c}\text{semantics \& types}\\ \text{category theory}\end{array}\right)$

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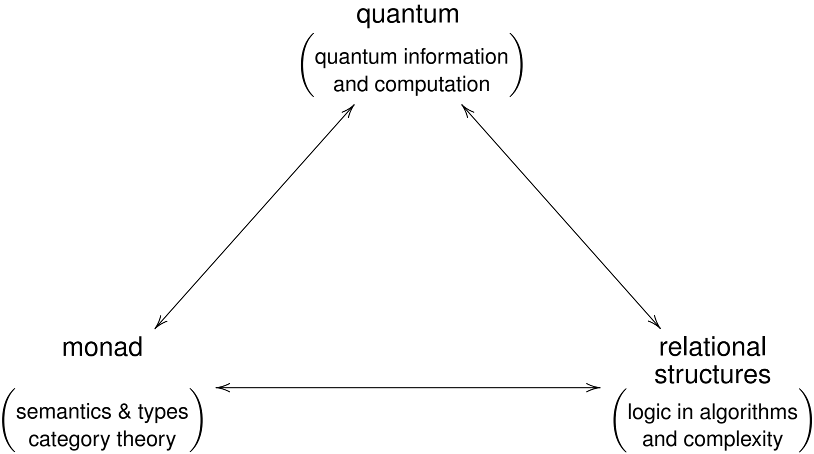
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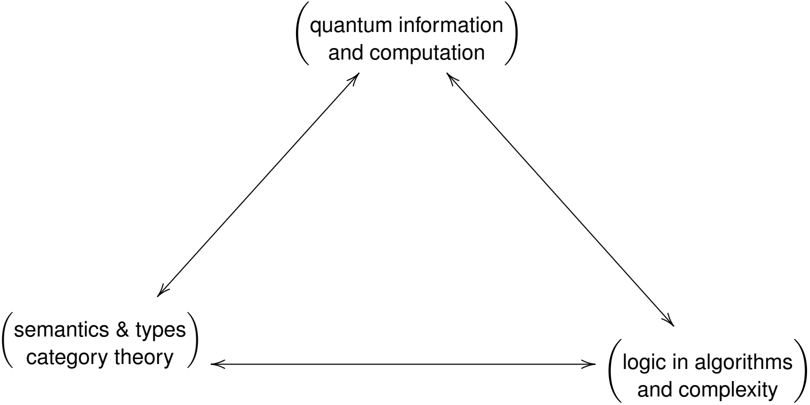

*Simons Institute semester ‘Logic in Computer Science’

1. Introduction

Motivation

Finite relational structures and homomorphisms

Pervasive notions in logic, computer science, combinatorics:

- constraint satisfaction
- finite model theory
- theory of relational databases
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Many relevant questions in these areas can be phrased in terms of (existence, number of, ...) homomorphisms between finite relational structures.

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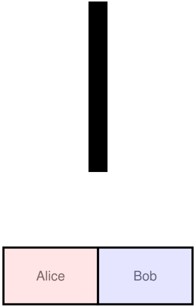
- delineate the scope of **quantum advantage**
- do this **uniformly**: quantum analogues *for free* for a whole range of classical notions from CS, logic, . . .
- Specfically, we formulate the task of constructing a homomorphism as a **non-local game**

Non-local games

Alice and Bob cooperate in solving a task set by Verifier

May share prior information,



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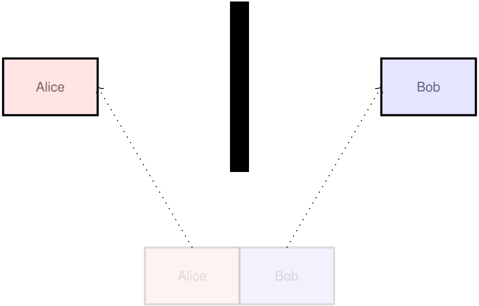

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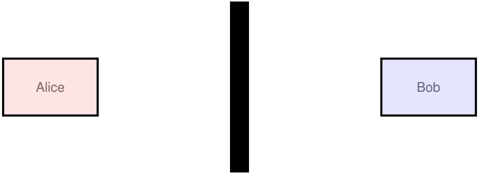

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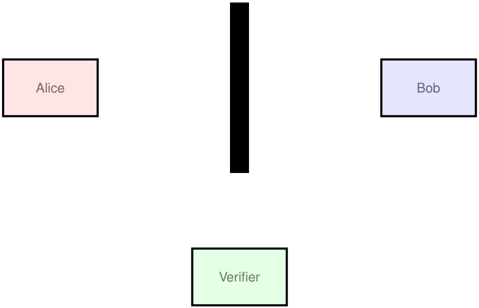

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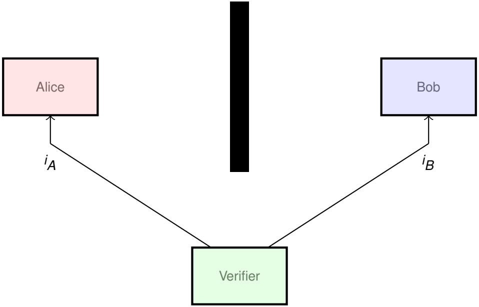

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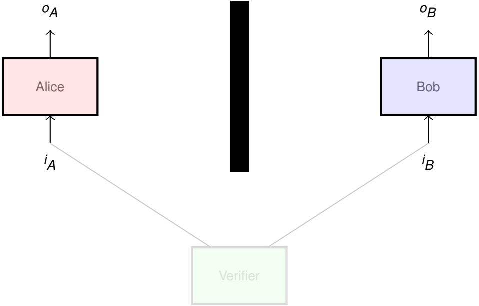

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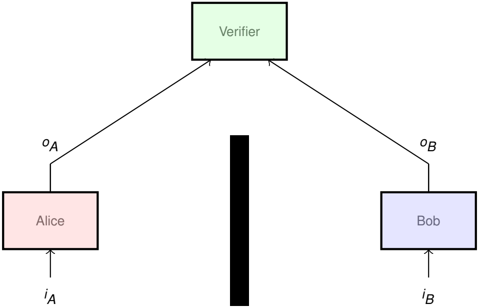

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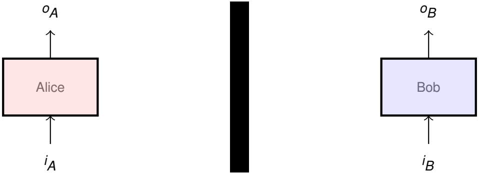

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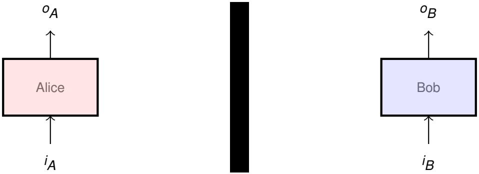

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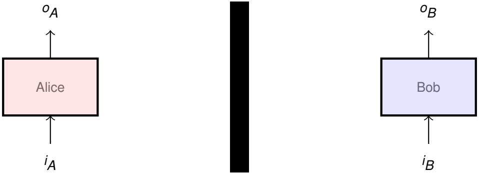

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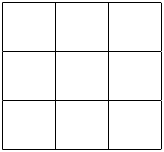
A **perfect strategy** is one that wins with probability 1.

E.g.: Binary constraint systems



Magic square:

- Fill with 0s and 1s
- rows and first two columns: even parity
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A	B	C
D	E	F
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System of linear equations over $\mathbb{Z}_2$:

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Clearly, this is not satisfiable in $\mathbb{Z}_2$.

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The system has a **quantum solution** but no classical solution!

Other

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Many of these works have some aspects in common. Our work aims to flesh this out by subsuming them under a common framework.

2. Homomorphisms game for relational structures

Relational structures and homomorphisms

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A σ**-structure** is $\mathcal{A} = (A; R_1^{\mathcal{A}}, \ldots R_p^{\mathcal{A}})$ where:

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$R(\sigma)$: category of σ-structures and homomorphisms.

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  - yield joint measurement $\mathcal{E} \otimes \mathcal{F} = \{\mathcal{E}_o \otimes \mathcal{F}_{o'}\}_{\langle o, o' \rangle \in O \times O'}$.
  - On a bipartite state $\psi \in \mathcal{H} \otimes \mathcal{K}$,
  - obtain joint outcome $\langle o, o' \rangle$ with probability $\psi^*(\mathcal{E}_o \otimes \mathcal{F}_{o'})\psi$.

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If only classical resources are allowed, there is a perfect strategy if and only if there exists a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

What about quantum resources?

The $(\mathcal{A}, \mathcal{B})$-homomorphism game

Given finite σ-structures $\mathcal{A}$ and $\mathcal{B}$, the players aim to convince the Verifier that there is a homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$.

- Verifier sends to Alice an index $l \in [p]$ and a tuple $\mathbf{x} \in R_l^{\mathcal{A}}$
- It sends to Bob an element $x \in A$

- Alice returns a tuple $\mathbf{y} \in B^{k_l}$
- Bob returns an element $y \in B$.

- They win this play if:
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(For simplicity, from now on consider a single relational symbol R of arity k)

Homomorphism game with quantum resources

Quantum resources:

- Finite-dimensional Hilbert spaces $\mathcal{H}$ (Alice's) and $\mathcal{K}$ (Bob's)
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These resources are used as follows:

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Perfect strategy conditions:

(QS1)	$\psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes I)\psi = 0$	if $\mathbf{y} \notin R^{\mathcal{B}}$
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3. From quantum perfect strategies to quantum homomorphisms

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

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The reduction proceeds in three steps:

1. The state and strategies are projected down to the support of the Schmidt decomposition of the state. This reduces the dimension of the Hilbert space and preserves the probabilities of the strategy exactly.

2. It is shown that this strategy must already satisfy strong properties (PVMs and $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{\mathbf{x}_i,y}$).

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N.B. In passing to the special form, the dimension is **reduced**; the process by which we obtain projective measurements is not at all akin to dilation.

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Moreover, these must be chosen so that $\mathcal{E}^i_{\mathbf{x},y}$ is independent of the context $\mathbf{x}$.

That is, we can define projectors $P_{x,y} := \mathcal{E}^i_{\mathbf{x},y}$ whenever $x = \mathbf{x}_i$.
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These $P_{x,y}$ are enough to determine the strategy!

Quantum homomorphisms

A quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a family of projectors $\{P_{x,y}\}_{x \in A, y \in B}$ in some dimension $d \in \mathbb{N}$ satisfying:

(QH1) For all $x \in A$, $\sum_{y \in B} P_{x,y} = I$.

(QH2) For all $\mathbf{x} \in R^{\mathcal{A}}$, $x = \mathbf{x}_i$, $x' = \mathbf{x}_j$,

$$[P_{x,y}, P_{x',y'}] = \mathbf{0} \quad \text{for any } y, y' \in B$$

Thus we can define a projective measurement $P_{\mathbf{x}} = \{P_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y}}$, where $P_{\mathbf{x},\mathbf{y}} := P_{\mathbf{x}_1,\mathbf{y}_1} \cdots P_{\mathbf{x}_k,\mathbf{y}_k}$.

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Theorem For finite structures $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent:

1. The ($\mathcal{A}$,$\mathcal{B}$)-homomorphism game has a quantum perfect strategy.
2. There is a quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$. ($\mathcal{A} \xrightarrow{q} \mathcal{B}$)

4. Quantum homomorphisms and the quantum monad

Quantum homomorphisms as Kleisli maps

For each $d \in \mathbb{N}$ and σ-structure $\mathcal{A}$, we can define a structure $\mathcal{Q}_d\mathcal{A}$ such that there is a one-to-one correspondence:[2]

$$\mathcal{A} \xrightarrow{q}_d \mathcal{B} \quad \cong \quad \mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$$

- quantum homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ of dimension d
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Universe of structure $\mathcal{Q}_d\mathcal{A}$: set of functions $p : A \longrightarrow \mathrm{Proj}(d)$ such that $\sum_{x \in A} p(x) = I$. (Projector-valued distributions on A in dimension d.)

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For R of arity k, $R^{\mathcal{Q}_d\mathcal{A}}$ is the set of tuples $\langle p_1, \ldots, p_k \rangle$ satisfying:

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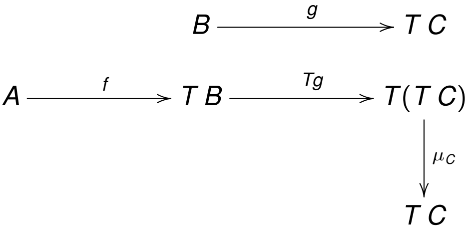

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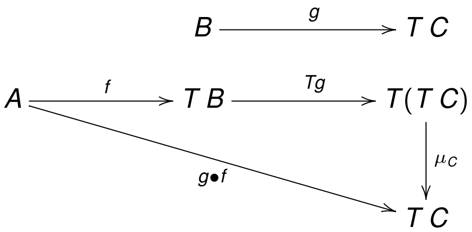

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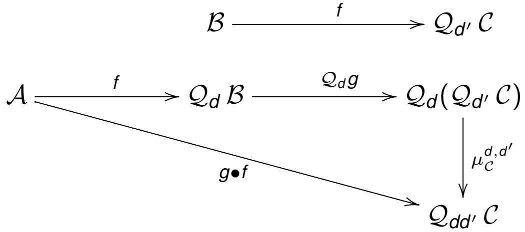

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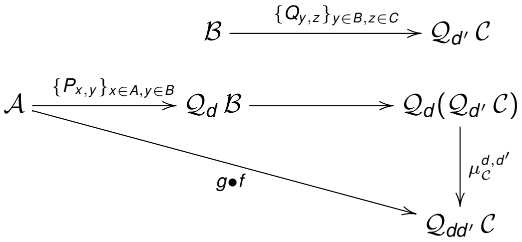

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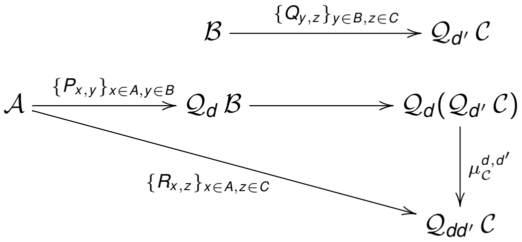

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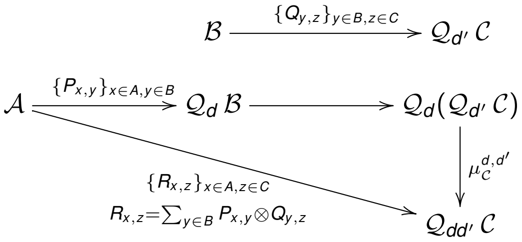
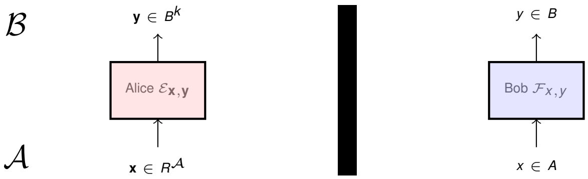
- $\mu_{\mathcal{A}}^{d,d'} : \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{A}) \longrightarrow \mathcal{Q}_{dd'}\,\mathcal{A}$



Composition of perfect strategies



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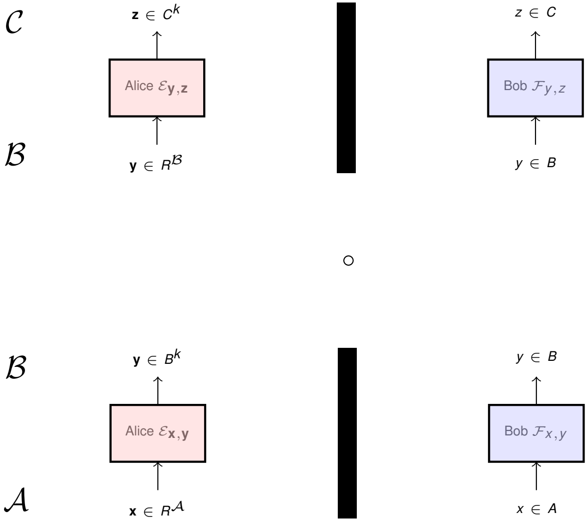

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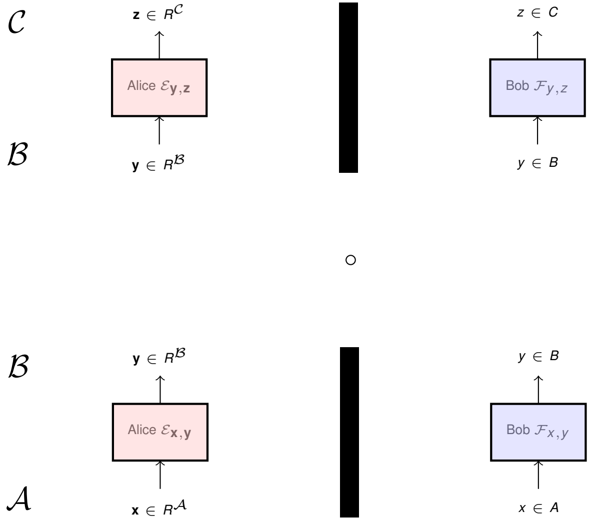

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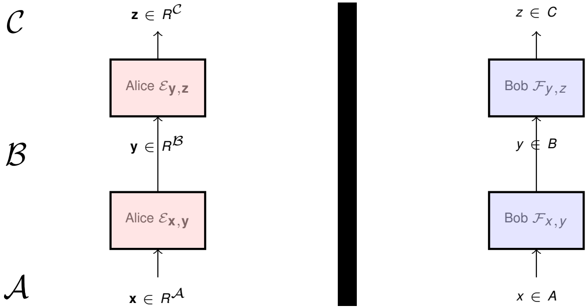

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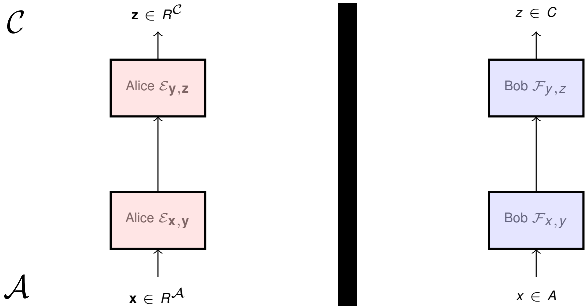

5. Quantum advantages

Unified framework for expressing quantum advantage in a wide range of information processing tasks

- Quantum advantage in constraint satisfaction
- Existence of quantum but not classical homomorphisms between relational structures
- State-independent strong contextuality

Classial correspondence

A CSP instance $\mathcal{K} = (V, D, C)$:

- V a set of variables
- D a domain of values
- C a set of constraints $(\mathbf{x}, r)$ with $\mathbf{x} \in V^k$ and $r \subseteq D^k$

A **solution** is an assignment $\alpha : V \longrightarrow D$ satisfying all constraints: for all $(\mathbf{x}, r) \in C$, $\alpha(\mathbf{x}) \in r$.

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As a homomorphism:

- σ has symbol $R_{(\mathbf{x},r)}$ of arity $k = |\mathbf{x}|$ for each constraint $(\mathbf{x}, r) \in C$
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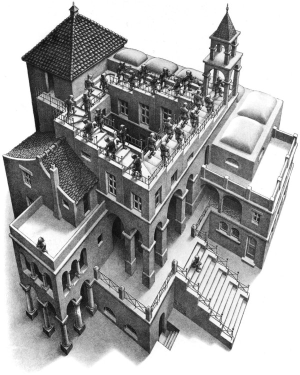
Immediate one-to-one correspondence between:

- solutions for $\mathcal{K}$
- homomorphisms $\mathcal{A}_\mathcal{K} \longrightarrow \mathcal{B}_\mathcal{K}$

Contextuality

Contextuality is a fundamental feature of quantum mechanics, which distinguishes it from classical physical theories.

It can be thought as saying that empirical predictions are inconsistent with all measurements having pre-determined outcomes.



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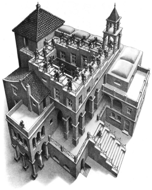
Recently linked to quantum advantage in information-processing tasks.

It is a property of the empirical data, and therefore should be studied at that appropriate level of generality.



Contextuality

Measurement scenario $(X, \mathcal{M}, O)$:

- X is a finite set of measurements
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- $\mathcal{M}$ is a cover of X, where $C \in \mathcal{M}$ is a set of compatible measurements (context)

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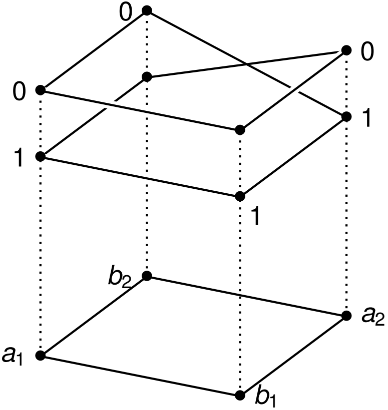
E.g.: GHZ, Kochen–Specker, (post-quantum) PR box

Strong contextuality

Strong Contextuality:
no consistent global assignment.

A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_1	b_1	✓	×	×	✓
a_1	b_2	✓	×	×	✓
a_2	b_1	✓	×	×	✓
a_2	b_2	×	✓	✓	×



CSP and strong contextuality

The support of an empirical model *e* can be described as a CSP $\mathcal{K}_e$

There is a one-to-one correspondence between:

- consistent global assignements for *e*
- solutions for $\mathcal{K}_e$

Hence, *e* is strongly contextual iff $\mathcal{K}_e$ has no (classical) solution.

Quantum correspondence

Ready-made notion of **quantum solution** to a CSP:

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- state φ
- PVM $P_x = \{P_{x,o}\}_{o \in O}$ for each $x \in X$ s.t.
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State-independent witness:
family of PVMs that yields such witness for any state φ.

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N.B. Provides a general way of turning state-independent contextuality proofs into Bell non-locality arguments!

6. Outlook

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- Homomorphisms are related to the existetial positive fragment: can this be extended to provide quantum vality for first-order formulae?
- Pebble games can be formulated via co-Kleisli maps $T_k \mathcal{A} \longrightarrow \mathcal{B}$. Can this be similarly *quantised*?
  Bi-Kleisli maps $T_k \mathcal{A} \longrightarrow \mathcal{Q}_d \mathcal{B}$ yield quantum pebble games?

Thank you!

Questions...

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