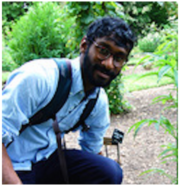
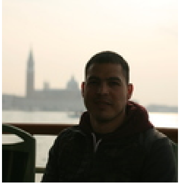
The quantum monad on relational structures

Samson Abramsky[1] Rui Soares Barbosa[1] Nadish de Silva[2] Octavio Zapata[2]

[1]University of Oxford

[2]University College London

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Keywords

quantum

monad

relational
structures

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$\left(\begin{array}{c}\text{semantics \& types}\\ \text{category theory}\end{array}\right)$

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(quantum information and computation)

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$$\begin{tikzcd}[column sep=small]
& \begin{array}{c}\text{quantum}\\ \left(\begin{array}{c}\text{quantum information}\\ \text{and computation}\end{array}\right)\end{array} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\begin{array}{c}\text{monad}\\ \left(\begin{array}{c}\text{semantics \& types}\\ \text{category theory}\end{array}\right)\end{array} \arrow[rr, leftrightarrow] & & \begin{array}{c}\text{relational}\\ \text{structures}\\ \left(\begin{array}{c}\text{finite model theory}\\ \text{CSP, databases, etc.}\end{array}\right)\end{array}
\end{tikzcd}$$

1. Introduction

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- use **quantum resources** for information-processing tasks
- delineate the scope of **quantum advantage**

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- A setting in which this has been explored is **non-local games**

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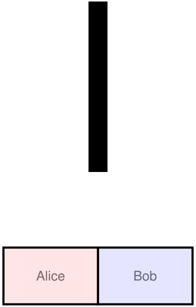
Alice and Bob cooperate in solving a task set by Verifier

May share prior information,



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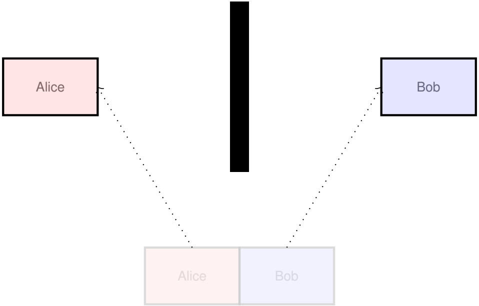

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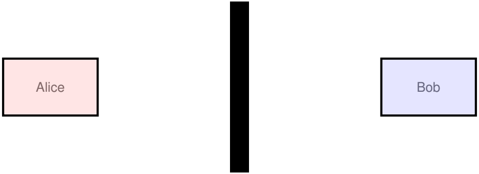

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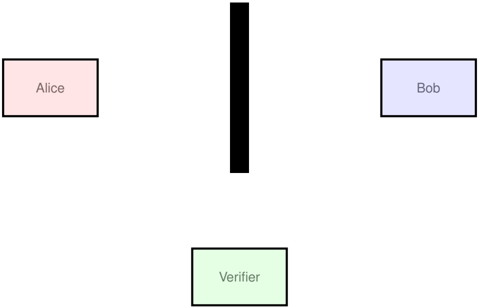

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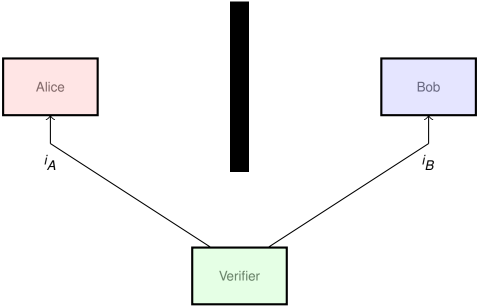

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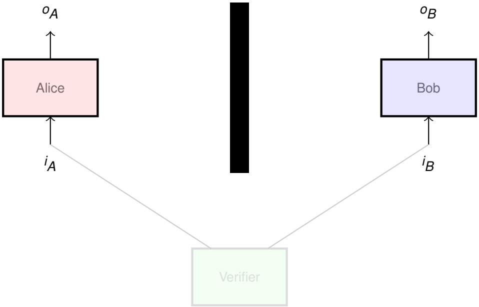

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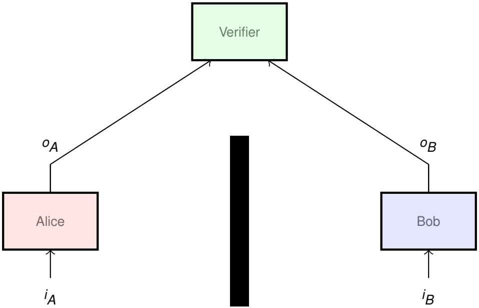

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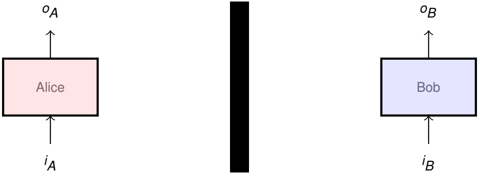

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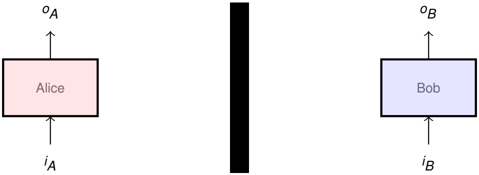

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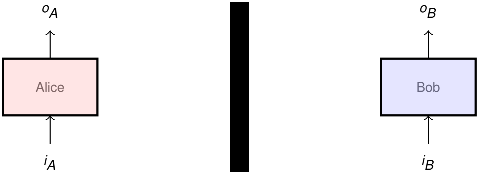

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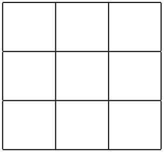
A **perfect strategy** is one that wins with probability 1.

E.g.: Binary constraint systems



Magic square:

- Fill with 0s and 1s
- rows and first two columns: even parity
- last column: odd parity

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A	B	C
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G	H	I

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System of linear equations over $\mathbb{Z}_2$:

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Clearly, this is not satisfiable in $\mathbb{Z}_2$.

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$\boxed{\text{The system has a } \textbf{quantum solution} \text{ but no classical solution!}}$

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Many of these works have some aspects in common. We aim to flesh this out by subsuming them under a common framework.

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Many relevant questions in these areas can be phrased in terms of (existence, number of, ...) homomorphisms between finite relational structures.

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- We formulate the task of constructing a homomorphism between two relational structures as a **non-local game**
- **uniformly** obtain quantum analogues *for free* for a whole range of classical notions from CS, logic, . . .

Motivation

What could it mean to quantise these fundamental structures?

- We formulate the task of constructing a homomorphism between two relational structures as a **non-local game**
- **uniformly** obtain quantum analogues *for free* for a whole range of classical notions from CS, logic, ...
- We then show that the use of quantum resources for information tasks is captured in a high-level way as **quantum homomorphisms**
- which can be integrated into a typed functional programming language through a **monadic** interface.

Outline of the talk

- Introduce homomorphism game for relational structures
- Arrive at the notion of quantum homomorphism,
  which removes the two-player aspect of the game

  (generalises Cleve & Mittal and Mančinska & Roberson)
- Quantum monad: capture quantum homomorphisms as classical homomorphisms to a *quantised* version of a relational structure

  (inspired on Mančinska & Roberson for graphs)
- Connection between non-locality and state-independent strong contextuality

2. Homomorphisms game for relational structures

Relational structures and homomorphisms

A relational vocabulary σ consists of relational symbols $R_1, \ldots, R_p$ where R_l has an arity $k_l \in \mathbb{N}$ for each $l \in [p] := \{1, \ldots, p\}$.

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A σ**-structure** is $\mathcal{A} = (A; R_1^{\mathcal{A}}, \ldots R_p^{\mathcal{A}})$ where:

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(For simplicity, from now on consider a single relational symbol R of arity k)

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Given finite σ-structures $\mathcal{A}$ and $\mathcal{B}$, the players aim to convince the Verifier that there is a homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$.

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$\boxed{\text{What about quantum resources?}}$

Homomorphism game with quantum resources

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These resources are used as follows:

- Given input $\mathbf{x} \in R^{\mathcal{A}}$, Alice measures $\mathcal{E}_{\mathbf{x}}$ on her part of ψ
- Given input $x \in A$, Bob measures $\mathcal{F}_x$ on his part of ψ
- Both output their respective measurement outcomes
- $P(\mathbf{y}, y \mid \mathbf{x}, x) = \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi$

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Perfect strategy conditions:

(QS1)	$\psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \quad I \quad)\psi = 0$	if $\mathbf{y} \notin R^{\mathcal{B}}$
(QS2)	$\psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi = 0$	if $x = \mathbf{x}_i$ and $y \neq \mathbf{y}_i$

3. From quantum perfect strategies to quantum homomorphisms

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

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$\psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes I)\psi = 0$	if $\mathbf{y} \notin R^{\mathcal{B}}$
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[1] This generalises Cleve & Mittal and Mančinska & Roberson.

Simplifying quantum strategies

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The reduction proceeds in three steps:

1. The state and strategies are projected down to the support of the Schmidt decomposition of the state. This reduces the dimension of the Hilbert space and preserves the probabilities of the strategy exactly.

2. It is shown that this strategy must already satisfy strong properties (PVMs and $\mathcal{E}^{i}_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{\mathbf{x}_i,y}$).

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N.B. In passing to the special form, the dimension is **reduced**; the process by which we obtain projective measurements is not at all akin to dilation.

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which must be chosen so that $\mathcal{E}^i_{\mathbf{x},y}$ is independent of the context $\mathbf{x}$.

That is, we can define projectors $P_{x,y} := \mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$ whenever $x = \mathbf{x}_i$.
If $\mathbf{x}_i = x = \mathbf{x}'_j$, then we have $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y} = \mathcal{E}^j_{\mathbf{x}',y}$, so $P_{x,y}$ is well-defined.

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These $P_{x,y}$ are enough to determine the strategy!

Quantum homomorphisms

A quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a family of projectors $\{P_{x,y}\}_{x \in A, y \in B}$ in some dimension $d \in \mathbb{N}$ satisfying:

(QH1) For all $x \in A$, $\sum_{y \in B} P_{x,y} \ = \ I$.

(QH2) For all $\mathbf{x} \in R^{\mathcal{A}}$, $x = \mathbf{x}_i$, $x' = \mathbf{x}_j$,

$$[P_{x,y}, P_{x',y'}] = \mathbf{0} \quad \text{for any } y, y' \in B$$

so we can define a projective measurement $P_{\mathbf{x}} = \{P_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y}}$, where $P_{\mathbf{x},\mathbf{y}} := P_{\mathbf{x}_1,\mathbf{y}_1} \cdots P_{\mathbf{x}_k,\mathbf{y}_k}$.

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Theorem For finite structures $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent:

1. The ($\mathcal{A}$,$\mathcal{B}$)-homomorphism game has a quantum perfect strategy.
2. There is a quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$. ($\mathcal{A} \xrightarrow{q} \mathcal{B}$)

4. Quantum homomorphisms and the quantum monad

Quantum homomorphisms as Kleisli maps

For each $d \in \mathbb{N}$ and σ-structure $\mathcal{A}$, we can define a structure $\mathcal{Q}_d\mathcal{A}$ such that there is a one-to-one correspondence:[2]

$$\mathcal{A} \xrightarrow{q}_d \mathcal{B} \quad \cong \quad \mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$$

- quantum homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ of dimension d
- (classical) homomorphisms from $\mathcal{A}$ to $\mathcal{Q}_d\mathcal{B}$

[2]Mančinska & Roberson: analogous construction for (their) graph homomorphisms.

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Universe of structure $\mathcal{Q}_d\mathcal{A}$: set of functions $p : A \longrightarrow \mathrm{Proj}(d)$ such that $\sum_{x \in A} p(x) = I$. (Projector-valued distributions on A in dimension d.)

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For R of arity k, $R^{\mathcal{Q}_d\mathcal{A}}$ is the set of tuples $\langle p_1, \ldots, p_k \rangle$ satisfying:

(QR1) For all $1 \leq i, j \leq k$ and $x, x' \in A$, $[p_i(x), p_j(x')] = \mathbf{0}$.

(QR2) For all $\mathbf{x} \in A^k$, if $\mathbf{x} \notin R^{\mathcal{A}}$, then $p_1(x_1) \cdots p_k(x_k) = \mathbf{0}$.

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$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

Monads play a major rôle in programming language theory, providing a uniform way of encapsulating various notions of computation:

- partiality
- exceptions
- non-determinism
- probabilistic
- state updates
- input/output
- …

Monads

Functor $T : \mathfrak{C} \longrightarrow \mathfrak{C}$ such that a T-program, a computation producing values of type B from values of type A with T-effects, is seen as a map $A \longrightarrow T\,B$ in the category $\mathfrak{C}$.

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Composition:

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\begin{tikzcd}
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$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

Graded by dimension:

$\rightsquigarrow$ composition of quantum homomorphisms, keeping track of the dimension of the resources

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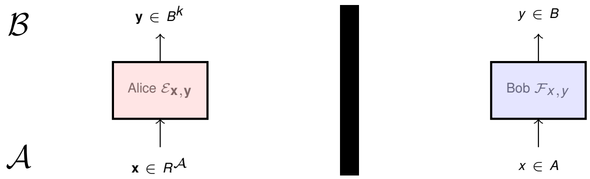
Composition of perfect strategies



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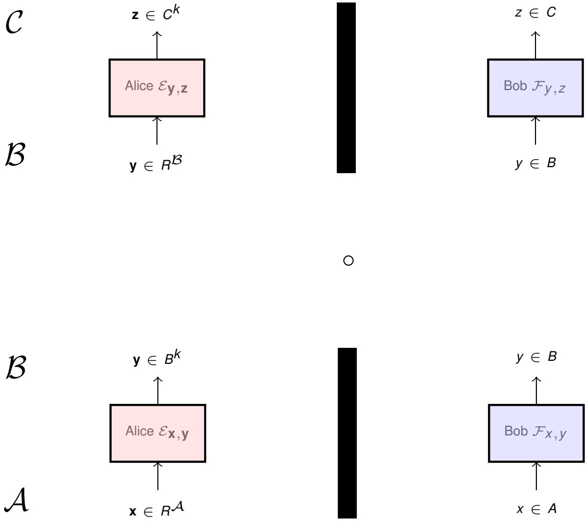

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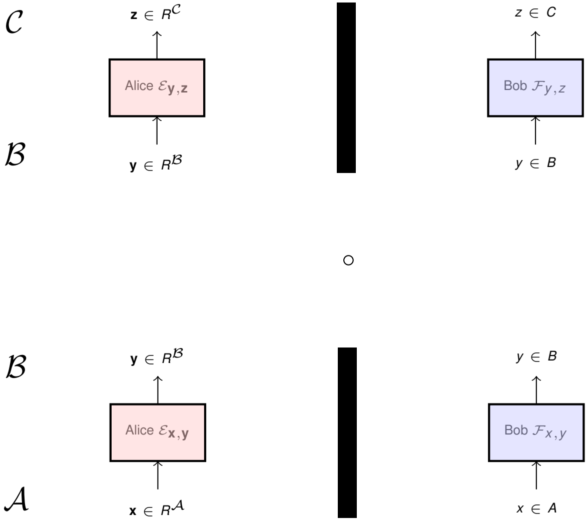

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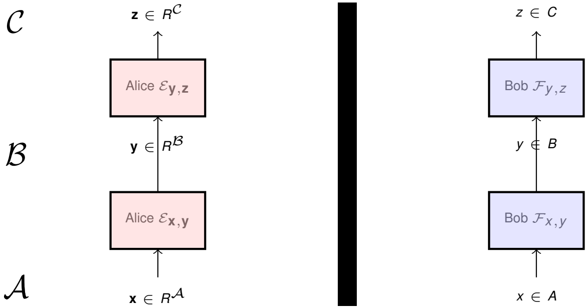

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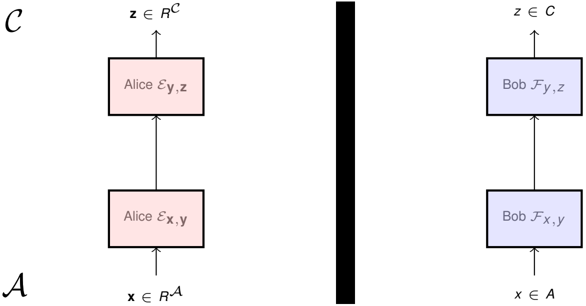
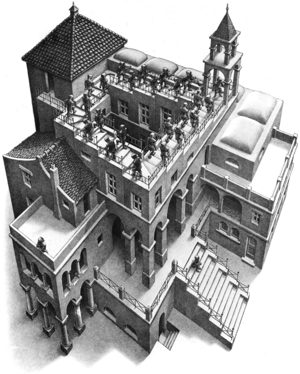


5. Contextuality and non-locality

Contextuality

Contextuality is a fundamental feature of quantum mechanics, which distinguishes it from classical physical theories.

It can be thought as saying that empirical predictions are inconsistent with all measurements having pre-determined outcomes.



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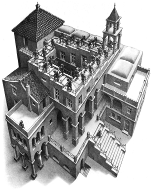
Non-locality is a particular case of contextuality for Bell scenarios

...but here we show that certain contextuality proofs can always be underwritten by non-locality arguments.



Contextuality

Measurement scenario $(X, \mathcal{M}, O)$:

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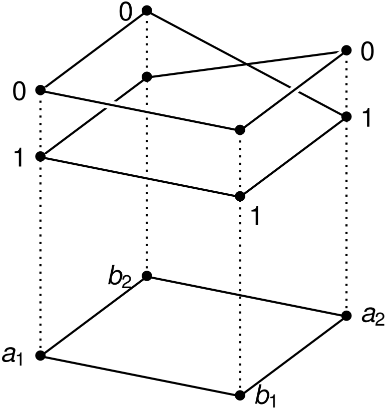
Strong contextuality: if there is no global assignment $g : X \longrightarrow O$ such that for all $C \in \mathcal{M}$, $e_C(g|_C) = 1$. That is, no global assignment is consistent with the model in the sense of yielding **possible** outcomes in all contexts.

E.g.: GHZ, Kochen–Specker, (post-quantum) PR box

Strong contextuality

Strong Contextuality:
no consistent global assignment.

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_1	b_1	✓	×	×	✓
a_1	b_2	✓	×	×	✓
a_2	b_1	✓	×	×	✓
a_2	b_2	×	✓	✓	×



Strong contextuality and constraint satisfaction

The support of e can be described as a CSP $\mathcal{K}_e$

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$\boxed{\text{Hence, } e \text{ is strongly contextual iff } \mathcal{K}_e \text{ has no (classical) solution.}}$

Quantum correspondence

Quantum witness for e:

- state φ
- PVM $P_x = \{P_{x,o}\}_{o \in O}$ for each $x \in X$
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General way of turning state-independent contextuality proofs into Bell non-locality arguments (generalising Heywood & Redhead's construction).

6. Outlook

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Thank you!

Questions...

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