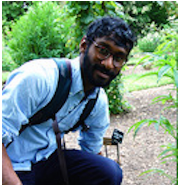
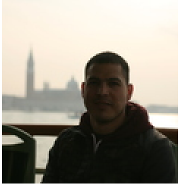
The quantum monad: towards quantum finite model theory


Samson Abramsky[1]


Rui Soares Barbosa[1]


Nadish de Silva[2]


Octavio Zapata[2]

[1]University of Oxford

[2]University College London

Quantum Physics & Logic (QPL 2018)
Dalhousie University, Halifax, 7th June 2018

- ‘The quantum monad on relational structures’ (MFCS’17, AQIS’17)
`arXiv:1705.07310 [cs.LO]`

- ‘The quantum monad on relational structures’
  (MFCS’17, AQIS’17)
  `arXiv:1705.07310 [cs.LO]`

- ‘The pebbling comonad in finite model theory’
  Samson Abramsky, Anuj Dawar, Pengming Wang
  (LICS’17)
  `arXiv:1704.05124 [cs.LO]`
- ‘Relating structure and power:
  Comonadic semantics for computational resources’
  Samson Abramsky, Nihil Shah

- ‘The quantum monad on relational structures’
  (MFCS’17, AQIS’17)
  `arXiv:1705.07310 [cs.LO]`

- ‘The pebbling comonad in finite model theory’
  Samson Abramsky, Anuj Dawar, Pengming Wang
  (LICS’17)
  `arXiv:1704.05124 [cs.LO]`
- ‘Relating structure and power:
  Comonadic semantics for computational resources’
  Samson Abramsky, Nihil Shah

- $\rightsquigarrow$ quantum finite model theory?

1. Introduction

Motivation

With the advent of quantum computation and information:

- use **quantum resources** for information-processing tasks
- delineate the scope of **quantum advantage**

Motivation

With the advent of quantum computation and information:

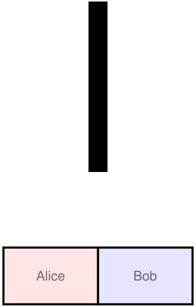
- use **quantum resources** for information-processing tasks
- delineate the scope of **quantum advantage**

- A setting in which this has been explored is **non-local games**

Non-local games

Alice and Bob cooperate in solving a task set by Verifier

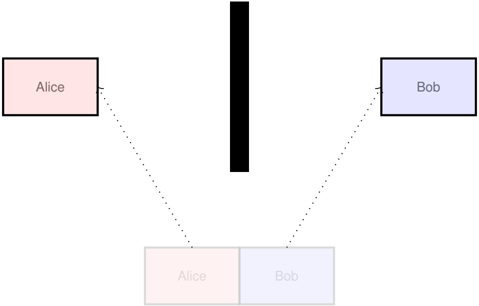
May share prior information,



Non-local games

Alice and Bob cooperate in solving a task set by Verifier

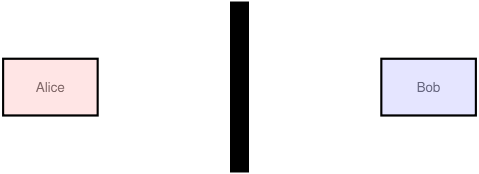
May share prior information,



Non-local games

Alice and Bob cooperate in solving a task set by Verifier

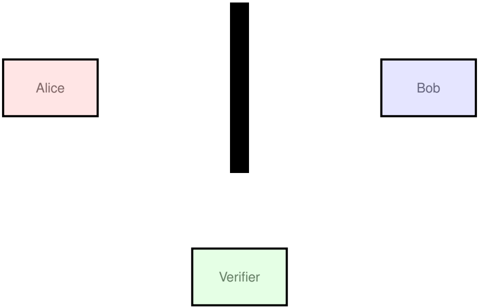
May share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier

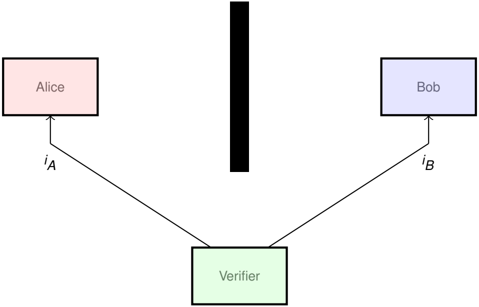
May share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier

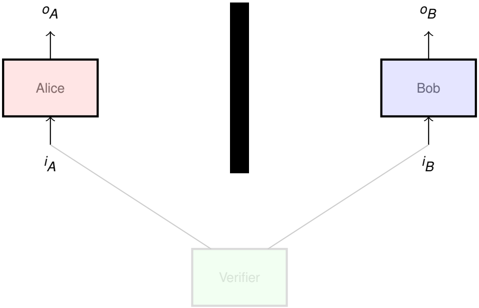
May share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier

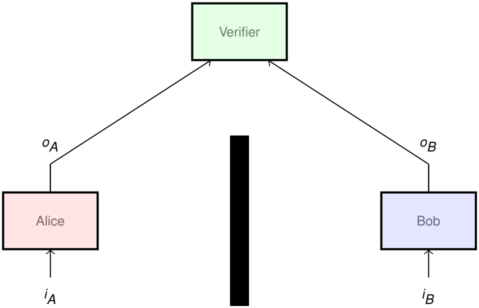
May share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier

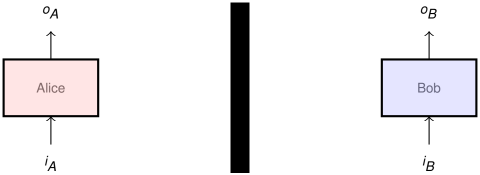
May share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier

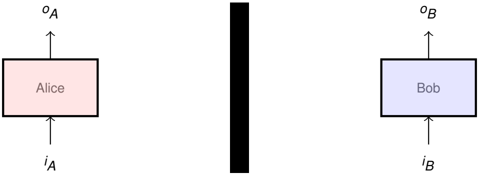
May share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier

May share prior information, but cannot communicate once game starts

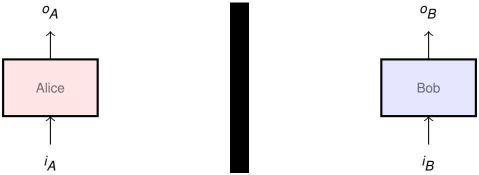


A strategy is described by the probabilities $P(\, o_A, o_B \mid i_A, i_B \,)$.

Non-local games

Alice and Bob cooperate in solving a task set by Verifier

May share prior information, but cannot communicate once game starts



A strategy is described by the probabilities $P(\, o_A, o_B \mid i_A, i_B \,)$.

A **perfect strategy** is one that wins with probability 1.

E.g.: Binary constraint systems

Magic square:

- Fill with 0s and 1s
- rows and first two columns: even parity
- last column: odd parity

E.g.: Binary constraint systems

A	B	C
D	E	F
G	H	I

Magic square:

- Fill with 0s and 1s
- rows and first two columns: even parity
- last column: odd parity

System of linear equations over $\mathbb{Z}_2$:

$$A \oplus B \oplus C = 0 \qquad A \oplus D \oplus G = 0$$

$$D \oplus E \oplus F = 0 \qquad B \oplus E \oplus H = 0$$

$$G \oplus H \oplus I = 0 \qquad C \oplus F \oplus I = 1$$

E.g.: Binary constraint systems

A	B	C
D	E	F
G	H	I

Magic square:

- Fill with 0s and 1s
- rows and first two columns: even parity
- last column: odd parity

System of linear equations over $\mathbb{Z}_2$:

$$
\begin{aligned}
A \oplus B \oplus C &= 0 \qquad & A \oplus D \oplus G &= 0 \\
D \oplus E \oplus F &= 0 \qquad & B \oplus E \oplus H &= 0 \\
G \oplus H \oplus I &= 0 \qquad & C \oplus F \oplus I &= 1
\end{aligned}
$$

Clearly, this is not satisfiable in $\mathbb{Z}_2$.

E.g.: Binary contraint satisfaction game

- Verifier sends an **equation** to Alice
- and a **variable** to Bob

E.g.: Binary contraint satisfaction game

- Verifier sends an **equation** to Alice
- and a **variable** to Bob

- Alice returns an **assignment** for the variables in her equation
- Bob returns a **value** for his variable

E.g.: Binary contraint satisfaction game

- Verifier sends an **equation** to Alice
- and a **variable** to Bob

- Alice returns an **assignment** for the variables in her equation
- Bob returns a **value** for his variable

- They win the play if:
  - Alice's assignment satisfies the equation
  - Bob's value is consistent with Alice's assigment

E.g.: Binary contraint satisfaction game

- Verifier sends an **equation** to Alice
- and a **variable** to Bob

- Alice returns an **assignment** for the variables in her equation
- Bob returns a **value** for his variable

- They win the play if:
  - Alice's assignment satisfies the equation
  - Bob's value is consistent with Alice's assigment

Classically, Alice and Bob have a perfect strategy if and only if there is an assignment to all variables satisfying the system of equations.

E.g.: Binary contraint satisfaction game

- Verifier sends an **equation** to Alice
- and a **variable** to Bob

- Alice returns an **assignment** for the variables in her equation
- Bob returns a **value** for his variable

- They win the play if:
  - Alice's assignment satisfies the equation
  - Bob's value is consistent with Alice's assigment

Classically, Alice and Bob have a perfect strategy if and only if there is an assignment to all variables satisfying the system of equations.

But using quantum resources, they can win the Magic Square game with probability 1, using Mermin's construction.

E.g.: Binary contraint satisfaction game

- Verifier sends an **equation** to Alice
- and a **variable** to Bob

- Alice returns an **assignment** for the variables in her equation
- Bob returns a **value** for his variable

- They win the play if:
  - Alice's assignment satisfies the equation
  - Bob's value is consistent with Alice's assigment

Classically, Alice and Bob have a perfect strategy if and only if there is an assignment to all variables satisfying the system of equations.

But using quantum resources, they can win the Magic Square game with probability 1, using Mermin's construction.

The system has a **quantum solution** but no classical solution!

Examples of non-local games

- Cleve, Mittal, Liu, Slofstra:
  games for binary constraint systems $\rightsquigarrow$ quantum solutions

Examples of non-local games

- Cleve, Mittal, Liu, Slofstra:
  games for binary constraint systems $\rightsquigarrow$ quantum solutions

- Cameron, Montanaro, Newman, Severini, Winter:
  game for graph colouring $\rightsquigarrow$ quantum chromatic number

- Mančinska & Roberson:
  generalised to a game for graph homomorphisms
  $\rightsquigarrow$ quantum graph homomorphisms

Examples of non-local games

- Cleve, Mittal, Liu, Slofstra:
  games for binary constraint systems $\rightsquigarrow$ quantum solutions

- Cameron, Montanaro, Newman, Severini, Winter:
  game for graph colouring $\rightsquigarrow$ quantum chromatic number

- Mančinska & Roberson:
  generalised to a game for graph homomorphisms
  $\rightsquigarrow$ quantum graph homomorphisms

Many of these works have some aspects in common. We aim to flesh this out by subsuming them under a common framework.

Motivation

Finite relational structures and homomorphisms

Motivation

Finite relational structures and homomorphisms

Pervasive notions in logic, computer science, combinatorics:

- constraint satisfaction
- finite model theory
- theory of relational databases
- graph theory

Motivation

Finite relational structures and homomorphisms

Pervasive notions in logic, computer science, combinatorics:

- constraint satisfaction
- finite model theory
- theory of relational databases
- graph theory

Many relevant questions in these areas can be phrased in terms of (existence, number of, . . .) homomorphisms between finite relational structures.

Motivation

What could it mean to quantise these fundamental structures?

Motivation

What could it mean to quantise these fundamental structures?

- We formulate the task of constructing a homomorphism between two relational structures as a **non-local game**
- **uniformly** obtain quantum analogues *for free* for a whole range of classical notions from CS, logic, . . .

Motivation

What could it mean to quantise these fundamental structures?

- We formulate the task of constructing a homomorphism between two relational structures as a **non-local game**
- **uniformly** obtain quantum analogues *for free* for a whole range of classical notions from CS, logic, . . .
- We then show that the use of quantum resources for information tasks is captured in a high-level way as **quantum homomorphisms**
- which can be integrated into a typed functional programming language through a **monadic** interface.

Outline

- Introduce homomorphism game for relational structures
- Arrive at the notion of quantum homomorphism,
  which removes the two-player aspect of the game

  (generalises Cleve & Mittal and Mančinska & Roberson)
- Quantum monad: capture quantum homomorphisms as classical homomorphisms to a *quantised* version of a relational structure

  (inspired on Mančinska & Roberson for graphs)
- Non-locality and state-independent strong contextuality
- Towards quantum finite model theory and descriptive complexity

2. Homomorphism game for relational structures

Relational structures and homomorphisms

A relational vocabulary σ consists of relational symbols $R_1, \ldots, R_p$ where R_l has an arity $k_l \in \mathbb{N}$ for each $l \in [p] := \{1, \ldots, p\}$.

Relational structures and homomorphisms

A relational vocabulary σ consists of relational symbols $R_1, \ldots, R_p$ where R_l has an arity $k_l \in \mathbb{N}$ for each $l \in [p] := \{1, \ldots, p\}$.

A σ**-structure** is $\mathcal{A} = (A; R_1^{\mathcal{A}}, \ldots R_p^{\mathcal{A}})$ where:

- A is a non-empty set,
- for each $l \in [p]$, $R_l^{\mathcal{A}} \subseteq A^{k_l}$ is a relation of arity k_l on A.

Relational structures and homomorphisms

A relational vocabulary σ consists of relational symbols $R_1, \ldots, R_p$ where R_l has an arity $k_l \in \mathbb{N}$ for each $l \in [p] := \{1, \ldots, p\}$.

A σ**-structure** is $\mathcal{A} = (A; R_1^{\mathcal{A}}, \ldots R_p^{\mathcal{A}})$ where:

- A is a non-empty set,
- for each $l \in [p]$, $R_l^{\mathcal{A}} \subseteq A^{k_l}$ is a relation of arity k_l on A.

A homomorphism of σ-structures $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a function $f : A \longrightarrow B$ such that for all $l \in [p]$ and $\mathbf{x} \in A^{k_l}$,

$$\mathbf{x} \in R_l^{\mathcal{A}} \quad \Longrightarrow \quad f(\mathbf{x}) \in R_l^{\mathcal{B}}$$

where $f(\mathbf{x}) = \langle f(x_1), \ldots, f(x_{k_l}) \rangle$ for $\mathbf{x} = \langle x_1, \ldots, x_{k_l} \rangle$.

Relational structures and homomorphisms

A relational vocabulary σ consists of relational symbols $R_1, \ldots, R_p$ where R_l has an arity $k_l \in \mathbb{N}$ for each $l \in [p] := \{1, \ldots, p\}$.

A σ**-structure** is $\mathcal{A} = (A; R_1^{\mathcal{A}}, \ldots R_p^{\mathcal{A}})$ where:

- A is a non-empty set,
- for each $l \in [p]$, $R_l^{\mathcal{A}} \subseteq A^{k_l}$ is a relation of arity k_l on A.

A homomorphism of σ-structures $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a function $f : A \longrightarrow B$ such that for all $l \in [p]$ and $\mathbf{x} \in A^{k_l}$,

$$\mathbf{x} \in R_l^{\mathcal{A}} \quad \Longrightarrow \quad f(\mathbf{x}) \in R_l^{\mathcal{B}}$$

where $f(\mathbf{x}) = \langle f(x_1), \ldots, f(x_{k_l}) \rangle$ for $\mathbf{x} = \langle x_1, \ldots, x_{k_l} \rangle$.

$R(\sigma)$: category of σ-structures and homomorphisms.

Relational structures and homomorphisms

A relational vocabulary σ consists of relational symbols $R_1, \ldots, R_p$ where R_l has an arity $k_l \in \mathbb{N}$ for each $l \in [p] := \{1, \ldots, p\}$.

A σ**-structure** is $\mathcal{A} = (A; R_1^{\mathcal{A}}, \ldots R_p^{\mathcal{A}})$ where:

- A is a non-empty set,
- for each $l \in [p]$, $R_l^{\mathcal{A}} \subseteq A^{k_l}$ is a relation of arity k_l on A.

A homomorphism of σ-structures $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a function $f : A \longrightarrow B$ such that for all $l \in [p]$ and $\mathbf{x} \in A^{k_l}$,

$$\mathbf{x} \in R_l^{\mathcal{A}} \quad \Longrightarrow \quad f(\mathbf{x}) \in R_l^{\mathcal{B}}$$

where $f(\mathbf{x}) = \langle f(x_1), \ldots, f(x_{k_l}) \rangle$ for $\mathbf{x} = \langle x_1, \ldots, x_{k_l} \rangle$.

$R(\sigma)$: category of σ-structures and homomorphisms.

(For simplicity, from now on consider a single relational symbol R of arity k)

The $(\mathcal{A}, \mathcal{B})$-homomorphism game

Given finite σ-structures $\mathcal{A}$ and $\mathcal{B}$, the players aim to convince the Verifier that there is a homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$.

The $(\mathcal{A}, \mathcal{B})$-homomorphism game

Given finite σ-structures $\mathcal{A}$ and $\mathcal{B}$, the players aim to convince the Verifier that there is a homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$.

- Verifier sends to Alice an index $l \in [p]$ and a tuple $\mathbf{x} \in R_l^{\mathcal{A}}$
- It sends to Bob an element $x \in A$

The $(\mathcal{A}, \mathcal{B})$-homomorphism game

Given finite σ-structures $\mathcal{A}$ and $\mathcal{B}$, the players aim to convince the Verifier that there is a homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$.

- Verifier sends to Alice an index $l \in [p]$ and a tuple $\mathbf{x} \in R_l^{\mathcal{A}}$
- It sends to Bob an element $x \in A$

- Alice returns a tuple $\mathbf{y} \in B^{k_l}$
- Bob returns an element $y \in B$.

The $(\mathcal{A}, \mathcal{B})$-homomorphism game

Given finite σ-structures $\mathcal{A}$ and $\mathcal{B}$, the players aim to convince the Verifier that there is a homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$.

- Verifier sends to Alice an index $l \in [p]$ and a tuple $\mathbf{x} \in R_l^{\mathcal{A}}$
- It sends to Bob an element $x \in A$
- Alice returns a tuple $\mathbf{y} \in B^{k_l}$
- Bob returns an element $y \in B$.
- They win this play if:
  - $\mathbf{y} \in R_l^{\mathcal{B}}$
  - $x = \mathbf{x}_i \implies y = \mathbf{y}_i$ for $1 \leq i \leq k_l$.

The $(\mathcal{A}, \mathcal{B})$-homomorphism game

Given finite σ-structures $\mathcal{A}$ and $\mathcal{B}$, the players aim to convince the Verifier that there is a homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$.

- Verifier sends to Alice an index $l \in [p]$ and a tuple $\mathbf{x} \in R_l^{\mathcal{A}}$
- It sends to Bob an element $x \in A$

- Alice returns a tuple $\mathbf{y} \in B^{k_l}$
- Bob returns an element $y \in B$.

- They win this play if:
  - $\mathbf{y} \in R_l^{\mathcal{B}}$
  - $x = \mathbf{x}_i \implies y = \mathbf{y}_i$ for $1 \leq i \leq k_l$.

If only classical resources are allowed, there is a perfect strategy if and only if there exists a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

The $(\mathcal{A}, \mathcal{B})$-homomorphism game

Given finite σ-structures $\mathcal{A}$ and $\mathcal{B}$, the players aim to convince the Verifier that there is a homomorphism $\mathcal{A} \longrightarrow \mathcal{B}$.

- Verifier sends to Alice an index $l \in [p]$ and a tuple $\mathbf{x} \in R_l^{\mathcal{A}}$
- It sends to Bob an element $x \in A$

- Alice returns a tuple $\mathbf{y} \in B^{k_l}$
- Bob returns an element $y \in B$.

- They win this play if:
  - $\mathbf{y} \in R_l^{\mathcal{B}}$
  - $x = \mathbf{x}_i \implies y = \mathbf{y}_i$ for $1 \leq i \leq k_l$.

If only classical resources are allowed, there is a perfect strategy if and only if there exists a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

What about quantum resources?

Homomorphism game with quantum resources

Quantum resources:

- Finite-dimensional Hilbert spaces $\mathcal{H}$ (Alice's) and $\mathcal{K}$ (Bob's)
- A bipartite pure state ψ on $\mathcal{H} \otimes \mathcal{K}$

Homomorphism game with quantum resources

Quantum resources:

- Finite-dimensional Hilbert spaces $\mathcal{H}$ (Alice's) and $\mathcal{K}$ (Bob's)
- A bipartite pure state ψ on $\mathcal{H} \otimes \mathcal{K}$
- For each tuple $\mathbf{x} \in R^{\mathcal{A}}$, Alice has POVM $\mathcal{E}_{\mathbf{x}} = \{\mathcal{E}_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y} \in B^k}$

Homomorphism game with quantum resources

Quantum resources:

- Finite-dimensional Hilbert spaces $\mathcal{H}$ (Alice's) and $\mathcal{K}$ (Bob's)
- A bipartite pure state ψ on $\mathcal{H} \otimes \mathcal{K}$
- For each tuple $\mathbf{x} \in R^{\mathcal{A}}$, Alice has POVM $\mathcal{E}_{\mathbf{x}} = \{\mathcal{E}_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y} \in B^k}$
- For each $x \in A$, Bob has POVM $\mathcal{F}_{\mathbf{x}} = \{\mathcal{F}_{x,y}\}_{y \in B}$

Homomorphism game with quantum resources

Quantum resources:

- Finite-dimensional Hilbert spaces $\mathcal{H}$ (Alice's) and $\mathcal{K}$ (Bob's)
- A bipartite pure state ψ on $\mathcal{H} \otimes \mathcal{K}$
- For each tuple $\mathbf{x} \in R^{\mathcal{A}}$, Alice has POVM $\mathcal{E}_{\mathbf{x}} = \{\mathcal{E}_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y} \in B^k}$
- For each $x \in A$, Bob has POVM $\mathcal{F}_{\mathbf{x}} = \{\mathcal{F}_{x,y}\}_{y \in B}$

These resources are used as follows:

- Given input $\mathbf{x} \in R^{\mathcal{A}}$, Alice measures $\mathcal{E}_{\mathbf{x}}$ on her part of ψ
- Given input $x \in A$, Bob measures $\mathcal{F}_x$ on his part of ψ
- Both output their respective measurement outcomes
- $P(\mathbf{y}, y \mid \mathbf{x}, x) = \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi$

Homomorphism game with quantum resources

Quantum resources:

- Finite-dimensional Hilbert spaces $\mathcal{H}$ (Alice's) and $\mathcal{K}$ (Bob's)
- A bipartite pure state ψ on $\mathcal{H} \otimes \mathcal{K}$
- For each tuple $\mathbf{x} \in R^{\mathcal{A}}$, Alice has POVM $\mathcal{E}_{\mathbf{x}} = \{\mathcal{E}_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y} \in B^k}$
- For each $x \in A$, Bob has POVM $\mathcal{F}_{\mathbf{x}} = \{\mathcal{F}_{x,y}\}_{y \in B}$

These resources are used as follows:

- Given input $\mathbf{x} \in R^{\mathcal{A}}$, Alice measures $\mathcal{E}_{\mathbf{x}}$ on her part of ψ
- Given input $x \in A$, Bob measures $\mathcal{F}_x$ on his part of ψ
- Both output their respective measurement outcomes
- $P(\mathbf{y}, y \mid \mathbf{x}, x) = \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi$

Perfect strategy conditions:

$$
\begin{array}{lll}
\text{(QS1)} & \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes I)\psi = 0 & \text{if } \mathbf{y} \notin R^{\mathcal{B}} \\
\text{(QS2)} & \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi = 0 & \text{if } x = \mathbf{x}_i \text{ and } y \neq \mathbf{y}_i
\end{array}
$$

3. From quantum perfect strategies to quantum homomorphisms

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

[1] This generalises Cleve & Mittal and Mančinska & Roberson.

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

- ψ is a maximally entangled state on $\mathbb{C}^d$, $\psi = {}^1/\sqrt{d} \sum_{i=1}^{d} e_i \otimes e_i$.

[1] This generalises Cleve & Mittal and Mančinska & Roberson.

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

- ψ is a maximally entangled state on $\mathbb{C}^d$, $\psi = {}^1/\sqrt{d} \sum_{i=1}^{d} e_i \otimes e_i$.
- The POVMs $\mathcal{E}_{\mathbf{x}}$ and $\mathcal{F}_x$ are projective.

[1] This generalises Cleve & Mittal and Mančinska & Roberson.

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

- ψ is a maximally entangled state on $\mathbb{C}^d$, $\psi = {}^1\!/\!\sqrt{d} \sum_{i=1}^{d} e_i \otimes e_i$.
- The POVMs $\mathcal{E}_{\mathbf{x}}$ and $\mathcal{F}_x$ are projective.
- If $x = \mathbf{x}_i$ then $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$, where $\mathcal{E}^i_{\mathbf{x},y} := \sum_{\mathbf{y}_i = y} \mathcal{E}_{\mathbf{x},\mathbf{y}}$.
- For $\mathbf{x} \in R^{\mathcal{A}}$, if $\mathbf{y} \notin R^{\mathcal{B}}$, then $\mathcal{E}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

[1] This generalises Cleve & Mittal and Mančinska & Roberson.

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

- ψ is a maximally entangled state on $\mathbb{C}^d$, $\psi = {}^1/\sqrt{d} \sum_{i=1}^{d} e_i \otimes e_i$.
- The POVMs $\mathcal{E}_{\mathbf{x}}$ and $\mathcal{F}_x$ are projective.
- If $x = \mathbf{x}_i$ then $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$, where $\mathcal{E}^i_{\mathbf{x},y} := \sum_{\mathbf{y}_i = y} \mathcal{E}_{\mathbf{x},\mathbf{y}}$.
- For $\mathbf{x} \in R^{\mathcal{A}}$, if $\mathbf{y} \notin R^{\mathcal{B}}$, then $\mathcal{E}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

$$\begin{array}{ll} \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes I)\psi = 0 & \text{if } \mathbf{y} \notin R^{\mathcal{B}} \\ \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi = 0 & \text{if } x = \mathbf{x}_i \text{ and } y \neq \mathbf{y}_i \end{array}$$

[1] This generalises Cleve & Mittal and Mančinska & Roberson.

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

- ψ is a maximally entangled state on $\mathbb{C}^d$, $\psi = 1/\sqrt{d} \sum_{i=1}^{d} e_i \otimes e_i$.
- The POVMs $\mathcal{E}_{\mathbf{x}}$ and $\mathcal{F}_x$ are projective.
- If $x = \mathbf{x}_i$ then $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$, where $\mathcal{E}^i_{\mathbf{x},y} := \sum_{\mathbf{y}_i = y} \mathcal{E}_{\mathbf{x},\mathbf{y}}$.
- For $\mathbf{x} \in R^{\mathcal{A}}$, if $\mathbf{y} \notin R^{\mathcal{B}}$, then $\mathcal{E}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

$\psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes I)\psi = 0$	if $\mathbf{y} \notin R^{\mathcal{B}}$
$\psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi = 0$	if $x = \mathbf{x}_i$ and $y \neq \mathbf{y}_i$

[1] This generalises Cleve & Mittal and Mančinska & Roberson.

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

- ψ is a maximally entangled state on $\mathbb{C}^d$, $\psi = {}^1\!/\!\sqrt{d} \sum_{i=1}^{d} e_i \otimes e_i$.
- The POVMs $\mathcal{E}_{\mathbf{x}}$ and $\mathcal{F}_x$ are projective.
- If $x = \mathbf{x}_i$ then $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$, where $\mathcal{E}^i_{\mathbf{x},y} := \sum_{\mathbf{y}_i = y} \mathcal{E}_{\mathbf{x},\mathbf{y}}$.
- For $\mathbf{x} \in R^{\mathcal{A}}$, if $\mathbf{y} \notin R^{\mathcal{B}}$, then $\mathcal{E}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

$\psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes I)\psi = 0$	if $\mathbf{y} \notin R^{\mathcal{B}}$
$\psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi = 0$	if $x = \mathbf{x}_i$ and $y \neq \mathbf{y}_i$

[1] This generalises Cleve & Mittal and Mančinska & Roberson.

Simplifying quantum strategies

The reduction proceeds in three steps:

1. The state and strategies are projected down to the support of the Schmidt decomposition of the state. This reduces the dimension of the Hilbert space and preserves the probabilities of the strategy exactly.

2. It is shown that this strategy must already satisfy strong properties (PVMs and $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{\mathbf{x}_i,y}$).

3. The state is changed but not the measurements. The probabilities change but the possibilities are preserved exactly.

Simplifying quantum strategies

The reduction proceeds in three steps:

1. The state and strategies are projected down to the support of the Schmidt decomposition of the state. This reduces the dimension of the Hilbert space and preserves the probabilities of the strategy exactly.

2. It is shown that this strategy must already satisfy strong properties (PVMs and $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{\mathbf{x}_i,y}$).

3. The state is changed but not the measurements. The probabilities change but the possibilities are preserved exactly.

N.B. In passing to the special form, the dimension is **reduced**; the process by which we obtain projective measurements is not at all akin to dilation.

Simplifying quantum strategies

Theorem The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

- ψ is a maximally entangled state on $\mathbb{C}^d$, $\psi = 1/\sqrt{d} \sum_{i=1}^{d} e_i \otimes e_i$.
- The POVMs $\mathcal{E}_{\mathbf{x}}$ and $\mathcal{F}_x$ are projective.
- If $x = \mathbf{x}_i$ then $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$, where $\mathcal{E}^i_{\mathbf{x},y} := \sum_{\mathbf{y}_i = y} \mathcal{E}_{\mathbf{x},\mathbf{y}}$.
- For $\mathbf{x} \in R^{\mathcal{A}}$, if $\mathbf{y} \notin R^{\mathcal{B}}$, then $\mathcal{E}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

Simplifying quantum strategies

Theorem The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

- ψ is a maximally entangled state on $\mathbb{C}^d$, $\psi = {}^1\!/\!\sqrt{d} \sum_{i=1}^{d} e_i \otimes e_i$.
- The POVMs $\mathcal{E}_{\mathbf{x}}$ and $\mathcal{F}_x$ are projective.
- If $x = \mathbf{x}_i$ then $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$, where $\mathcal{E}^i_{\mathbf{x},y} := \sum_{\mathbf{y}_i = y} \mathcal{E}_{\mathbf{x},\mathbf{y}}$.
- For $\mathbf{x} \in R^{\mathcal{A}}$, if $\mathbf{y} \notin R^{\mathcal{B}}$, then $\mathcal{E}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

All the information determining the strategy is in Alice's (or Bob's) operators.

Simplifying quantum strategies

Theorem The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

- ψ is a maximally entangled state on $\mathbb{C}^d$, $\psi = 1/\sqrt{d} \sum_{i=1}^{d} e_i \otimes e_i$.
- The POVMs $\mathcal{E}_{\mathbf{x}}$ and $\mathcal{F}_x$ are projective.
- If $x = \mathbf{x}_i$ then $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$, where $\mathcal{E}^i_{\mathbf{x},y} := \sum_{\mathbf{y}_i = y} \mathcal{E}_{\mathbf{x},\mathbf{y}}$.
- For $\mathbf{x} \in R^{\mathcal{A}}$, if $\mathbf{y} \notin R^{\mathcal{B}}$, then $\mathcal{E}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

All the information determining the strategy is in Alice's (or Bob's) operators.

which must be chosen so that $\mathcal{E}^i_{\mathbf{x},y}$ is independent of the context $\mathbf{x}$.

That is, we can define projectors $P_{x,y} := \mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$ whenever $x = \mathbf{x}_i$.
If $\mathbf{x}_i = x = \mathbf{x}'_j$, then we have $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y} = \mathcal{E}^j_{\mathbf{x}',y}$, so $P_{x,y}$ is well-defined.

Simplifying quantum strategies

Theorem The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

- ψ is a maximally entangled state on $\mathbb{C}^d$, $\psi = 1/\sqrt{d} \sum_{i=1}^{d} e_i \otimes e_i$.
- The POVMs $\mathcal{E}_{\mathbf{x}}$ and $\mathcal{F}_x$ are projective.
- If $x = \mathbf{x}_i$ then $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$, where $\mathcal{E}^i_{\mathbf{x},y} := \sum_{\mathbf{y}_i = y} \mathcal{E}_{\mathbf{x},\mathbf{y}}$.
- For $\mathbf{x} \in R^{\mathcal{A}}$, if $\mathbf{y} \notin R^{\mathcal{B}}$, then $\mathcal{E}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

All the information determining the strategy is in Alice's (or Bob's) operators.

which must be chosen so that $\mathcal{E}^i_{\mathbf{x},y}$ is independent of the context $\mathbf{x}$.

That is, we can define projectors $P_{x,y} := \mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y}$ whenever $x = \mathbf{x}_i$.
If $\mathbf{x}_i = x = \mathbf{x}'_j$, then we have $\mathcal{E}^i_{\mathbf{x},y} = \mathcal{F}^{\mathsf{T}}_{x,y} = \mathcal{E}^j_{\mathbf{x}',y}$, so $P_{x,y}$ is well-defined.

These $P_{x,y}$ are enough to determine the strategy!

Quantum homomorphisms

A quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a family of projectors $\{P_{x,y}\}_{x \in A, y \in B}$ in some dimension $d \in \mathbb{N}$ satisfying:

(QH1) For all $x \in A$, $\sum_{y \in B} P_{x,y} = I$.

(QH2) For all $\mathbf{x} \in R^{\mathcal{A}}$, $x = \mathbf{x}_i$, $x' = \mathbf{x}_j$,

$$[P_{x,y}, P_{x',y'}] = \mathbf{0} \quad \text{for any } y, y' \in B$$

so we can define a projective measurement $P_{\mathbf{x}} = \{P_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y}}$, where $P_{\mathbf{x},\mathbf{y}} := P_{\mathbf{x}_1,\mathbf{y}_1} \cdots P_{\mathbf{x}_k,\mathbf{y}_k}$.

(QH3) If $\mathbf{x} \in R^{\mathcal{A}}$ and $\mathbf{y} \notin R^{\mathcal{B}}$, then $P_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

Quantum homomorphisms

A quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a family of projectors $\{P_{x,y}\}_{x \in A, y \in B}$ in some dimension $d \in \mathbb{N}$ satisfying:

(QH1) For all $x \in A$, $\sum_{y \in B} P_{x,y} = I$.

(QH2) For all $\mathbf{x} \in R^{\mathcal{A}}$, $x = \mathbf{x}_i$, $x' = \mathbf{x}_j$,

$$[P_{x,y}, P_{x',y'}] = \mathbf{0} \quad \text{for any } y, y' \in B$$

so we can define a projective measurement $P_{\mathbf{x}} = \{P_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y}}$, where $P_{\mathbf{x},\mathbf{y}} := P_{\mathbf{x}_1,\mathbf{y}_1} \cdots P_{\mathbf{x}_k,\mathbf{y}_k}$.

(QH3) If $\mathbf{x} \in R^{\mathcal{A}}$ and $\mathbf{y} \notin R^{\mathcal{B}}$, then $P_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

Theorem For finite structures $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent:

1. The ($\mathcal{A}$,$\mathcal{B}$)-homomorphism game has a quantum perfect strategy.
2. There is a quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$. ($\mathcal{A} \xrightarrow{q} \mathcal{B}$)

4. Quantum homomorphisms and the quantum monad

Quantum homomorphisms as Kleisli maps

For each $d \in \mathbb{N}$ and σ-structure $\mathcal{A}$, we can define a structure $\mathcal{Q}_d\mathcal{A}$ such that there is a one-to-one correspondence:[2]

$$\mathcal{A} \xrightarrow{q}_d \mathcal{B} \quad \cong \quad \mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$$

- quantum homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ of dimension d
- (classical) homomorphisms from $\mathcal{A}$ to $\mathcal{Q}_d\mathcal{B}$

[2] Mančinska & Roberson: analogous construction for (their) graph homomorphisms.

Quantum homomorphisms as Kleisli maps

For each $d \in \mathbb{N}$ and σ-structure $\mathcal{A}$, we can define a structure $\mathcal{Q}_d\mathcal{A}$ such that there is a one-to-one correspondence:[2]

$$\mathcal{A} \xrightarrow{q}_d \mathcal{B} \quad \cong \quad \mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$$

- quantum homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ of dimension d
- (classical) homomorphisms from $\mathcal{A}$ to $\mathcal{Q}_d\mathcal{B}$

Universe of structure $\mathcal{Q}_d\mathcal{A}$: set of functions $p : A \longrightarrow \mathrm{Proj}(d)$ such that $\sum_{x \in A} p(x) = I$. (Projector-valued distributions on A in dimension d.)

[2] Mančinska & Roberson: analogous construction for (their) graph homomorphisms.

Quantum homomorphisms as Kleisli maps

For each $d \in \mathbb{N}$ and σ-structure $\mathcal{A}$, we can define a structure $\mathcal{Q}_d\mathcal{A}$ such that there is a one-to-one correspondence:[2]

$$\mathcal{A} \xrightarrow{q}_d \mathcal{B} \quad \cong \quad \mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$$

- quantum homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ of dimension d
- (classical) homomorphisms from $\mathcal{A}$ to $\mathcal{Q}_d\mathcal{B}$

Universe of structure $\mathcal{Q}_d\mathcal{A}$: set of functions $p : A \longrightarrow \mathrm{Proj}(d)$ such that $\sum_{x \in A} p(x) = I$. (Projector-valued distributions on A in dimension d.)

For R of arity k, $R^{\mathcal{Q}_d\mathcal{A}}$ is the set of tuples $\langle p_1, \ldots, p_k \rangle$ satisfying:

(QR1) For all $1 \leq i, j \leq k$ and $x, x' \in A$, $[p_i(x), p_j(x')] = \mathbf{0}$.

(QR2) For all $\mathbf{x} \in A^k$, if $\mathbf{x} \notin R^{\mathcal{A}}$, then $p_1(x_1) \cdots p_k(x_k) = \mathbf{0}$.

[2] Mančinska & Roberson: analogous construction for (their) graph homomorphisms.

Quantum homomorphisms as Kleisli maps

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

Monads play a major rôle in programming language theory, providing a uniform way of encapsulating various notions of computation:

- partiality
- exceptions
- non-determinism
- probabilistic
- state updates
- input/output
- ...

Monads

Functor $T : \mathfrak{C} \longrightarrow \mathfrak{C}$ such that a T-program, a computation producing values of type B from values of type A with T-effects, is seen as a map $A \longrightarrow T\,B$ in the category $\mathfrak{C}$.

Monads

Functor $T : \mathfrak{C} \longrightarrow \mathfrak{C}$ such that a T-program, a computation producing values of type B from values of type A with T-effects, is seen as a map $A \longrightarrow T\,B$ in the category $\mathfrak{C}$.

- $\eta_A : A \longrightarrow T\,A$
- $\mu_A : T(T\,A) \longrightarrow T\,A$

Monads

Functor $T : \mathfrak{C} \longrightarrow \mathfrak{C}$ such that a T-program, a computation producing values of type B from values of type A with T-effects, is seen as a map $A \longrightarrow T\,B$ in the category $\mathfrak{C}$.

- $\eta_A : A \longrightarrow T\,A$
- $\mu_A : T(T\,A) \longrightarrow T\,A$

Composition:

$$B \xrightarrow{\quad g \quad} T\,C$$

$$A \xrightarrow{\quad f \quad} T\,B$$

Kleisli morphisms form a category.

Monads

Functor $T : \mathfrak{C} \longrightarrow \mathfrak{C}$ such that a T-program, a computation producing values of type B from values of type A with T-effects, is seen as a map $A \longrightarrow T\,B$ in the category $\mathfrak{C}$.

- $\eta_A : A \longrightarrow T\,A$
- $\mu_A : T(T\,A) \longrightarrow T\,A$

Composition:

$$B \xrightarrow{\quad g \quad} T\,C$$

$$A \xrightarrow{\quad f \quad} T\,B \xrightarrow{\quad Tg \quad} T(T\,C)$$

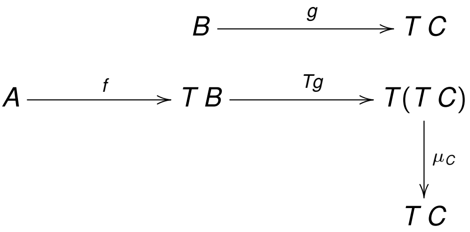
Kleisli morphisms form a category.

Monads

Functor $T : \mathfrak{C} \longrightarrow \mathfrak{C}$ such that a T-program, a computation producing values of type B from values of type A with T-effects, is seen as a map $A \longrightarrow T\,B$ in the category $\mathfrak{C}$.

- $\eta_A : A \longrightarrow T\,A$
- $\mu_A : T(T\,A) \longrightarrow T\,A$

Composition:



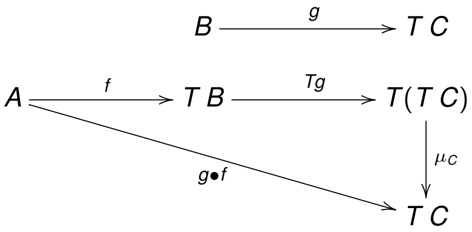
Kleisli morphisms form a category.

Monads

Functor $T : \mathfrak{C} \longrightarrow \mathfrak{C}$ such that a T-program, a computation producing values of type B from values of type A with T-effects, is seen as a map $A \longrightarrow T\,B$ in the category $\mathfrak{C}$.

- $\eta_A : A \longrightarrow T\,A$
- $\mu_A : T(T\,A) \longrightarrow T\,A$

Composition:



Kleisli morphisms form a category.

Composition of quantum homomorphsms

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

Composition of quantum homomorphisms

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

The quantum monad is graded by dimension

- $\mu_{\mathcal{A}}^{d,d'} : \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{A}) \longrightarrow \mathcal{Q}_{dd'}\,\mathcal{A}$

$\rightsquigarrow$ composition of quantum homomorphisms
keeps track of the dimension of the resources

$$\mathcal{B} \xrightarrow{\quad f \quad} \mathcal{Q}_{d'}\,\mathcal{C}$$

$$\mathcal{A} \xrightarrow{\quad f \quad} \mathcal{Q}_d\,\mathcal{B}$$

Composition of quantum homomorphisms

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

The quantum monad is graded by dimension

- $\mu_{\mathcal{A}}^{d,d'} : \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{A}) \longrightarrow \mathcal{Q}_{dd'}\,\mathcal{A}$

$\rightsquigarrow$ composition of quantum homomorphisms
keeps track of the dimension of the resources

$$\mathcal{B} \xrightarrow{\quad f \quad} \mathcal{Q}_{d'}\,\mathcal{C}$$

$$\mathcal{A} \xrightarrow{\quad f \quad} \mathcal{Q}_d\,\mathcal{B}$$

Composition of quantum homomorphisms

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

The quantum monad is graded by dimension

- $\mu_{\mathcal{A}}^{d,d'} : \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{A}) \longrightarrow \mathcal{Q}_{dd'}\,\mathcal{A}$

$\rightsquigarrow$ composition of quantum homomorphisms
keeps track of the dimension of the resources

$$\mathcal{B} \xrightarrow{\quad f \quad} \mathcal{Q}_{d'}\,\mathcal{C}$$

$$\mathcal{A} \xrightarrow{\quad f \quad} \mathcal{Q}_d\,\mathcal{B} \xrightarrow{\quad \mathcal{Q}_d g \quad} \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{C})$$

Composition of quantum homomorphisms

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

The quantum monad is graded by dimension

- $\mu^{d,d'}_{\mathcal{A}} : \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{A}) \longrightarrow \mathcal{Q}_{dd'}\,\mathcal{A}$

$\rightsquigarrow$ composition of quantum homomorphisms
keeps track of the dimension of the resources

$$\mathcal{B} \xrightarrow{\quad f \quad} \mathcal{Q}_{d'}\,\mathcal{C}$$

$$\mathcal{A} \xrightarrow{\quad f \quad} \mathcal{Q}_d\,\mathcal{B} \xrightarrow{\quad \mathcal{Q}_d g \quad} \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{C}) \xrightarrow{\quad \mu^{d,d'}_{\mathcal{C}} \quad} \mathcal{Q}_{dd'}\,\mathcal{C}$$

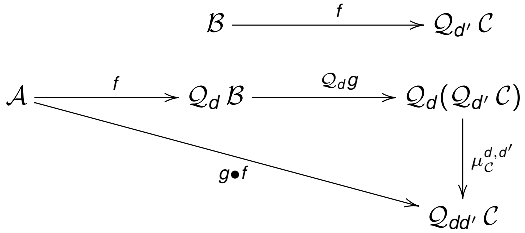
Composition of quantum homomorphisms

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

The quantum monad is graded by dimension

- $\mu_{\mathcal{A}}^{d,d'} : \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{A}) \longrightarrow \mathcal{Q}_{dd'}\,\mathcal{A}$

$\rightsquigarrow$ composition of quantum homomorphisms
keeps track of the dimension of the resources



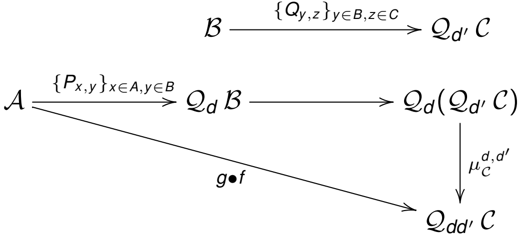
Composition of quantum homomorphisms

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

The quantum monad is graded by dimension

- $\mu_{\mathcal{A}}^{d,d'} : \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{A}) \longrightarrow \mathcal{Q}_{dd'}\,\mathcal{A}$

$\rightsquigarrow$ composition of quantum homomorphisms
keeps track of the dimension of the resources



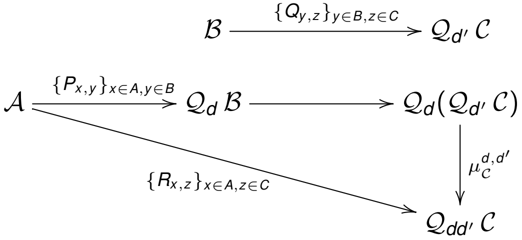
Composition of quantum homomorphisms

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

The quantum monad is graded by dimension

- $\mu_{\mathcal{A}}^{d,d'} : \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{A}) \longrightarrow \mathcal{Q}_{dd'}\,\mathcal{A}$

$\rightsquigarrow$ composition of quantum homomorphisms
keeps track of the dimension of the resources



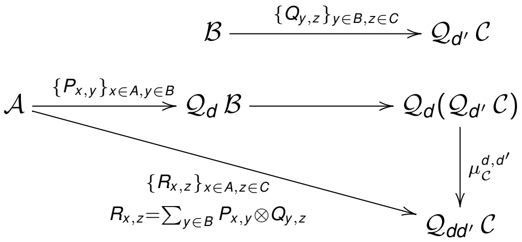
Composition of quantum homomorphisms

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$ of relational structures and (classical) homomorphisms.

The quantum monad is graded by dimension

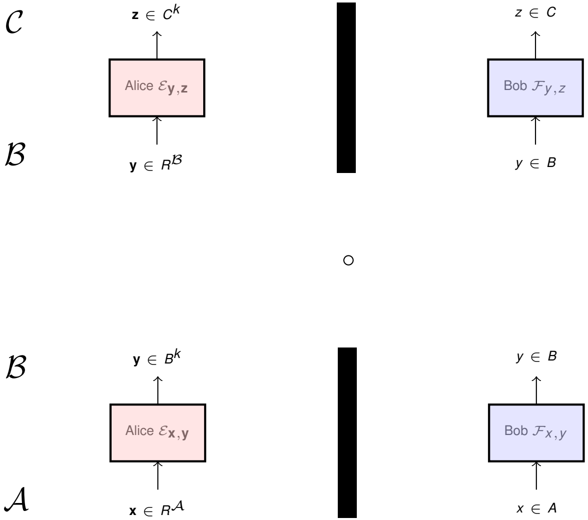
- $\mu^{d,d'}_{\mathcal{A}} : \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{A}) \longrightarrow \mathcal{Q}_{dd'}\,\mathcal{A}$

$\rightsquigarrow$ composition of quantum homomorphisms
keeps track of the dimension of the resources

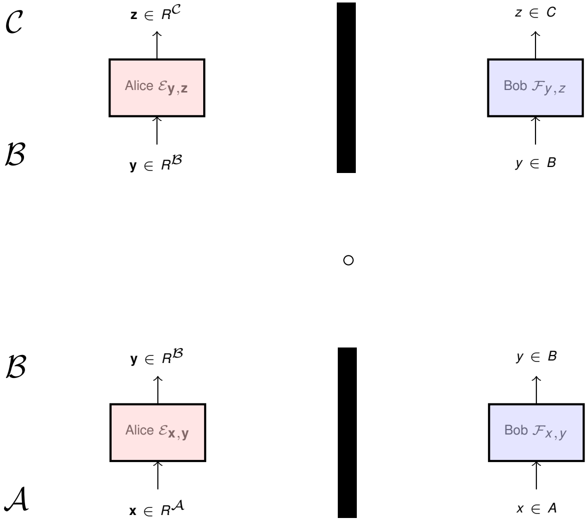


Composition of perfect strategies

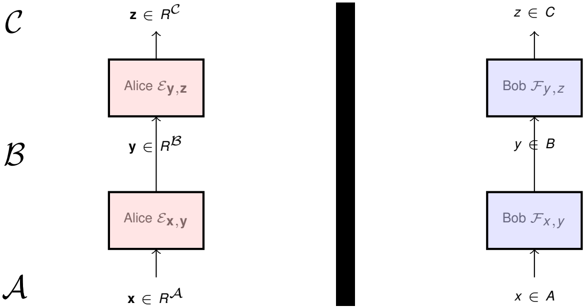
Composition of perfect strategies



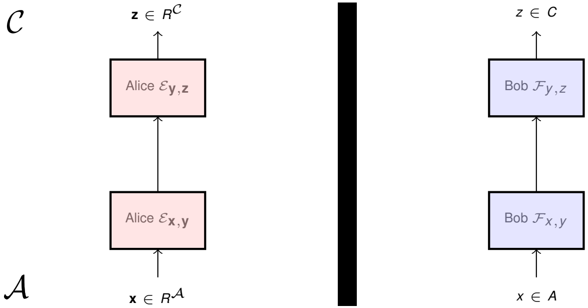
Composition of perfect strategies



Composition of perfect strategies



Composition of perfect strategies

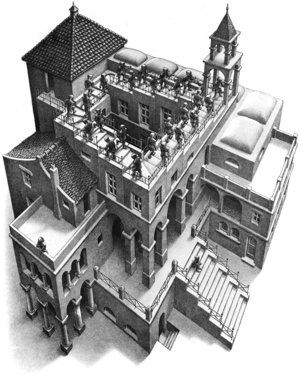


5. Contextuality and non-locality

Contextuality

Contextuality is a fundamental feature of quantum mechanics, which distinguishes it from classical physical theories.

It can be thought as saying that empirical predictions are inconsistent with all measurements having pre-determined outcomes.



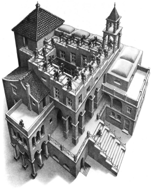
Contextuality

Contextuality is a fundamental feature of quantum mechanics, which distinguishes it from classical physical theories.

It can be thought as saying that empirical predictions are inconsistent with all measurements having pre-determined outcomes.

Non-locality is a particular case of contextuality for Bell scenarios

…but here we show that certain contextuality proofs can always be underwritten by non-locality arguments.



Contextuality

Measurement scenario $(X, \mathcal{M}, O)$:

- X is a finite set of measurements
- O is a finite set of outcomes
- $\mathcal{M}$ is a cover of X, where $C \in \mathcal{M}$ is a set of compatible measurements (context)

Contextuality

Measurement scenario $(X, \mathcal{M}, O)$:

- X is a finite set of measurements
- O is a finite set of outcomes
- $\mathcal{M}$ is a cover of X, where $C \in \mathcal{M}$ is a set of compatible measurements (context)

Empirical model: probability distributions on joint outcomes of measurements in a context C.

Contextuality

Measurement scenario $(X, \mathcal{M}, O)$:

- X is a finite set of measurements
- O is a finite set of outcomes
- $\mathcal{M}$ is a cover of X, where $C \in \mathcal{M}$ is a set of compatible measurements (context)

Empirical model: probability distributions on joint outcomes of measurements in a context C.

Possibilistic information: for $C \in \mathcal{M}$ and $s \in O^C$, $e_C(s) \in \{0, 1\}$ indicates if joint outcome s for measurements C is possible or not.

Contextuality

Measurement scenario $(X, \mathcal{M}, O)$:

- X is a finite set of measurements
- O is a finite set of outcomes
- $\mathcal{M}$ is a cover of X, where $C \in \mathcal{M}$ is a set of compatible measurements (context)

Empirical model: probability distributions on joint outcomes of measurements in a context C.

Possibilistic information: for $C \in \mathcal{M}$ and $s \in O^C$, $e_C(s) \in \{0, 1\}$ indicates if joint outcome s for measurements C is possible or not.

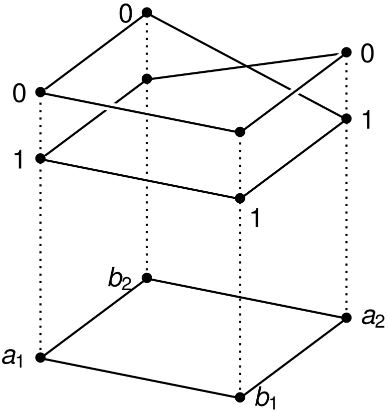
Strong contextuality: if there is no global assignment $g : X \longrightarrow O$ such that for all $C \in \mathcal{M}$, $e_C(g|_C) = 1$. That is, no global assignment is consistent with the model in the sense of yielding **possible** outcomes in all contexts.

E.g.: GHZ, Kochen–Specker, (post-quantum) PR box

Strong contextuality

Strong Contextuality:
no consistent global assignment.

A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_1	b_1	✓	×	×	✓
a_1	b_2	✓	×	×	✓
a_2	b_1	✓	×	×	✓
a_2	b_2	×	✓	✓	×



Strong contextuality and constraint satisfaction

The support of e can be described as a CSP $\mathcal{K}_e$

There is a one-to-one correspondence between:

- solutions for $\mathcal{K}_e$
- (homomorphisms $\mathcal{A}_{\mathcal{K}_e} \longrightarrow \mathcal{B}_{\mathcal{K}_e}$)

Strong contextuality and constraint satisfaction

The support of e can be described as a CSP $\mathcal{K}_e$

There is a one-to-one correspondence between:

- solutions for $\mathcal{K}_e$
- (homomorphisms $\mathcal{A}_{\mathcal{K}_e} \longrightarrow \mathcal{B}_{\mathcal{K}_e}$)
- consistent global assignements for e

Hence, e is strongly contextual iff $\mathcal{K}_e$ has no (classical) solution.

Quantum correspondence

Quantum witness for e:

- state φ
- PVM $P_x = \{P_{x,o}\}_{o \in O}$ for each $x \in X$
- $[P_{x,o}, P_{x',o'}] = \mathbf{0}$ whenever $x, x' \in C \in M$
- For all $C \in \mathcal{M}, s \in O^C, \quad e_C(s) = 0 \implies \varphi^* P_{\mathbf{x},s(\mathbf{x})} \varphi = 0$

State-independent witness: family of PVMs yielding witness for any φ.

Quantum correspondence

Quantum witness for e:

- state φ
- PVM $P_x = \{P_{x,o}\}_{o \in O}$ for each $x \in X$
- $[P_{x,o}, P_{x',o'}] = \mathbf{0}$ whenever $x, x' \in C \in M$
- For all $C \in \mathcal{M}, s \in O^C, \quad e_C(s) = 0 \implies \varphi^* P_{\mathbf{x},s(\mathbf{x})} \varphi = 0$

State-independent witness: family of PVMs yielding witness for any φ.

There is one-to-one correspondence between:

- quantum solutions for the CSP $\mathcal{K}_e$
- (i.e. quantum homomorphisms $\mathcal{A}_{\mathcal{K}_e} \xrightarrow{q} \mathcal{B}_{\mathcal{K}_e}$)

Quantum correspondence

Quantum witness for e:

- state φ
- PVM $P_x = \{P_{x,o}\}_{o \in O}$ for each $x \in X$
- $[P_{x,o}, P_{x',o'}] = \mathbf{0}$ whenever $x, x' \in C \in M$
- For all $C \in \mathcal{M}, s \in O^C, \quad e_C(s) = 0 \implies \varphi^* P_{\mathbf{x}, s(\mathbf{x})} \varphi = 0$

State-independent witness: family of PVMs yielding witness for any φ.

There is one-to-one correspondence between:

- quantum solutions for the CSP $\mathcal{K}_e$
- (i.e. quantum homomorphisms $\mathcal{A}_{\mathcal{K}_e} \xrightarrow{q} \mathcal{B}_{\mathcal{K}_e}$)
- state-independent witnesses for e

Quantum state-independent strong contextuality:
$\mathcal{K}_e$ has quantum solution but no classical solution!

Quantum correspondence

Quantum witness for e:

- state φ
- PVM $P_x = \{P_{x,o}\}_{o \in O}$ for each $x \in X$
- $[P_{x,o}, P_{x',o'}] = \mathbf{0}$ whenever $x, x' \in C \in M$
- For all $C \in \mathcal{M}, s \in O^C, \quad e_C(s) = 0 \implies \varphi^* P_{\mathbf{x},s(\mathbf{x})}\varphi = 0$

State-independent witness: family of PVMs yielding witness for any φ.

There is one-to-one correspondence between:

- quantum solutions for the CSP $\mathcal{K}_e$
- (i.e. quantum homomorphisms $\mathcal{A}_{\mathcal{K}_e} \xrightarrow{q} \mathcal{B}_{\mathcal{K}_e}$)
- state-independent witnesses for e

Quantum state-independent strong contextuality: $\mathcal{K}_e$ has quantum solution but no classical solution!

General way of turning state-independent contextuality proofs into Bell non-locality arguments (generalising Heywood & Redhead's construction).

6. Outlook

Outlook

- Quantum graph isomorphisms, and isomorphisms of relational structures. How does it fit? Other similar generalisations?

Outlook

- Quantum graph isomorphisms, and isomorphisms of relational structures. How does it fit? Other similar generalisations?
- What about state-dependent contextuality?

Outlook

- Quantum graph isomorphisms, and isomorphisms of relational structures. How does it fit? Other similar generalisations?
- What about state-dependent contextuality?
- A strategy has a winning probability. Can we adapt this to deal with quantitative aspects (contextual fraction, ...)?

Outlook

- Quantum graph isomorphisms, and isomorphisms of relational structures. How does it fit? Other similar generalisations?
- What about state-dependent contextuality?
- A strategy has a winning probability. Can we adapt this to deal with quantitative aspects (contextual fraction, . . .)?
- Quantising classical notions in the framework of relational structures

Outlook: Towards quantum finite model theory

Model theory:

- Logics give **limited access** to structures they're interpreted over.
- One 'sees' a structure up to **definable properties**.

Outlook: Towards quantum finite model theory

Model theory:

- Logics give **limited access** to structures they're interpreted over.
- One 'sees' a structure up to **definable properties**.
- $\rightsquigarrow$ equivalences coarser than isomorphism:

$$\mathcal{A} \equiv^{\mathcal{L}} \mathcal{B} \quad := \quad \forall \phi \in \mathcal{L}.\, \mathcal{A} \models \phi \Leftrightarrow \mathcal{B} \models \phi$$

Outlook: Towards quantum finite model theory

Descriptive complexity:

- Characterise a complexity class (for decision problems) as those classes of structures expressible in a certain logic

Outlook: Towards quantum finite model theory

Descriptive complexity:

- Characterise a complexity class (for decision problems) as those classes of structures expressible in a certain logic
- $\phi \in \mathcal{L}$ determines class $\mathcal{K} = \{\mathcal{A} \mid \mathcal{A} \models \phi\}$.

Outlook: Towards quantum finite model theory

Descriptive complexity:

- Characterise a complexity class (for decision problems) as those classes of structures expressible in a certain logic
- $\phi \in \mathcal{L}$ determines class $\mathcal{K} = \{\mathcal{A} \mid \mathcal{A} \models \phi\}$.
- computational complexity $\leftrightarrow$ expressive power of logics

E.g. PH	$\leftrightarrow$ SO	(second-order logic)
NP	$\leftrightarrow$ $\exists$SO	(existential second-order logic)
AC^0	$\leftrightarrow$ FO$(+, \times)$	(first-order logic with $+$ and $\times$)

Outlook: Towards quantum finite model theory

- Spoiler–Duplicator games are the main tool for characterising elementary equivalences.
- E.g. pebble games capture the idea of limited access to a structure through a 'moving window' of fixed size k (number of pebbles), corresponding to what is expressible in k-variable logic.

Outlook: Towards quantum finite model theory

- Spoiler–Duplicator games are the main tool for characterising elementary equivalences.
- E.g. pebble games capture the idea of limited access to a structure through a 'moving window' of fixed size k (number of pebbles), corresponding to what is expressible in k-variable logic.

Model theory without syntax:

- Capture model-theoretic notions (equivalences and definable classes) using game comonads.
- Pebble games can be formulated via co-Kleisli maps $T_k \mathcal{A} \longrightarrow \mathcal{B}$.
- Key combinatorial invariants arise as the coalgebra number.

Outlook: Towards quantum finite model theory

- Spoiler–Duplicator games are the main tool for characterising elementary equivalences.
- E.g. pebble games capture the idea of limited access to a structure through a 'moving window' of fixed size *k* (number of pebbles), corresponding to what is expressible in *k*-variable logic.

Model theory without syntax:

- Capture model-theoretic notions (equivalences and definable classes) using game comonads.
- Pebble games can be formulated via co-Kleisli maps $T_k\mathcal{A} \longrightarrow \mathcal{B}$.
- Key combinatorial invariants arise as the coalgebra number.

Can this be similarly *quantised*?

Outlook: Towards quantum finite model theory

Can this be similarly *quantised*?

Outlook: Towards quantum finite model theory

Can this be similarly *quantised*?

- Do Bi-Kleisli maps $T_k\mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$ yield quantum pebble games?

Outlook: Towards quantum finite model theory

Can this be similarly *quantised*?

- Do Bi-Kleisli maps $T_k\mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$ yield quantum pebble games?
- $\rightsquigarrow$ quantum version of these logical equivalences
  $\rightsquigarrow$ quantum finite model theory? (without quantum logic)

Outlook: Towards quantum finite model theory

Can this be similarly *quantised*?

- Do Bi-Kleisli maps $T_k\mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$ yield quantum pebble games?
- $\rightsquigarrow$ quantum version of these logical equivalences
  $\rightsquigarrow$ quantum finite model theory? (without quantum logic)
- Quantum validity?
  Homomorphisms are related to the existential positive fragment: can this be extended to provide quantum vality for FO formulae?

Outlook: Towards quantum finite model theory

Can this be similarly *quantised*?

- Do Bi-Kleisli maps $T_k\mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$ yield quantum pebble games?
- $\rightsquigarrow$ quantum version of these logical equivalences
  $\rightsquigarrow$ quantum finite model theory? (without quantum logic)
- Quantum validity?
  Homomorphisms are related to the existential positive fragment: can this be extended to provide quantum vality for FO formulae?
- Quantum descriptive complexity?
  Do quantised versions of these logical equivalences correspond to quantum computational complexity classes?

Thank you!

Questions...

?

The quantum monad on relational structures (MFCS'17, AQIS'17)

`arXiv:1705.07310`