Contextuality as a resource yielding quantum advantage

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Joint work with:

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and also:

- Kohei Kishida (Dalhousie)
- Giovanni Carù (Oxford)
- Nadish de Silva (UCL)
- Octavio Zapata (UCL)

Motivation

- **Contextuality and non-locality**:
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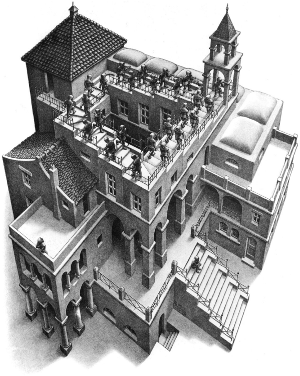

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    Howard–Wallman–Veith–Emerson (2014)
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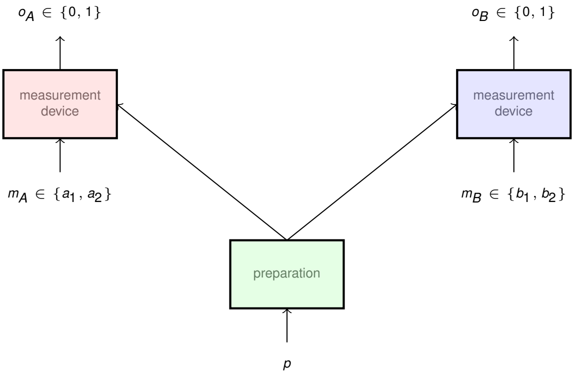
  - Quantifiable advantages in QC and QIP tasks

Contextuality

Empirical data



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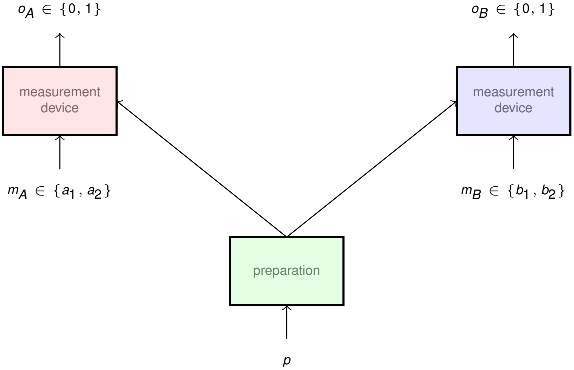

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- Hence, $\sum_{i=1}^{N} p_i \leq N - 1$.

Analysis of the Bell table

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The inequality is violated by $1/4$.

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- **Local consistency** *vs* **global inconsistency**.

Abramsky–Brandenburger framework

Measurement scenario $\langle X, \mathcal{M}, O \rangle$:

- X is a finite set of measurements or variables
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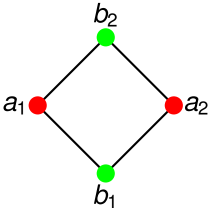
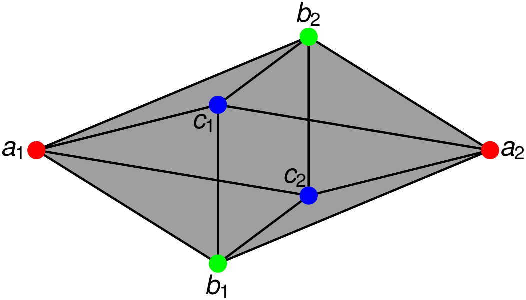
Example: (2,2,2) Bell scenario

- The set of variables is $X = \{a_1, a_2, b_1, b_2\}$.
- The outcomes are $O = \{0, 1\}$.
- The measurement contexts are:

$$\{\ \{a_1, b_1\}, \quad \{a_1, b_2\}, \quad \{a_2, b_1\}, \quad \{a_2, b_2\}\ \}.$$

Measurement scenarios





Examples: Bell-type scenarios, KS configurations, and more.

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- A set of 18 variables, $X = \{A, \ldots, O\}$
- A set of outcomes $O = \{0, 1\}$
- A measurement cover $\mathcal{M} = \{C_1, \ldots, C_9\}$, whose contexts C_i correspond to the columns in the following table:

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9
A	A	H	H	B	I	P	P	Q
B	E	I	K	E	K	Q	R	R
C	F	C	G	M	N	D	F	M
D	G	J	L	N	O	J	L	O

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In multipartite scenarios, compatibility $=$ the **no-signalling** principle.

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Note: existence of a global probability distribution is equivalent to existence of a factorisable hidden-variable model (more familiar in the case of Bell locality).

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Logical contextuality: Hardy model

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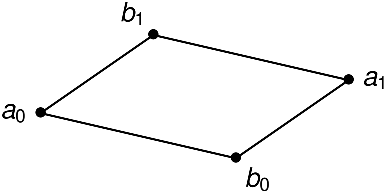

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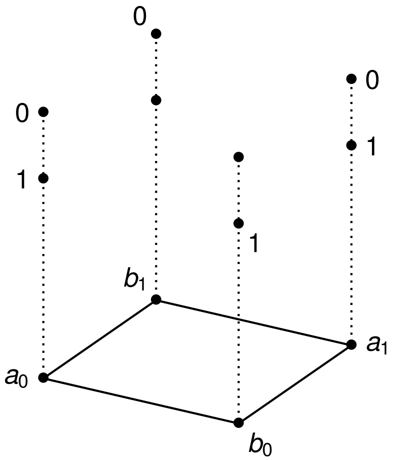

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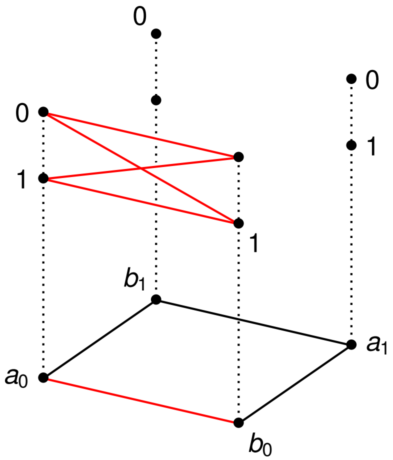

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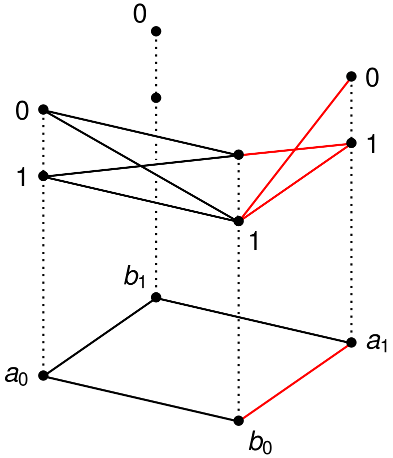

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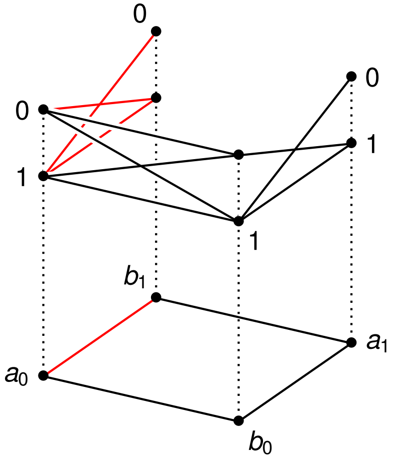

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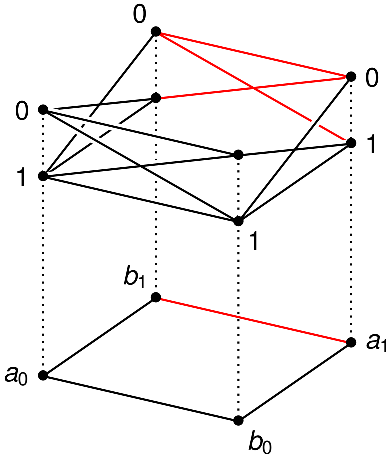

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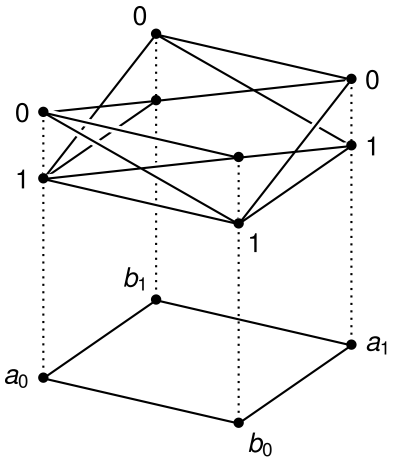

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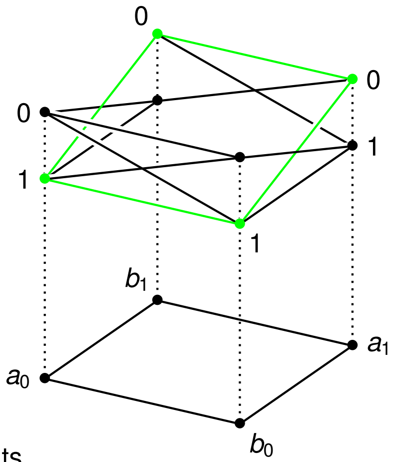

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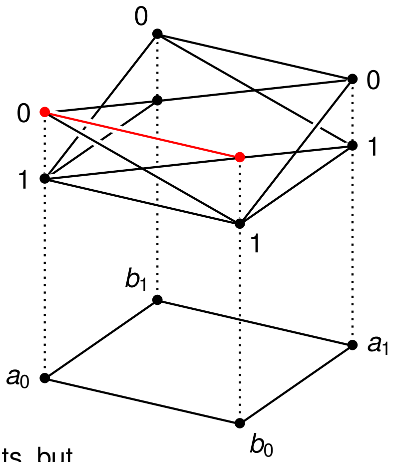

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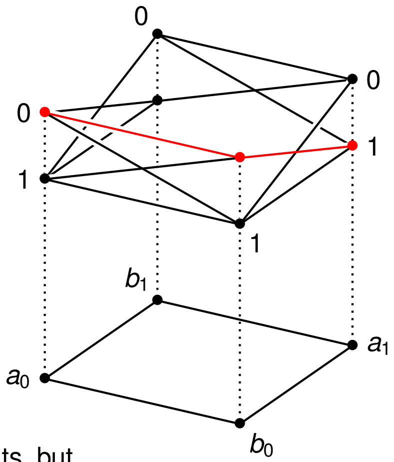

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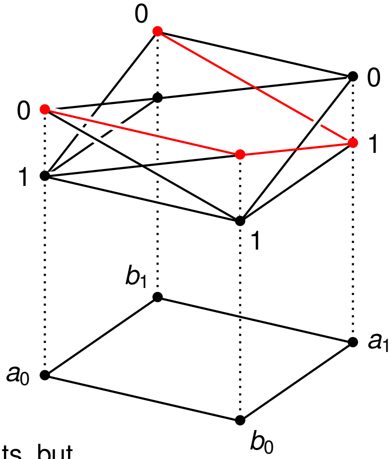

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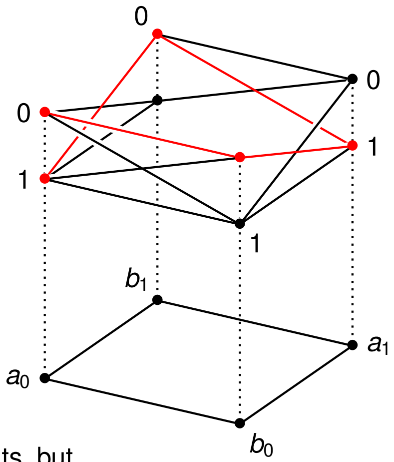

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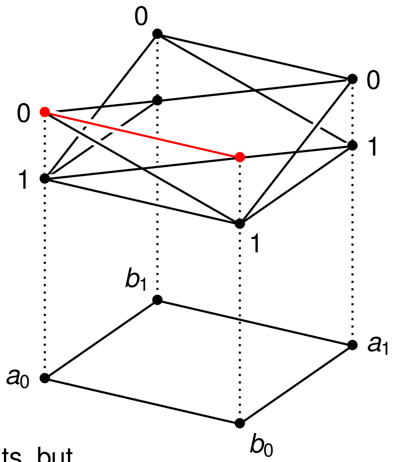

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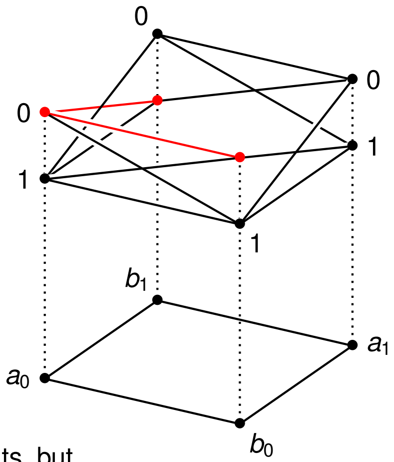

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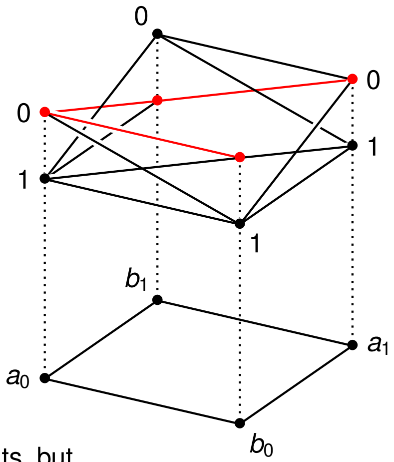

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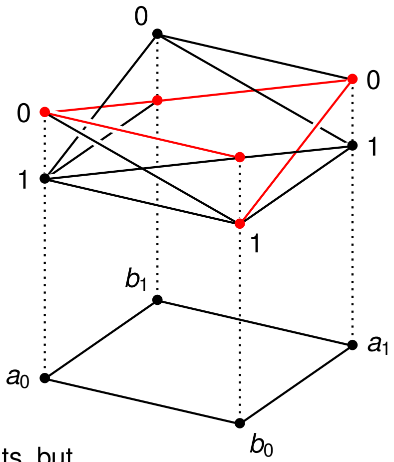

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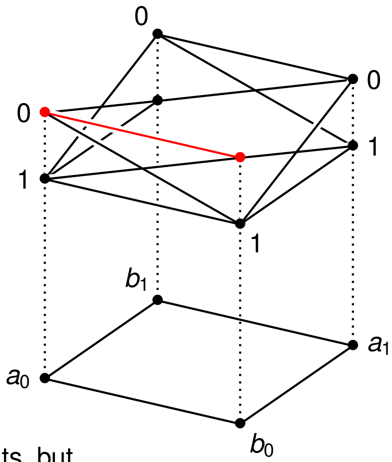

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Logical contextuality: Not all assignments extend to global ones.

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no event can be extended to a global assignment.

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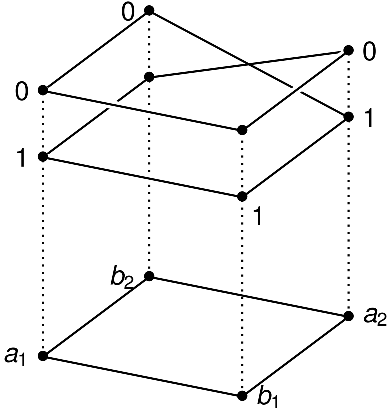

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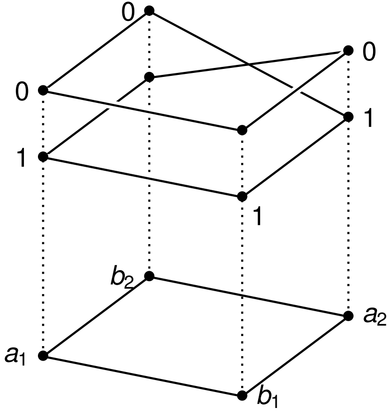

Cohomological witnesses of contextuality
(Abramsky–B–Mansfield, ABM–Kishida–Lal, Carù, Raussendorf et al.)

Measuring Contextuality

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- Relates to quantifiable **advantages** in QC and QIP tasks

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$$\mathrm{NCF}(e) = \lambda \qquad \mathrm{CF}(e) = 1 - \lambda$$

(Non-)contextual fraction via linear programming

Checking contextuality of e corresponds to solving

$$\begin{array}{ll} \texttt{Find} & \mathbf{d} \in \mathbb{R}^n \\ \texttt{such that} & \mathbf{M}\,\mathbf{d} = \mathbf{v}^e \\ \texttt{and} & \mathbf{d} \geq \mathbf{0} \end{array} \quad .$$

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E.g. Equatorial measurements on GHZ(n)

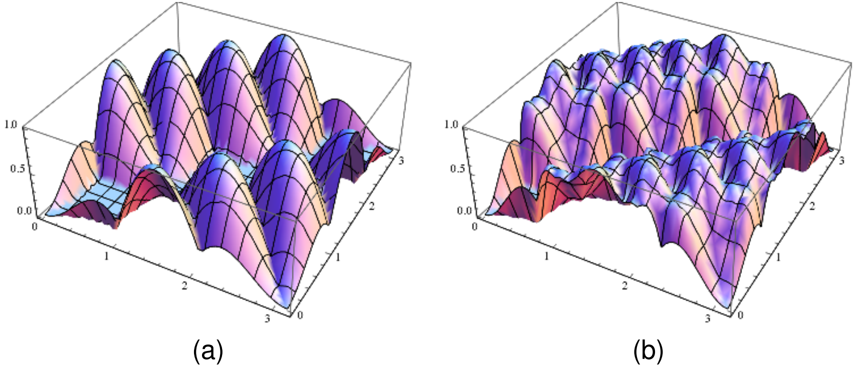

Figure: Contextual fraction of empirical models obtained with equatorial measurements at ϕ_1 and ϕ_2 on each qubit of $|\psi_{\mathrm{GHZ}(n)}\rangle$ with: (a) $n = 3$; (b) $n = 4$.

Violations of Bell inequalities

Generalised Bell inequalities

An **inequality** for a scenario $\langle X, \mathcal{M}, O \rangle$ is given by:

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NB: Complete set of inequalities can be derived from logical consistency.

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A Bell inequality establishes a bound for the value of $\mathcal{B}_\alpha(e)$ amongst NC models.

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The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by an empirical model e is the value

$$\frac{\max\{0, \mathcal{B}_\alpha(e) - R\}}{\|\alpha\| - R} .$$

Bell inequality violation and the contextual fraction

Proposition

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- Moreover, this Bell inequality is tight at “the” non-contextual model e^{NC} and maximally violated by “the” strongly contextual model e^{SC} for any decomposition:

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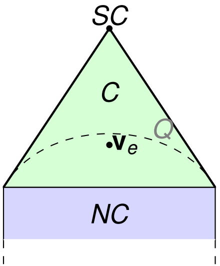

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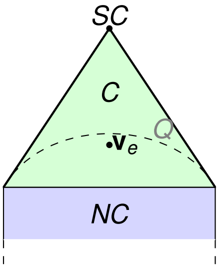

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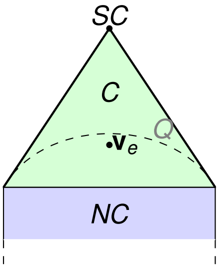

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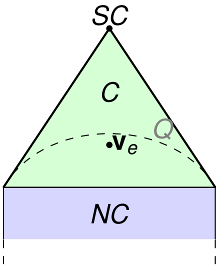

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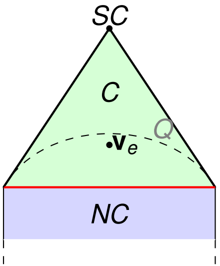

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computes tight Bell inequality (separating hyperplane)

Operations on empirical models

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- Towards a resource theory
  as for entanglement (e.g. LOCC), non-locality, ...

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- The operations remind one of process algebras.

Operations

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For $C' \in M', s : C' \longrightarrow O$, $(e \restriction \mathcal{M}')_{C'}(s) := e_C|_{C'}(s)$
with any $C \in \mathcal{M}$ s.t. $C' \subseteq C$

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$$e : \langle X, \mathcal{M}, O\rangle \quad \alpha : (X, \mathcal{M}) \cong (X', M') \quad \rightsquigarrow \quad e[\alpha] : \langle X', \mathcal{M}', O\rangle$$

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For $C \in M, s : C \longrightarrow O'$, $(e/f)_C(s) := \sum_{t:C \longrightarrow O, f \circ t = s} e_C(t)$

Operations

Mixing

$$\begin{array}{l} e, e' : \langle X, \mathcal{M}, O \rangle \\ \lambda \in [0,1] \end{array} \quad \rightsquigarrow \quad e +_\lambda e' : \langle X, \mathcal{M}, O \rangle$$

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Operations and the contextual fraction

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  - leaving a model on scenario $\mathsf{lk}_\sigma \mathcal{M}$

Contextual fraction and quantum advantages

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- Measure of contextuality $\rightsquigarrow$ quantify such advantages.

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  (cf. Abramsky & Hardy, "Logical Bell inequalities")
- We have

$$\boxed{1 - \bar{p}_S \quad \geq \quad \mathsf{NCF}\,\frac{n-k}{n}}$$

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- Probabilistic version: non-linear function computed with sufficently large probability of success implies contextuality.

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(average distance between f and closest $\mathbb{Z}_2$-linear function)

where for Boolean functions f and g, $d(f,g) := 2^{-m}|\{\mathbf{i} \in 2^m \mid f(\mathbf{i}) \neq g(\mathbf{i})\}$.

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- Then,

$$\boxed{1 - \bar{p}_S \;\; \geq \;\; \mathsf{NCF}(e)\, \nu(f)}$$

Questions...

?

“The contextual fraction as a measure of contextuality”
Samson Abramsky, RSB, Shane Mansfield
PRL 119:050504 (2017), `arXiv:1705.07918[quant-ph]`