Simulations of quantum resources and the degrees of contextuality


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8th Conference on Algebra and Coalgebra in Computer Science (CALCO 2019)

London, 3rd June 2019

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  Identify **non-classical** features of quantum mechanics responsible for quantum advantage.
- Here, we will focus on **non-local** and **contextual** behaviours as quantum resources.
- Contextuality is a key signature of non-classicality.

Overview: Contextuality as a resource for quantum computation

- Measurement-based quantum computation (MBQC)

  '*Contextuality in measurement-based quantum computation*'
  Raussendorf, Physical Review A, 2013.

  '*Contextual fraction as a measure of contextuality*'
  Abramsky, B, Mansfield, Physical Review Letters, 2017.

- Magic state distillation

  '*Contextuality supplies the 'magic' for quantum computation*'
  Howard, Wallman, Veitch, Emerson, Nature, 2014.

- Shallow circuits

  '*Quantum advantage with shallow circuits*'
  Bravyi, Gossett, Koenig, Science, 2018.

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- Simulations or reducibility (coming from Computer Science)

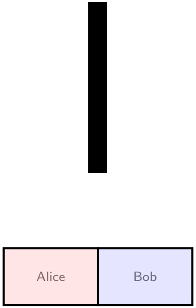
  Notion of **simulation** between behaviours of systems.
  One resource can be reduced to another if it can be simulated by it.

  Examples: (in)computability, degrees of unsolvability, complexity classes.

Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

They may share prior information,



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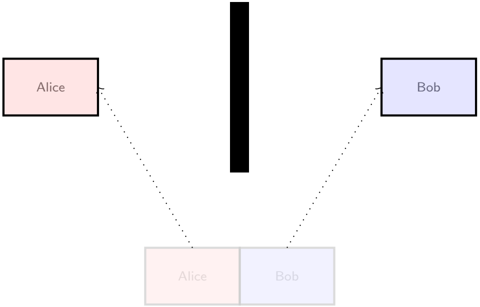

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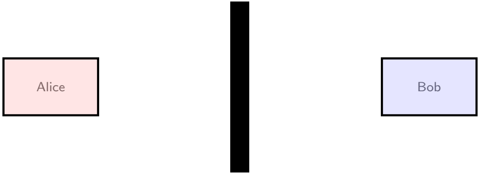

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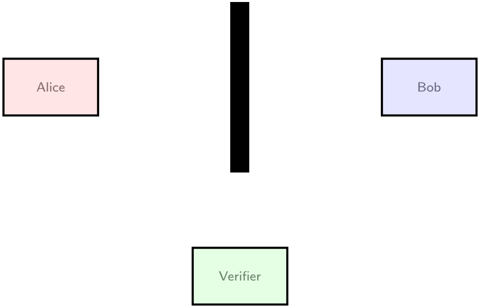

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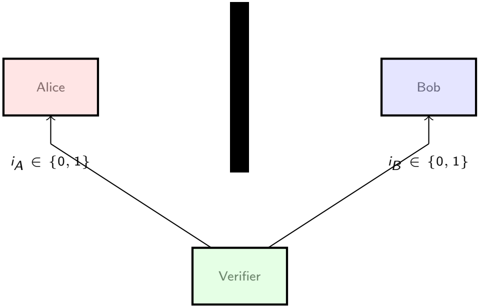

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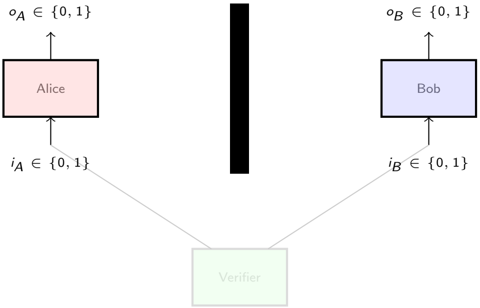

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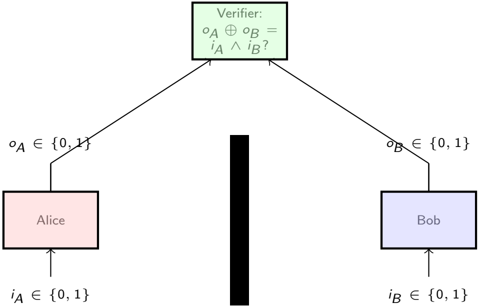

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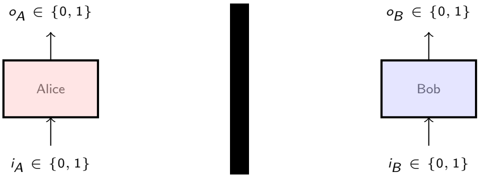

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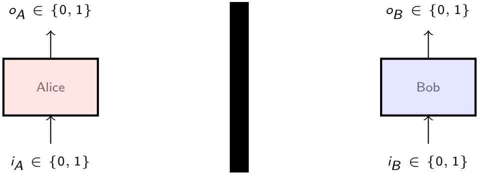

They win a play if $o_A \oplus o_B = i_A \wedge i_B$.

A strategy is described by the probabilities $P(\, o_A, o_B \mid i_A, i_B \,)$.

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This quantum advantage is related to the fact that these probabilities do not arise from a probability distribution on global assignments in $\{a_0, a_1, b_0, b_1\} \longrightarrow \{0, 1\}$.

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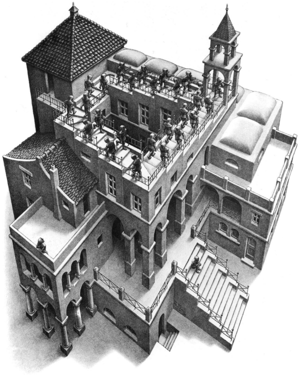

M. C. Escher, *Ascending and Descending*

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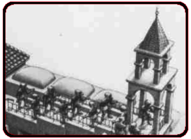

Local consistency

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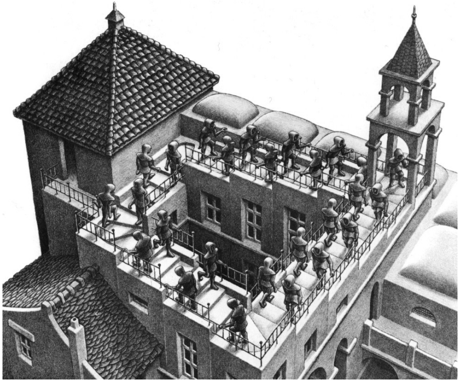

Local consistency *but* **Global inconsistency**

Formalising contextuality: Measurement scenarios

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- Σ is an abstract simplicial complex on X,
  i.e. a down-closed family of subsets of X containing singletons
  Faces of Σ are the **contexts**, indicating **joint measurability**.

Formalising contextuality: Measurement scenarios

Example: (2,2,2) Bell scenario

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- The set of variables is $X = \{a_0, a_1, b_0, b_1\}$.
- The outcomes are $O_x = \{0, 1\}$ for all $x \in X$.
- The measurement contexts are:

$$\Sigma \;=\; \downarrow \{ \, \{a_0, b_0\}, \quad \{a_0, b_1\}, \quad \{a_1, b_0\}, \quad \{a_1, b_1\} \, \}.$$

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We require **compatibility**: if $\tau \subseteq \sigma$, then $e_\sigma|_\tau = e_\tau$.
i.e. for any $\sigma, \sigma' \in \Sigma$, $e_\sigma|_{\sigma \cap \sigma'} = e_{\sigma'}|_{\sigma \cap \sigma'}$.

This is the **no-signalling** or **no-disturbance** condition.

Contextuality

An empirical model $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on a measurement scenario (X, Σ, O) is **non-contextual** if there is a distribution $d \in \mathsf{Prob}(\prod_{x \in X} O_x)$ such that, for all $\sigma \in \Sigma$:

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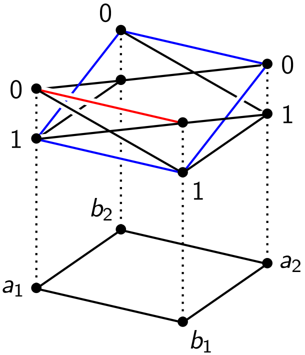
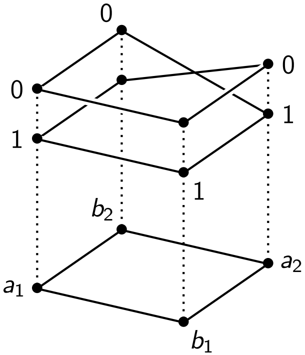
The import of Bell's and Kochen–Specker's theorems is that there are behaviours arising from quantum mechanics that are contextual.

Visualising Contextuality





The Hardy table and the PR box as bundles

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- a way of **simulating** e using d.

Free operations

Free operations generate terms typed by measurement scenarios:

$$\begin{aligned} \mathsf{Terms} \ni t \quad ::= \;& v \in \mathsf{Var} \;\mid\; \mathsf{z} \;\mid\; \mathsf{u} \;\mid\; f^* t \;\mid\; t/h \\ \mid\;& t +_\lambda t \;\mid\; t \otimes t \;\mid\; t[x?y] \end{aligned}$$

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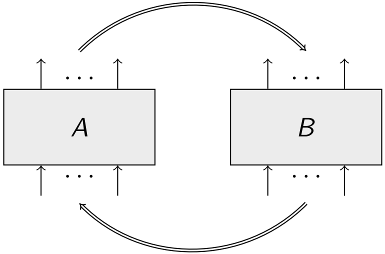
Can d be transformed to e?

Formally: is there a typed term $v : \langle Y, \Delta, P \rangle \vdash t : \langle X, \Sigma, O \rangle$ such that $t[d/v] = e$?

Basic simulations



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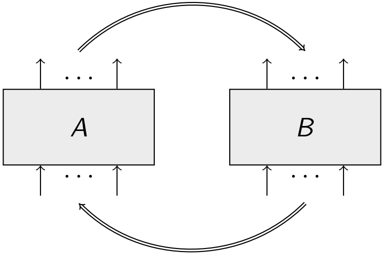

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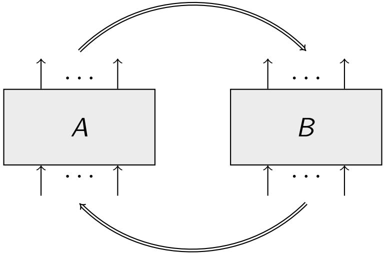

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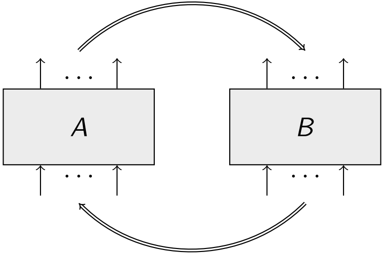

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- Akin to the Hom functor being **contravariant** in its first argument, **covariant** in its second.
- Logically, to reduce one implication to another, one must **weaken** the antecedent and **strengthen** the consequent.

Formalising simulations

A morphism of scenarios $(\pi, h) : \langle X, \Sigma, O \rangle \rightarrow \langle Y, \Delta, P \rangle$ is given by:

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Simpliciality of π means that contexts in Δ are mapped to contexts in Σ.

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$$((\pi, h)^* e)_\tau = \mathsf{Prob}(\gamma)(e_{\pi(\tau)})$$

the push-forward of the probability measure $e_{\pi(\tau)}$ along the map

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This gives a category **Emp**, with:

- objects are empirical models $e\colon \langle X, \Sigma, O\rangle$,
- morphisms $e \to e'$ are simulations $(\pi, h) : \langle X, \Sigma, O\rangle \to \langle Y, \Delta, P\rangle$ such that $(\pi, h)^* e = e'$.

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Formally, we construct a **comonad** MP on the category of empirical models, where $\mathsf{MP}(e\colon \langle X, \Sigma, O\rangle)$ is the model obtained by taking all measurement protocols over the given scenario.

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MP defines a comonoidal comonad on the category **Emp** of empirical models.

Roughly: comultiplication $\mathrm{MP}(\mathbf{X}) \to \mathrm{MP}^2(\mathbf{X})$ by "flattening", unit $\mathrm{MP}(\mathbf{X}) \to \mathbf{X}$, and $\mathrm{MP}(\mathbf{X} \otimes \mathbf{Y}) \to \mathrm{MP}(\mathbf{X}) \otimes \mathrm{MP}(\mathbf{Y})$

General simulations

Given empirical models e and d, a **simulation** of e by d is a map

$$d \otimes c \to e$$

in $\mathbf{Emp}_{\mathrm{MP}}$, the coKleisli category of MP, *i.e.* a map

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We denote the existence of a simulation of e by d as $d \rightsquigarrow e$, read "d simulates e".

Results

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- Using these normal forms, we show that the two viewpoints agree:

  Let $e : \mathbf{X}$ and $d : \mathbf{Y}$ be empirical models. Then $d \rightsquigarrow e$ if and only if there is a typed term $v : \mathbf{Y} \vdash t : \mathbf{X}$ such that $t[d/v] \simeq e$.

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- We also prove a number of further results, including a form of no-cloning theorem at the abstract level of simulations:

  $e \rightsquigarrow e \otimes e$ if and only if e is noncontextual

Degrees of Contextuality

The relation $d \rightsquigarrow e$ is a preorder on empirical models. The induced equivalence classes are the **degrees of contextuality**. They are partially ordered by the existence of simulations between representatives.

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The order contains both infinite strict chains and infinite antichains.

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Theorem

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This is just a preliminary observation. Many questions arise, and there are natural variations and refinements.

Generalized Vorob'ev theory

The property of (non)contextuality itself can be equivalently formulated as the existence of a simulation by an empirical model over the empty scenario. (N.B. $\mathsf{MP}(\mathbf{0}) = \mathbf{1}$).

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This can be formulated as characterizing those scenarios such that every model over them can be simulated by a (the) model over the empty scenario.

More generally, we can ask for conditions on scenarios (X, Σ, O) and (Y, Δ, P) such that every empirical model over (Y, Δ, P) can be simulated by some empirical model over (X, Σ, O).

Variations on simulations

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Examples:

- depth of adaptivity
- width of adaptivity
- classical randomness (grading by the noncontextual auxilliary resource c)
- $d^{\otimes n} \rightsquigarrow e$ (number of copies of d)
- $d^{\otimes \omega} \rightsquigarrow e$ (make d free)
- $d \rightsquigarrow_{\epsilon} e$ (approximate simulation)

Questions...

?