A comonadic view of simulation and quantum resources




Samson Abramsky[1] Rui Soares Barbosa[1] Shane Mansfield[2] Martti Karvonen[3]

UNIVERSITY OF OXFORD

SORBONNE UNIVERSITÉ

THE UNIVERSITY of EDINBURGH

34th Annual ACM/IEEE Symposium on Logic in Computer Science (LiCS 2019)
Vancouver, 25th June 2019

Overview

- Central object of study of quantum information and computation theory: the **advantage** afforded by **quantum resources** in information-processing tasks.

Overview

- Central object of study of quantum information and computation theory: the **advantage** afforded by **quantum resources** in information-processing tasks.
- A range of examples are known and have been studied ... but a systematic understanding of the scope and structure of quantum advantage is lacking.

Overview

- Central object of study of quantum information and computation theory:
  the **advantage** afforded by **quantum resources** in information-processing tasks.

- A range of examples are known and have been studied ... but a systematic understanding of the scope and structure of quantum advantage is lacking.

- This is related to **non-classical** features of quantum mechancics.

Overview

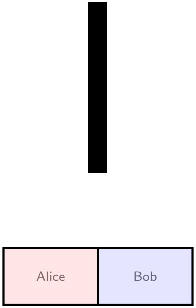
- Central object of study of quantum information and computation theory:
  the **advantage** afforded by **quantum resources** in information-processing tasks.
- A range of examples are known and have been studied ... but a systematic understanding of the scope and structure of quantum advantage is lacking.
- This is related to **non-classical** features of quantum mechancics.
- In this talk, we focus on **non-local** and **contextual** behaviours as quantum resources.
- Contextuality is a feature of **empirical data** that is a key signature of non-classicality.

Contextuality

Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

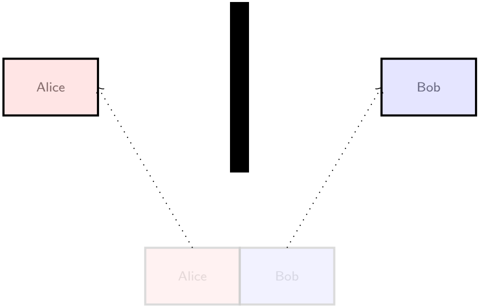
They may share prior information,



Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

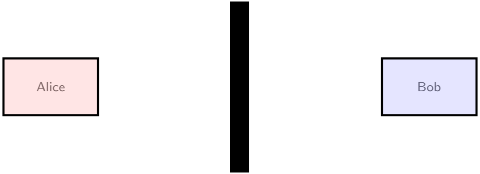
They may share prior information,



Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

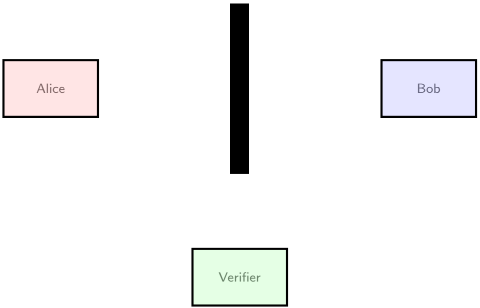
They may share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

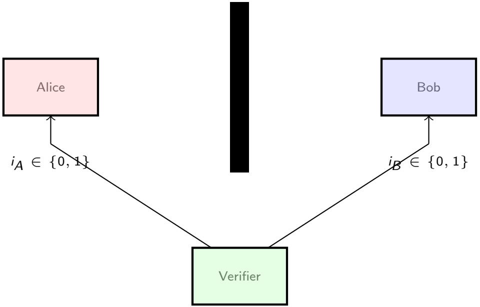
They may share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

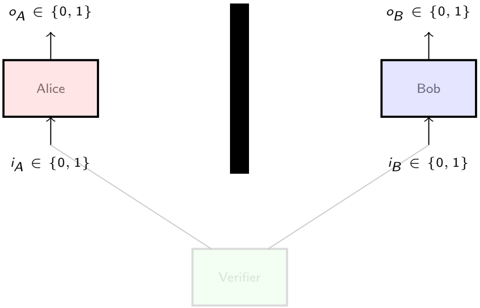
They may share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

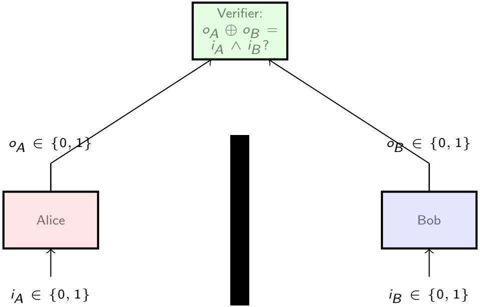
They may share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

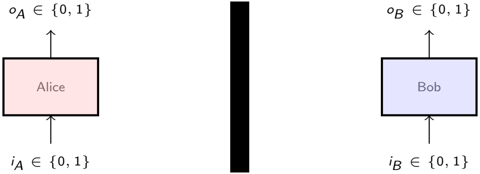
They may share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

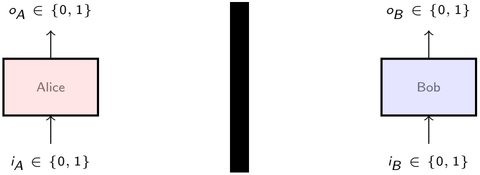
They may share prior information, but cannot communicate once game starts



Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

They may share prior information, but cannot communicate once game starts



They win a play if $o_A \oplus o_B = i_A \wedge i_B$.

A strategy is described by the probabilities $P(\, o_A, o_B \mid i_A, i_B \,)$.

Non-local games

- Classically, Alice and Bob's optimal winning probability is 0.75.

Non-local games

- Classically, Alice and Bob's optimal winning probability is 0.75.

- Sharing a pair of qubits and performing quantum measurements, Alice and Bob can realise:

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_0	b_0	$1/2$	0	0	$1/2$
a_0	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_1	b_0	$3/8$	$1/8$	$1/8$	$3/8$
a_1	b_1	$1/8$	$3/8$	$3/8$	$1/8$

Non-local games

- Classically, Alice and Bob's optimal winning probability is 0.75.

- Sharing a pair of qubits and performing quantum measurements, Alice and Bob can realise:

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_0	b_0	$1/2$	0	0	$1/2$
a_0	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_1	b_0	$3/8$	$1/8$	$1/8$	$3/8$
a_1	b_1	$1/8$	$3/8$	$3/8$	$1/8$

This gives a winning probability $3.25/4 \approx 0.81$!

Non-local games

- Classically, Alice and Bob's optimal winning probability is 0.75.
- Sharing a pair of qubits and performing quantum measurements, Alice and Bob can realise:

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_0	b_0	$1/2$	0	0	$1/2$
a_0	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_1	b_0	$3/8$	$1/8$	$1/8$	$3/8$
a_1	b_1	$1/8$	$3/8$	$3/8$	$1/8$

This gives a winning probability $3.25/4 \approx 0.81$!

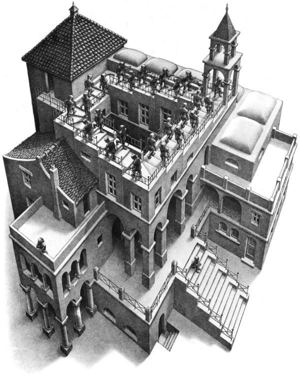
This quantum advantage is related to the fact that these probabilities do not arise from a probability distribution on global assignments in $\{a_0, a_1, b_0, b_1\} \longrightarrow \{0, 1\}$.

The essence of contextuality

- Not all properties may be observed at once.
- Jointly observable properties provide **partial snapshots**.

The essence of contextuality

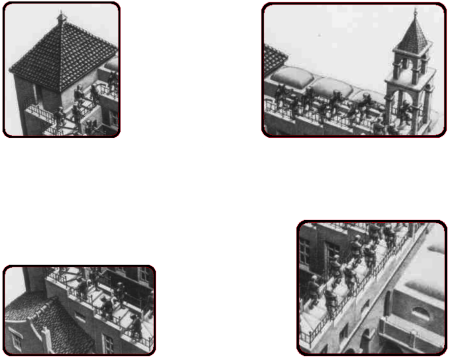
- Not all properties may be observed at once.
- Jointly observable properties provide **partial snapshots**.



M. C. Escher, *Ascending and Descending*

The essence of contextuality

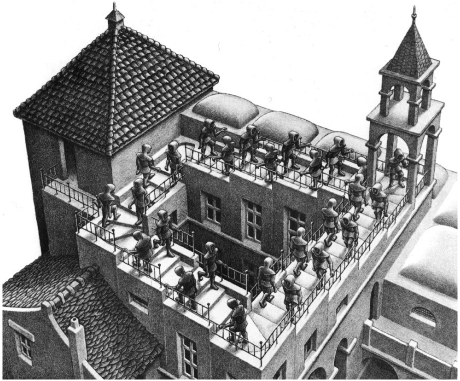
- Not all properties may be observed at once.
- Jointly observable properties provide **partial snapshots**.



Local consistency

The essence of contextuality

- Not all properties may be observed at once.
- Jointly observable properties provide **partial snapshots**.



Local consistency *but* **Global inconsistency**

Formalising empirical data

A **measurement scenario** $\mathbf{X} = \langle X, \Sigma, O \rangle$:

- X – a finite set of measurements
- Σ – a simplicial complex on X
  faces are called the **measurement contexts**
- $O = (O_x)_{x \in X}$ – for each $x \in X$ a non-empty set of possible outcomes O_x

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_0	b_0	$1/2$	0	0	$1/2$
a_0	b_1	$1/2$	0	0	$1/2$
a_1	b_0	$1/2$	0	0	$1/2$
a_1	b_1	0	$1/2$	$1/2$	0

$X = \{a_0, a_1, b_0, b_1\}, \;\; O_x = \{0, 1\}$

$\Sigma = \downarrow \{ \{a_0, b_0\}, \{a_0, b_1\}, \{a_1, b_0\}, \{a_1, b_1\} \}.$

Formalising empirical data

A **measurement scenario** $\mathbf{X} = \langle X, \Sigma, O \rangle$:

- X – a finite set of measurements
- Σ – a simplicial complex on X
  faces are called the **measurement contexts**
- $O = (O_x)_{x \in X}$ – for each $x \in X$ a non-empty set of possible outcomes O_x

A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_0	b_0	$1/2$	0	0	$1/2$
a_0	b_1	$1/2$	0	0	$1/2$
a_1	b_0	$1/2$	0	0	$1/2$
a_1	b_1	0	$1/2$	$1/2$	0

$X = \{a_0, a_1, b_0, b_1\}, \ O_x = \{0, 1\}$

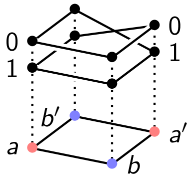
$\Sigma = \downarrow \{ \{a_0, b_0\}, \{a_0, b_1\}, \{a_1, b_0\}, \{a_1, b_1\} \}.$

An **empirical model** $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on $\mathbf{X}$:

- each $e_\sigma \in \mathrm{Prob}\left(\prod_{x \in \sigma} O_x\right)$ is a probability distribution over joint outcomes for σ.
- *generalised no-signalling* holds: for any $\sigma, \tau \in \Sigma$, if $\tau \subseteq \sigma$,

$$e_\sigma|_\tau = e_\tau$$

(i.e. marginals are well-defined)



Formalising empirical data

A **measurement scenario** $\mathbf{X} = \langle X, \Sigma, O \rangle$:

- X – a finite set of measurements
- Σ – a simplicial complex on X
  faces are called the **measurement contexts**
- $O = (O_x)_{x \in X}$ – for each $x \in X$ a non-empty set of possible outcomes O_x

An **empirical model** $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on $\mathbf{X}$:

- each $e_\sigma \in \mathrm{Prob}\left(\prod_{x \in \sigma} O_x\right)$ is a probability distribution over joint outcomes for σ.
- *generalised no-signalling* holds: for any $\sigma, \tau \in \Sigma$, if $\tau \subseteq \sigma$,

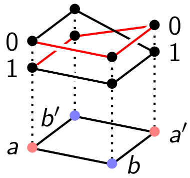
$$e_\sigma|_\tau = e_\tau$$

(i.e. marginals are well-defined)

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_0	b_0	$1/2$	0	0	$1/2$
a_0	b_1	$1/2$	0	0	$1/2$
a_1	b_0	$1/2$	0	0	$1/2$
a_1	b_1	0	$1/2$	$1/2$	0

$X = \{a_0, a_1, b_0, b_1\}, \;\; O_x = \{0, 1\}$

$\Sigma = \downarrow \{\, \{a_0, b_0\}, \{a_0, b_1\}, \{a_1, b_0\}, \{a_1, b_1\} \,\}.$



Contextuality

An empirical model $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on a measurement scenario (X, Σ, O) is **non-contextual** if there is a distribution d on $\prod_{x \in X} O_x$ such that, for all $\sigma \in \Sigma$:

$$d|_\sigma = e_\sigma .$$

Contextuality

An empirical model $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on a measurement scenario (X, Σ, O) is **non-contextual** if there is a distribution d on $\prod_{x \in X} O_x$ such that, for all $\sigma \in \Sigma$:

$$d|_\sigma \;=\; e_\sigma .$$

That is, we can **glue** all the local information together into a global consistent description from which the local information can be recovered.

Contextuality

An empirical model $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on a measurement scenario (X, Σ, O) is **non-contextual** if there is a distribution d on $\prod_{x \in X} O_x$ such that, for all $\sigma \in \Sigma$:

$$d|_\sigma \;=\; e_\sigma .$$

That is, we can **glue** all the local information together into a global consistent description from which the local information can be recovered.

If no such global distribution exists, the empirical model is **contextual**.

Contextuality

An empirical model $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on a measurement scenario (X, Σ, O) is **non-contextual** if there is a distribution d on $\prod_{x \in X} O_x$ such that, for all $\sigma \in \Sigma$:

$$d|_\sigma = e_\sigma .$$

That is, we can **glue** all the local information together into a global consistent description from which the local information can be recovered.

If no such global distribution exists, the empirical model is **contextual**.

Contextuality: family of data that is **locally consistent** but **globally inconsistent**.

Contextuality

An empirical model $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on a measurement scenario (X, Σ, O) is **non-contextual** if there is a distribution d on $\prod_{x \in X} O_x$ such that, for all $\sigma \in \Sigma$:

$$d|_\sigma \;=\; e_\sigma .$$

That is, we can **glue** all the local information together into a global consistent description from which the local information can be recovered.

If no such global distribution exists, the empirical model is **contextual**.

Contextuality: family of data that is **locally consistent** but **globally inconsistent**.

The import of Bell's and Kochen–Spekker's theorems is that there are behaviours arising from quantum mechanics that are contextual.

Contextuality as a resource

Contextuality and advantage in quantum computation

- Measurement-based quantum computation (MBQC)

  '*Contextuality in measurement-based quantum computation*'
  Raussendorf, Physical Review A, 2013.

  '*Contextual fraction as a measure of contextuality*'
  Abramsky, B, Mansfield, Physical Review Letters, 2017.

- Magic state distillation

  '*Contextuality supplies the 'magic' for quantum computation*'
  Howard, Wallman, Veitch, Emerson, Nature, 2014.

- Shallow circuits

  '*Quantum advantage with shallow circuits*'
  Bravyi, Gossett, Koenig, Science, 2018.

  - Contextuality analysis: Aasnæss, Forthcoming, 2019.

Overview: Contextuality as a resource

- Our focus is on contextuality as a **resource**:
  - how can we use it, what can we do with it?

Overview: Contextuality as a resource

- Our focus is on contextuality as a **resource**:
  - how can we use it, what can we do with it?
- From this perspective, we want to compare contextual behaviours:
  - When is one a more powerful resource than another?
  - When are two behaviours essentially the same?

Overview: Contextuality as a resource

- Our focus is on contextuality as a **resource**:
  - how can we use it, what can we do with it?
- From this perspective, we want to compare contextual behaviours:
  - When is one a more powerful resource than another?
  - When are two behaviours essentially the same?

Example

'*Popescu-Rohrlich correlations as a unit of nonlocality*'
Barrett, Pironio, Physical Review Letters, 2005.

- PR boxes simulate all 2-outcome bipartite boxes
- A tripartite quantum box that cannot be simulated from PR boxes

Structure of resources

Two perspectives:

Structure of resources

Two perspectives:

1. **Resource theories** (coming from Physics):
   Algebraic theory of '**free operations**' that do not introduce more of the resource in question.

   Resource B can be obtained from resource A if it can be built from A using free operations.

   '*Contextual fraction as a measure of contextuality*', Abramsky, B, Mansfield, PRL, 2017.
   '*Noncontextual wirings*', Amaral, Cabello, Terra Cunha, Aolita, PRL, 2018.

Structure of resources

Two perspectives:

1. **Resource theories** (coming from Physics):
   Algebraic theory of '**free operations**' that do not introduce more of the resource in question.

   Resource B can be obtained from resource A if it can be built from A using free operations.

   '*Contextual fraction as a measure of contextuality*', Abramsky, B, Mansfield, PRL, 2017.
   '*Noncontextual wirings*', Amaral, Cabello, Terra Cunha, Aolita, PRL, 2018.

2. **Simulations or reducibility** (coming from Computer Science):
   Notion of **simulation** between behaviours of systems.

   One resource can be reduced to another if it can be simulated by it.
   Cf. (in)computability, degrees of unsolvability, complexity classes.

   '*Categories of empirical models*', Karvonen, QPL 2018.

Morphisms between empirical models

We have defined our mathematical objects of interest:
Empirical models describe behaviours that may be used as resources.

Morphisms between empirical models

We have defined our mathematical objects of interest:
Empirical models describe behaviours that may be used as resources.

What are the morphisms?

Morphisms between empirical models

We have defined our mathematical objects of interest:
Empirical models describe behaviours that may be used as resources.

What are the morphisms?

The idea is to capture **convertibility** of resources.

Morphisms between empirical models

We have defined our mathematical objects of interest:
Empirical models describe behaviours that may be used as resources.

What are the morphisms?

The idea is to capture **convertibility** of resources.

Given $e\colon \langle X, \Sigma, O\rangle$ and $d\colon \langle Y, \Delta, P\rangle$, a morphism $d \to e$ is:

Morphisms between empirical models

We have defined our mathematical objects of interest:
Empirical models describe behaviours that may be used as resources.

What are the morphisms?

The idea is to capture **convertibility** of resources.

Given $e \colon \langle X, \Sigma, O \rangle$ and $d \colon \langle Y, \Delta, P \rangle$, a morphism $d \to e$ is:

- a way of transforming d into e using **free operations**.

Morphisms between empirical models

We have defined our mathematical objects of interest:
Empirical models describe behaviours that may be used as resources.

What are the morphisms?

The idea is to capture **convertibility** of resources.

Given $e \colon \langle X, \Sigma, O \rangle$ and $d \colon \langle Y, \Delta, P \rangle$, a morphism $d \to e$ is:

- a way of transforming d into e using **free operations**.
- a way of **simulating** e using d.

Free operations

- **Zero model** z: unique empirical model on the empty measurement scenario

$$\langle \emptyset, \Delta_0 = \{\emptyset\}, () \rangle \ .$$

- **Singleton model** u: unique empirical model on the 1-outcome 1-measurement scenario

$$\langle \mathbf{1} = \{\star\}, \Delta_1 = \{\emptyset, \mathbf{1}\}, (O_\star = \mathbf{1}) \rangle \ .$$

- **Probabilistic mixing**: Given empirical models e and d in $\langle X, \Sigma, O \rangle$ and $\lambda \in [0,1]$, the model $e +_\lambda d : \langle X, \Sigma, O \rangle$ is given by the mixture $\lambda e + (1-\lambda)d$.

Free operations

- **Tensor**: Let $e : \langle X, \Sigma, O \rangle$ and $d : \langle Y, \Delta, P \rangle$. Then

$$e \otimes d : \langle X \sqcup Y, \Sigma * \Delta, [O, P] \rangle$$

  where $\Sigma * \Theta := \{\sigma \cup \tau | \sigma \in \Sigma, \tau \in \Delta\}$. *Runs e and d independently and in parallel.*

- **Coarse-graining**: Given $e : \langle X, \Sigma, O \rangle$ and a family of functions $h = (h_x : O_x \longrightarrow O'_x)_{x \in X}$, get a coarse-grained model

$$e/h : \langle X, \Sigma, O' \rangle$$

  .

- **Measurement translation**: Given $e : \langle X, \Sigma, O \rangle$ and a simplicial map $f : \Sigma' \longrightarrow \Sigma$, the model $f^* e : \langle X', \Sigma', O \rangle$ is defined by pulling e back along the map f.

New free operation

- **Conditioning on a measurement**: Given $e : \langle X, \Sigma, O \rangle$, $x \in X$ and a family of measurements $(y_o)_{o \in O_x}$ with $y_o \in \mathsf{Vert}(\mathsf{lk}_x \Sigma)$. Consider a new measurement $x?(y_o)_{o \in O_x}$, abbreviated $x?y$. Get

$$e[x?y] : \langle X \cup \{x?y\}, \Sigma[x?y], O[x?y \mapsto O_{x?y}] \rangle$$

that results from adding $x?y$ to e.

If Σ is a simplicial complex and a $\sigma \in \Sigma$ is a face, the **link** of σ in Σ is the subcomplex of Σ whose faces are

$$\mathsf{lk}_\sigma \Sigma := \{\tau \in \Sigma \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Sigma\} \ .$$

What contexts are still available once the measurements in σ have been performed.

Free operations

Free operations generate terms typed by measurement scenarios:

$$\begin{aligned} \mathsf{Terms} \ni t \quad ::= \quad & v \in \mathsf{Var} \;\mid\; \mathsf{z} \;\mid\; \mathsf{u} \;\mid\; f^* t \;\mid\; t/h \\ \mid\; & t +_\lambda t \;\mid\; t \otimes t \;\mid\; t[x?y] \end{aligned}$$

Free operations

Free operations generate terms typed by measurement scenarios:

$$\begin{aligned}\mathsf{Terms} \ni t \quad ::= \;& v \in \mathsf{Var} \;\mid\; \mathsf{z} \;\mid\; \mathsf{u} \;\mid\; f^* t \;\mid\; t/h \\ & \mid\; t +_\lambda t \;\mid\; t \otimes t \;\mid\; t[x?y]\end{aligned}$$

Terms without variables represent noncontextual empirical models.

Conversely, every noncontextual model can be represented by a term without variables.

Free operations

Free operations generate terms typed by measurement scenarios:

$$
\begin{aligned}
\text{Terms} \ni t \quad ::= \;& v \in \text{Var} \;\mid\; \mathsf{z} \;\mid\; \mathsf{u} \;\mid\; f^* t \;\mid\; t/h \\
\mid\;& t +_\lambda t \;\mid\; t \otimes t \;\mid\; t[x?y]
\end{aligned}
$$

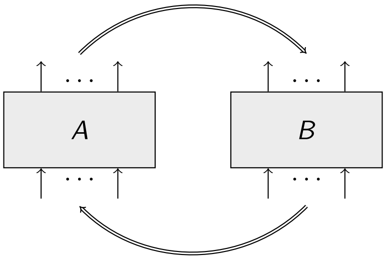
Terms without variables represent noncontextual empirical models.

Conversely, every noncontextual model can be represented by a term without variables.

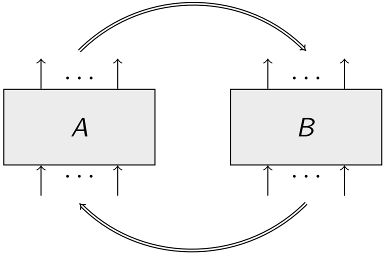
Can d be transformed to e?

Formally: is there a typed term $v : \langle Y, \Delta, P\rangle \vdash t : \langle X, \Sigma, O\rangle$ such that $t[d/v] = e$?

Basic simulations

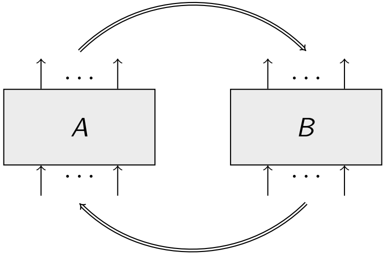


Basic simulations



To simulate B using A:

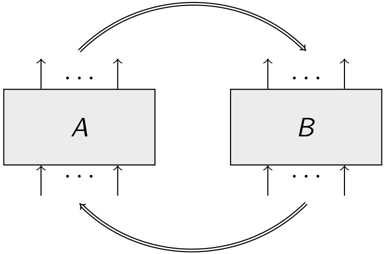
Basic simulations



To simulate B using A:

- map inputs of B (measurements) to inputs of A

Basic simulations



To simulate B using A:

- map inputs of B (measurements) to inputs of A
- map outputs of A (measurement outcomes) to outputs of B

Basic simulations

This is a simple notion of simulation, but already covers several things:

Basic simulations

This is a simple notion of simulation, but already covers several things:

- It allows for relabelling of measurements and outcomes (**isomorphisms**).

Basic simulations

This is a simple notion of simulation, but already covers several things:

- It allows for relabelling of measurements and outcomes (**isomorphisms**).
- Since the map on measurements needn't be surjective, it allows for simulating B using only part of A (**restrictions**).

Basic simulations

This is a simple notion of simulation, but already covers several things:

- It allows for relabelling of measurements and outcomes (**isomorphisms**).
- Since the map on measurements needn't be surjective, it allows for simulating B using only part of A (**restrictions**).
- Since the map on outcomes needn't be injective, it allows for **coarse-graining** of outcomes.

Basic simulations

This is a simple notion of simulation, but already covers several things:

- It allows for relabelling of measurements and outcomes (**isomorphisms**).
- Since the map on measurements needn't be surjective, it allows for simulating B using only part of A (**restrictions**).
- Since the map on outcomes needn't be injective, it allows for **coarse-graining** of outcomes.

Note that mappings of inputs go backward, of outputs forward:

Basic simulations

This is a simple notion of simulation, but already covers several things:

- It allows for relabelling of measurements and outcomes (**isomorphisms**).
- Since the map on measurements needn't be surjective, it allows for simulating B using only part of A (**restrictions**).
- Since the map on outcomes needn't be injective, it allows for **coarse-graining** of outcomes.

Note that mappings of inputs go backward, of outputs forward:

- Akin to the Hom functor being **contravariant** in its first argument, **covariant** in its second.

Basic simulations

This is a simple notion of simulation, but already covers several things:

- It allows for relabelling of measurements and outcomes (**isomorphisms**).
- Since the map on measurements needn't be surjective, it allows for simulating B using only part of A (**restrictions**).
- Since the map on outcomes needn't be injective, it allows for **coarse-graining** of outcomes.

Note that mappings of inputs go backward, of outputs forward:

- Akin to the Hom functor being **contravariant** in its first argument, **covariant** in its second.
- Logically, to reduce one implication to another, one must **weaken** the antecedent and **strengthen** the consequent.

Formalising simulations

A morphism of scenarios $(\pi, h) : \langle X, \Sigma, O \rangle \to \langle Y, \Delta, P \rangle$ is given by:

Formalising simulations

A morphism of scenarios $(\pi, h) : \langle X, \Sigma, O \rangle \rightarrow \langle Y, \Delta, P \rangle$ is given by:

- A simplicial map $\pi : \Delta \rightarrow \Sigma$.

Formalising simulations

A morphism of scenarios $(\pi, h) : \langle X, \Sigma, O \rangle \to \langle Y, \Delta, P \rangle$ is given by:

- A simplicial map $\pi : \Delta \to \Sigma$.
- For each $y \in Y$, a map $h_y : O_{\pi(y)} \to P_y$.

Formalising simulations

A morphism of scenarios $(\pi, h) : \langle X, \Sigma, O\rangle \to \langle Y, \Delta, P\rangle$ is given by:

- A simplicial map $\pi : \Delta \to \Sigma$.
- For each $y \in Y$, a map $h_y : O_{\pi(y)} \to P_y$.

Simpliciality of π means that contexts in Δ are mapped to contexts in Σ.

Basic simulations

A morphism of scenarios induces a natural action on empirical models:

Basic simulations

A morphism of scenarios induces a natural action on empirical models:

Gien $e\colon \langle X, \Sigma, O\rangle$, then $(\pi, h)^* e\colon \langle Y, \Delta, P\rangle$ is given by, for $\tau \in \Delta$:

$$((\pi, h)^* e)_\tau = \mathrm{Prob}(\gamma)(e_{\pi(\tau)})$$

the push-forward of the probability measure $e_{\pi(\tau)}$ along the map

$$\gamma : \prod_{x \in \pi(\tau)} O_x \longrightarrow \prod_{y \in \tau} P_y$$

given by $\gamma(s)_y = h_y \left(s_{\pi(y)}\right)$.

Basic simulations

A morphism of scenarios induces a natural action on empirical models:

Gien $e\colon \langle X, \Sigma, O\rangle$, then $(\pi, h)^* e\colon \langle Y, \Delta, P\rangle$ is given by, for $\tau \in \Delta$:

$$((\pi, h)^* e)_\tau = \mathrm{Prob}(\gamma)(e_{\pi(\tau)})$$

the push-forward of the probability measure $e_{\pi(\tau)}$ along the map

$$\gamma : \prod_{x \in \pi(\tau)} O_x \longrightarrow \prod_{y \in \tau} P_y$$

given by $\gamma(s)_y = h_y\left(s_{\pi(y)}\right)$.

This gives a category **Emp**, with:

- objects are empirical models $e\colon \langle X, \Sigma, O\rangle$,
- morphisms $e \to e'$ are simulations $(\pi, h) : \langle X, \Sigma, O\rangle \to \langle Y, \Delta, P\rangle$ such that $(\pi, h)^* e = e'$.

Beyond deterministic simulations

Basic simulations are useful, but limited.

Beyond deterministic simulations

Basic simulations are useful, but limited.

To allow **adaptive** use of the resource, we introduce **measurement protocols**.

Beyond deterministic simulations

Basic simulations are useful, but limited.

To allow **adaptive** use of the resource, we introduce **measurement protocols**.

These protocols proceed iteratively by first performing a measurement over the given scenario, and then conditioning their further measurements on the observed outcome.

Beyond deterministic simulations

Basic simulations are useful, but limited.

To allow **adaptive** use of the resource, we introduce **measurement protocols**.

These protocols proceed iteratively by first performing a measurement over the given scenario, and then conditioning their further measurements on the observed outcome.

Note that different paths can lead into different, **incompatible** contexts.

Beyond deterministic simulations

Basic simulations are useful, but limited.

To allow **adaptive** use of the resource, we introduce **measurement protocols**.

These protocols proceed iteratively by first performing a measurement over the given scenario, and then conditioning their further measurements on the observed outcome.

Note that different paths can lead into different, **incompatible** contexts.

Thus they incorporate adaptive classical processing, of the kind used e.g. in Measurement-Based Quantum Computing.

Beyond deterministic simulations

Basic simulations are useful, but limited.

To allow **adaptive** use of the resource, we introduce **measurement protocols**.

These protocols proceed iteratively by first performing a measurement over the given scenario, and then conditioning their further measurements on the observed outcome.

Note that different paths can lead into different, **incompatible** contexts.

Thus they incorporate adaptive classical processing, of the kind used e.g. in Measurement-Based Quantum Computing.

Previously considered in:

‘*A combinatorial approach to nonlocality and contextuality*’
Acin, Fritz, Leverrier, Sainz, Communications in Mathematical Physics, 2015.

Beyond deterministic simulations

Basic simulations are useful, but limited.

To allow **adaptive** use of the resource, we introduce **measurement protocols**.

These protocols proceed iteratively by first performing a measurement over the given scenario, and then conditioning their further measurements on the observed outcome.

Note that different paths can lead into different, **incompatible** contexts.

Thus they incorporate adaptive classical processing, of the kind used e.g. in Measurement-Based Quantum Computing.

Previously considered in:

'*A combinatorial approach to nonlocality and contextuality*'
Acin, Fritz, Leverrier, Sainz, Communications in Mathematical Physics, 2015.

Formally, we construct a **comonad** MP on the category of empirical models, where $\mathrm{MP}(e \colon \langle X, \Sigma, O \rangle)$ is the model obtained by taking all measurement protocols over the given scenario.

The MP construction

Given a scenario $\mathbf{X} = \langle X, \Sigma, O \rangle$ we build a new scenario $\mathsf{MP}(\mathbf{X})$, where:

The MP construction

Given a scenario $\mathbf{X} = \langle X, \Sigma, O \rangle$ we build a new scenario $\mathsf{MP}(\mathbf{X})$, where:

- measurements are the measurement protocols on $\mathbf{X}$

$$\mathsf{MP}(\langle X, \Sigma, O \rangle) ::= \emptyset \mid (x, f) \quad \text{where } x \in X \text{ and } f \colon O_x \to \mathsf{MP}(\langle X \setminus \{x\}, \mathsf{lk}_x \Sigma, O \rangle).$$

The MP construction

Given a scenario $\mathbf{X} = \langle X, \Sigma, O \rangle$ we build a new scenario $\mathrm{MP}(\mathbf{X})$, where:

- measurements are the measurement protocols on $\mathbf{X}$

$$\mathrm{MP}(\langle X, \Sigma, O \rangle) ::= \emptyset \mid (x, f) \quad \text{where } x \in X \text{ and } f \colon O_x \to \mathrm{MP}(\langle X \setminus \{x\}, \mathrm{lk}_x \Sigma, O \rangle).$$

- outcomes are the joint outcomes observed during a run of the protocol

The MP construction

Given a scenario $\mathbf{X} = \langle X, \Sigma, O \rangle$ we build a new scenario $\mathrm{MP}(\mathbf{X})$, where:

- measurements are the measurement protocols on $\mathbf{X}$

$$\mathrm{MP}(\langle X, \Sigma, O \rangle) ::= \emptyset \mid (x, f) \quad \text{where } x \in X \text{ and } f\colon O_x \to \mathrm{MP}(\langle X \setminus \{x\}, \mathrm{lk}_x \Sigma, O \rangle).$$

- outcomes are the joint outcomes observed during a run of the protocol
- measurement protocols are compatible if they can be combined consistently

The MP construction

Given a scenario $\mathbf{X} = \langle X, \Sigma, O \rangle$ we build a new scenario $\mathsf{MP}(\mathbf{X})$, where:

- measurements are the measurement protocols on $\mathbf{X}$

$$\mathsf{MP}(\langle X, \Sigma, O \rangle) ::= \emptyset \mid (x, f) \quad \text{where } x \in X \text{ and } f \colon O_x \to \mathsf{MP}(\langle X \setminus \{x\}, \mathrm{lk}_x \Sigma, O \rangle).$$

- outcomes are the joint outcomes observed during a run of the protocol

- measurement protocols are compatible if they can be combined consistently
  - A run is a sequence $\bar{x} = (x_i, o_i)_{i=1}^{l}$ with $x_i \in X$, $o_i \in O_{x_i}$
  - $\sigma_{\bar{x}} = \{x_1, x_2, \ldots, x_l\} \in \Sigma$.
  - Two runs (of different protocols) are consistent if they agree on common measurements
  - Protocols $\{Q_1, \ldots, Q_n\}$ are compatible if for any choice of pairwise consistent runs $\bar{x}_i$ from Q_i, we have $\bigcup_i \sigma_{\bar{x}_i} \in \Sigma$

The MP construction

Given a scenario $\mathbf{X} = \langle X, \Sigma, O \rangle$ we build a new scenario $\mathrm{MP}(\mathbf{X})$, where:

- measurements are the measurement protocols on $\mathbf{X}$

$$\mathrm{MP}(\langle X, \Sigma, O \rangle) ::= \emptyset \mid (x, f) \quad \text{where } x \in X \text{ and } f : O_x \to \mathrm{MP}(\langle X \setminus \{x\}, \mathrm{lk}_x \Sigma, O \rangle).$$

- outcomes are the joint outcomes observed during a run of the protocol
- measurement protocols are compatible if they can be combined consistently
  - A run is a sequence $\bar{x} = (x_i, o_i)_{i=1}^{l}$ with $x_i \in X$, $o_i \in O_{x_i}$
  - $\sigma_{\bar{x}} = \{x_1, x_2, \ldots, x_l\} \in \Sigma$.
  - Two runs (of different protocols) are consistent if they agree on common measurements
  - Protocols $\{Q_1, \ldots, Q_n\}$ are compatible if for any choice of pairwise consistent runs $\bar{x}_i$ from Q_i, we have $\bigcup_i \sigma_{\bar{x}_i} \in \Sigma$

The MP construction

Empirical models in $\mathbf{X}$ are then naturally lifted to this scenario $\mathsf{MP}(\mathbf{X})$.

The MP construction

Empirical models in $\mathbf{X}$ are then naturally lifted to this scenario $\mathsf{MP}(\mathbf{X})$.

Proposition

MP defines a comonoidal comonad on the category **Emp** of empirical models.

Roughly: comultiplication $\mathsf{MP}(\mathbf{X}) \to \mathsf{MP}^2(\mathbf{X})$ by "flattening", unit $\mathsf{MP}(\mathbf{X}) \to \mathbf{X}$, and $\mathsf{MP}(\mathbf{X} \otimes \mathbf{Y}) \to \mathsf{MP}(\mathbf{X}) \otimes \mathsf{MP}(\mathbf{Y})$

General simulations

Given empirical models e and d, a **simulation** of e by d is a map

$$d \otimes c \to e$$

in $\mathbf{Emp}_{\mathrm{MP}}$, the coKleisli category of MP, *i.e.* a map

$$\mathrm{MP}(d \otimes c) \to e$$

in **Emp**, for some noncontextual model c.

General simulations

Given empirical models e and d, a **simulation** of e by d is a map

$$d \otimes c \to e$$

in $\mathbf{Emp}_{\mathrm{MP}}$, the coKleisli category of MP, *i.e.* a map

$$\mathrm{MP}(d \otimes c) \to e$$

in **Emp**, for some noncontextual model c.

The use of the noncontextual model c is to allow for classical randomness in the simulation.

General simulations

Given empirical models e and d, a **simulation** of e by d is a map

$$d \otimes c \to e$$

in $\mathbf{Emp}_{\mathrm{MP}}$, the coKleisli category of MP, *i.e.* a map

$$\mathrm{MP}(d \otimes c) \to e$$

in **Emp**, for some noncontextual model c.

The use of the noncontextual model c is to allow for classical randomness in the simulation.

We denote the existence of a simulation of e by d as $d \rightsquigarrow e$, read "d simulates e".

Results

Results

Theorem [Viewpoints agree]

Let $e : \mathbf{X}$ and $d : \mathbf{Y}$ be empirical models. Then $d \rightsquigarrow e$ if and only if there is a typed term $a : \mathbf{Y} \vdash t : \mathbf{X}$ such that $t[d/a] \simeq e$.

Roughly: We develop the equational theory of free operations, and use this to obtain normal forms. These provide a means of decomposing morphisms into operations.

Results

Theorem [Viewpoints agree]

Let $e : \mathbf{X}$ and $d : \mathbf{Y}$ be empirical models. Then $d \rightsquigarrow e$ if and only if there is a typed term $a : \mathbf{Y} \vdash t : \mathbf{X}$ such that $t[d/a] \simeq e$.

Roughly: We develop the equational theory of free operations, and use this to obtain normal forms. These provide a means of decomposing morphisms into operations.

Theorem [Generalised no-cloning]

$e \rightsquigarrow e \otimes e$ if and only if e is noncontextual.

Roughly: Use the monotonicity properties of the **contextual fraction** under free operations

Some directions

- '$\rightsquigarrow$' defines a preorder on empirical models. How rich is this order?

Some directions

- '$\rightsquigarrow$' defines a preorder on empirical models. How rich is this order?
  - Existing results in the non-locality literature can be leveraged to prove that it contains both infinite strict chains and infinite antichains.

Some directions

- '$\rightsquigarrow$' defines a preorder on empirical models. How rich is this order?
  - Existing results in the non-locality literature can be leveraged to prove that it contains both infinite strict chains and infinite antichains.

- "Relative" forms of contextuality
  - Noncontextuality can be equivalently formulated as the existence of a simulation by the empirical model over the empty scenario. (N.B. $\mathsf{MP}(\mathbf{0}) = \mathbf{1}$).
  - This suggests that much of contextuality theory can be generalized to a "relative" form.
  - E.g. Vorob'ev's theorem

Some directions

- '$\rightsquigarrow$' defines a preorder on empirical models. How rich is this order?
  - Existing results in the non-locality literature can be leveraged to prove that it contains both infinite strict chains and infinite antichains.
- "Relative" forms of contextuality
  - Noncontextuality can be equivalently formulated as the existence of a simulation by the empirical model over the empty scenario. (N.B. $\mathsf{MP}(\mathbf{0}) = \mathbf{1}$).
  - This suggests that much of contextuality theory can be generalized to a "relative" form.
  - E.g. Vorob'ev's theorem
- Graded versions of simulability
  - e.g. by width or depth of adaptivity, auxiliary classical randomness, numbers of copies of resource, approximate simulations, ...

Questions...

?