Resource theory of contextual behaviours

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THE UNIVERSITY of EDINBURGH

Workshop on Contextuality as a Resource in Quantum Computation
Oxford, 4th July 2019

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  the **advantage** afforded by **quantum resources** in information-processing tasks.

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- A hypothesis: this is related to **non-classical** features of quantum mechancics.
- In this talk, we focus on **non-local** and **contextual** behaviours as quantum resources.
- Contextuality is a feature of **empirical data** that is a key signature of non-classicality.

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- Unified, theory-independent framework for non-locality and contextuality

  '*The sheaf-theoretic structure of non-locality and contextuality*'
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- A resource theory for contextuality:
  - Measure of contextuality
  - Combine and transform contextual blackboxes
  - Quantifiable advantages in QC and QIP tasks

'*Contextual fraction as a measure of contextuality*'
Abramsky, B, Mansfield, Physical Review Letters, 2017.

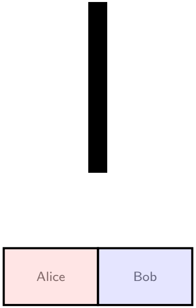
'*A comonadic view of simulation and quantum resources*'
Abramsky, B, Karvonen, Mansfield, LiCS 2019.

Contextuality

Non-local games

Alice and Bob cooperate in solving a task set by Verifier.

They may share prior information,



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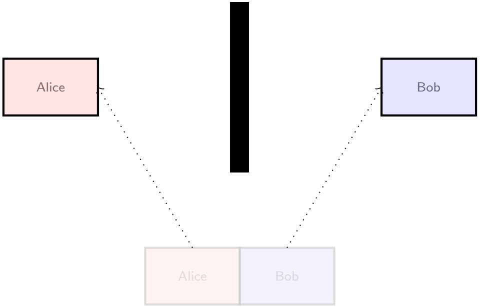

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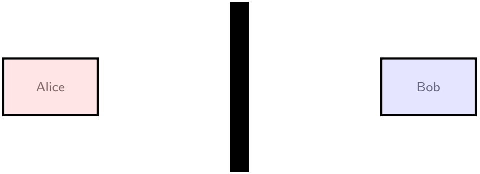

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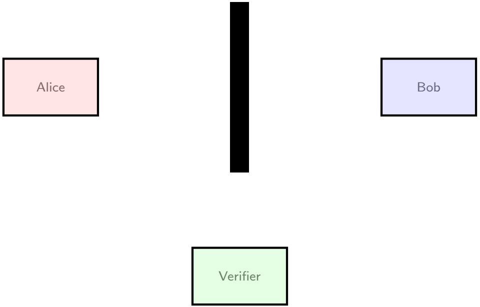

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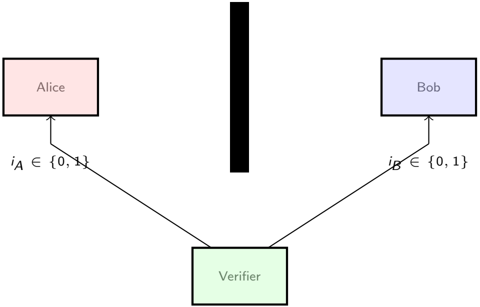

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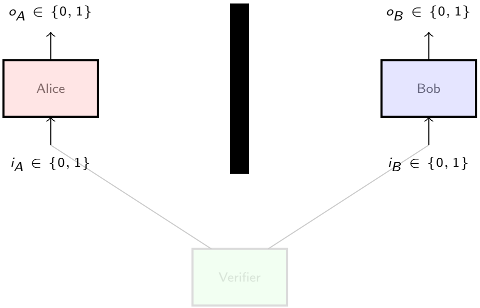

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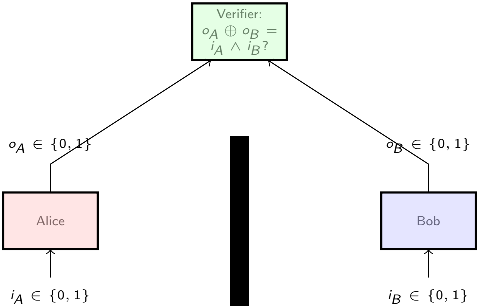

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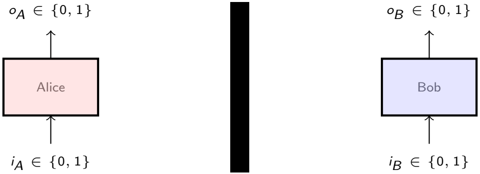

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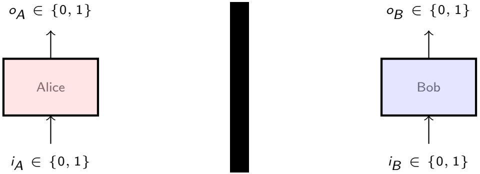

They win a play if $o_A \oplus o_B = i_A \wedge i_B$.

A strategy is described by the probabilities $P(o_A, o_B \mid i_A, i_B)$.

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- Sharing a pair of qubits and performing quantum measurements, Alice and Bob can realise:

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Assuming a uniform distribution on inputs,

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P(o_A \oplus o_B = i_A \wedge i_B) = & \frac{1}{4} P(o_A = o_B \mid i_A = 0, i_B = 0) + \frac{1}{4} P(o_A = o_B \mid i_A = 0, i_B = 1) \\
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- Classically, Alice and Bob's optimal winning probability is 0.75.

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- Hence,

$$\sum_{i=1}^{N} p_i \leq N - 1 \quad .$$

Analysis of the Bell table

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All Bell inequalities arise in this way.

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- The Bell table can be realised in the real world.
- So, what was our unwarranted assumption?
- That all variables could *in principle* be observed simultaneously.

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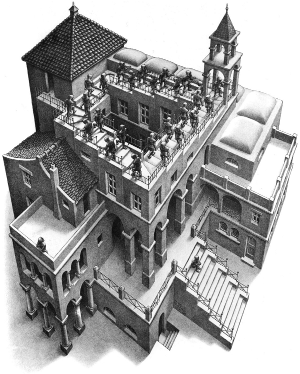

M. C. Escher, *Ascending and Descending*

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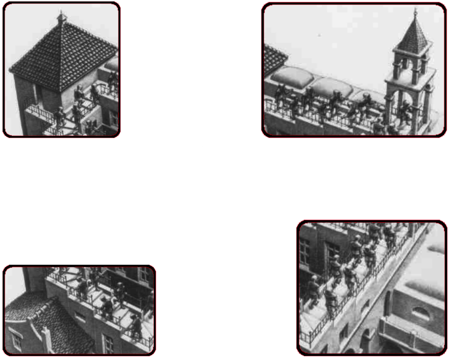

Local consistency

The essence of contextuality

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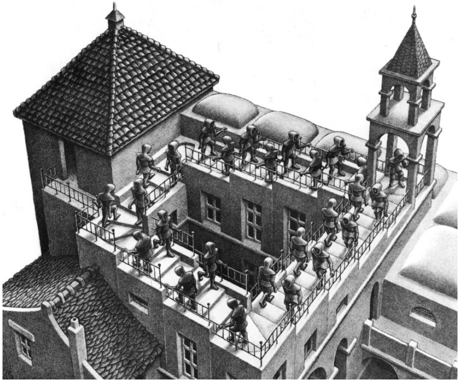

Local consistency *but* **Global inconsistency**

Formalising empirical data

A **measurement scenario** $\mathbf{X} = \langle X, \Sigma, O \rangle$:

- X – a finite set of measurements
- Σ – a simplicial complex on X
  faces are called the **measurement contexts**
- $O = (O_x)_{x \in X}$ – for each $x \in X$ a non-empty set of possible outcomes O_x

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$X = \{a_0, a_1, b_0, b_1\}, \;\; O_x = \{0, 1\}$

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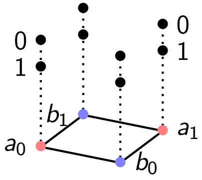

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- each $e_\sigma \in \mathrm{Prob}\left(\prod_{x \in \sigma} O_x\right)$ is a probability distribution over joint outcomes for σ.
- *generalised no-signalling* holds: for any $\sigma, \tau \in \Sigma$, if $\tau \subseteq \sigma$,

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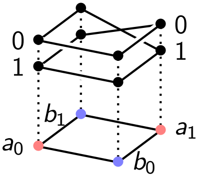

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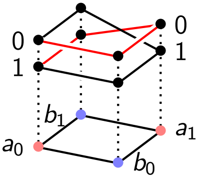

Contextuality

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The import of Bell's and Kochen–Spekker's theorems is that there are behaviours arising from quantum mechanics that are contextual.

Hierarchy of contextuality

Possibilistic collapse

- Given an empirical model e, define possibilistic model $\mathsf{poss}(e)$ by taking the support of each distributions.

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$\longmapsto$

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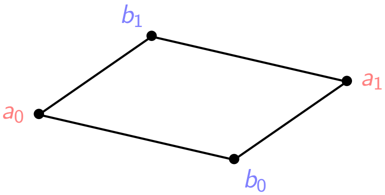

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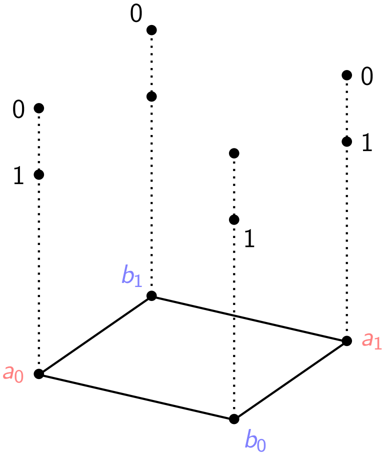

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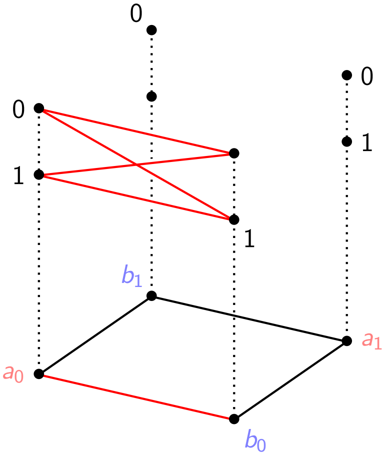

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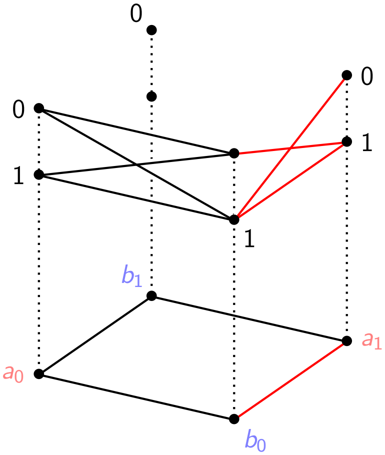

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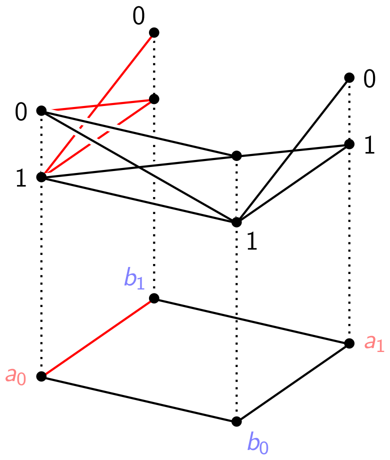

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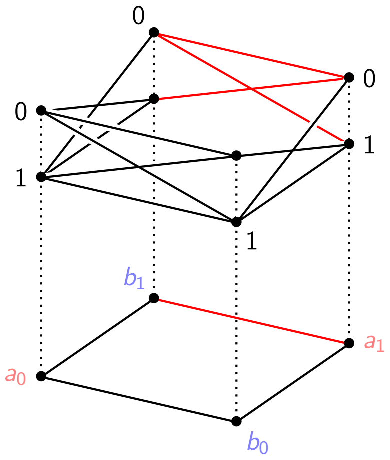

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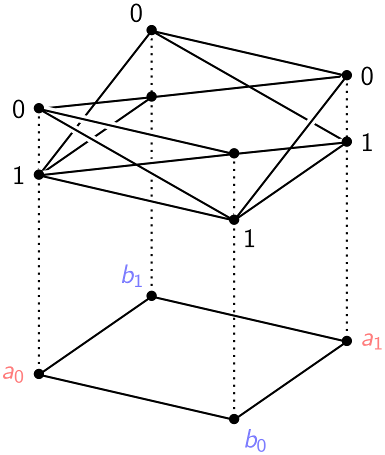

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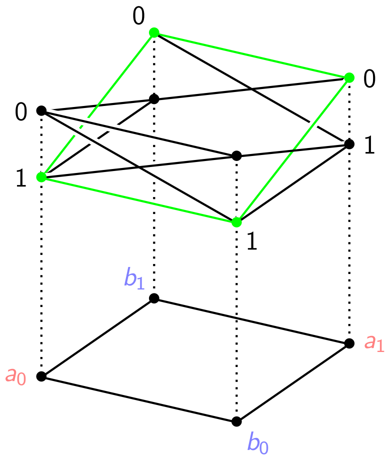

There are some global sections,

Classical assignment: $[a_0 \mapsto 1, a_1 \mapsto 0, b_0 \mapsto 1, b_1 \mapsto 0]$

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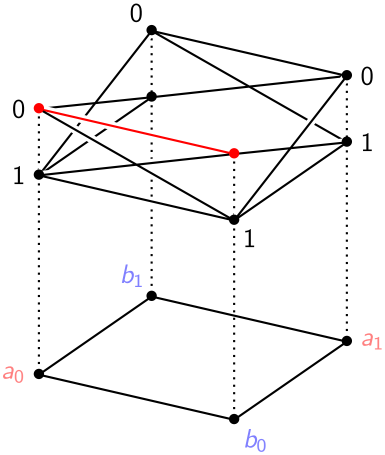

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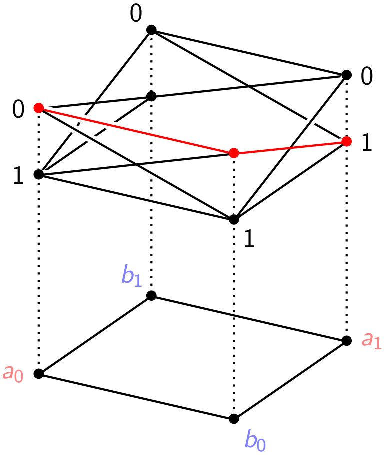

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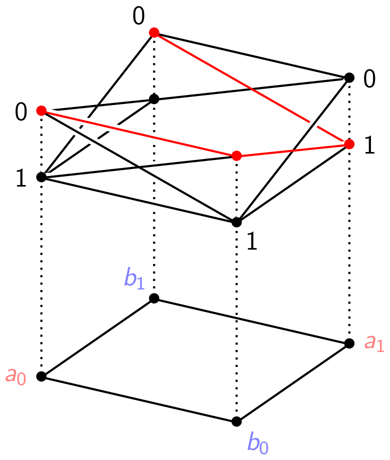

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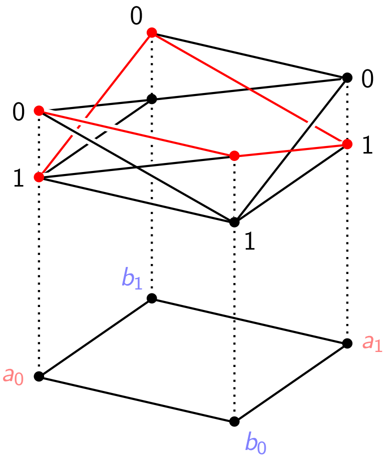

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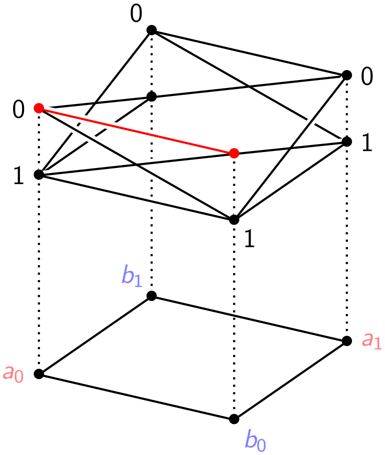

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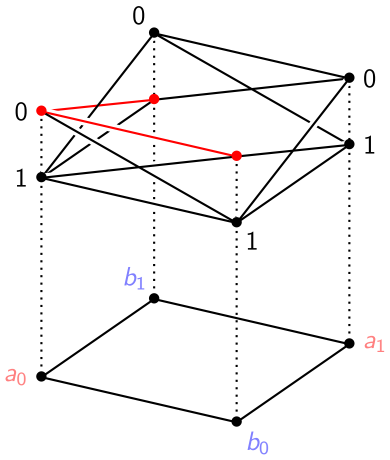

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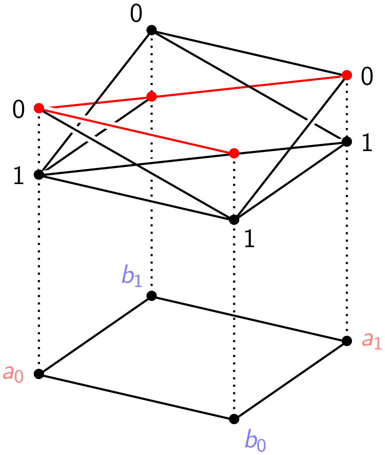

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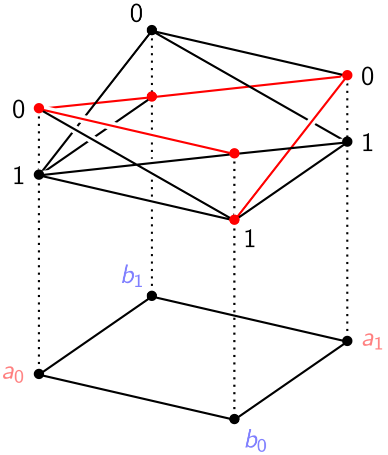

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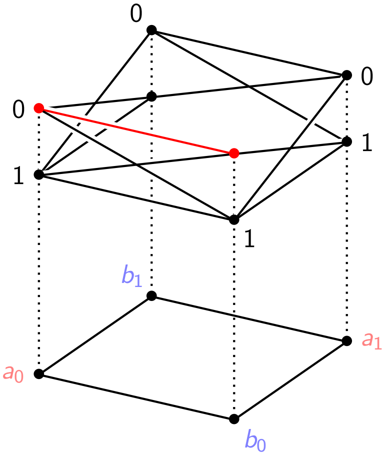

There are some global sections, but . . .

Logical contextuality: Not all sections extend to global ones.

Hierarchy of contextuality

Popescu–Rohrlich box

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a_0	b_0	1	0	0	1
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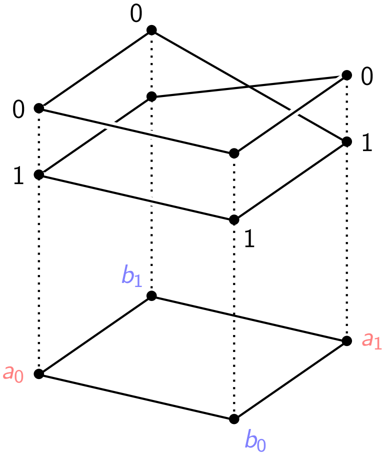

Strong contextuality:
no event can be extended to a global assignment.

$a_0 \leftrightarrow b_0 \quad a_0 \leftrightarrow b_1 \quad a_1 \leftrightarrow b_0 \quad a_1 \oplus b_1$

Measuring Contextuality

The contextual fraction

Non-contextuality: global distribution $d \in \mathsf{Prob}(O^X)$ such that:

$$\forall_{C \in \mathcal{M}}.\ d|_C \ = \ e_C \ .$$

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Non-contextual fraction: maximum weight of such a subdistribution.

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Non-contextual fraction: maximum weight of such a subdistribution.

Equivalently, maximum weight λ over all convex decompositions

$$e = \lambda e^{NC} + (1 - \lambda) e'$$

where e^{NC} is a non-contextual model.

The contextual fraction

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$$\text{NCF}(e) = \lambda \qquad \text{CF}(e) = 1 - \lambda$$

Generalised Bell inequalities

An **inequality** for a scenario $\langle X, \mathcal{M}, O \rangle$ is given by:

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NB: A complete set of inequalities can be derived from the logical approach.

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The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by an empirical model e is the value

$$\frac{\max\{0, \mathcal{B}_\alpha(e) - R\}}{\|\alpha\| - R} \ .$$

Bell inequality violation and the contextual fraction

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Proposition

Let e be an empirical model.

- The normalised violation by e of any Bell inequality is at most $\mathsf{CF}(e)$.
- This bound is attained: there exists a Bell inequality whose normalised violation by e is exactly $\mathsf{CF}(e)$.
- Moreover, this Bell inequality is tight at "the" non-contextual model e^{NC} and maximally violated by "the" strongly contextual model e^{SC} for any decomposition:

$$e = \mathsf{NCF}(e)e^{NC} + \mathsf{CF}(e)e^{SC} \; .$$

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Quantifying Contextuality LP:

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\begin{array}{ll}
\texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\
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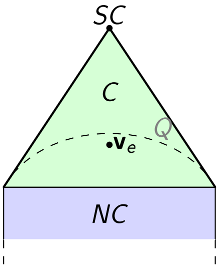

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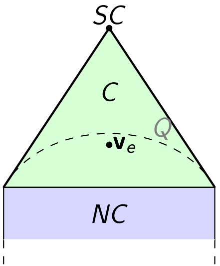

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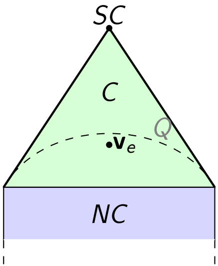

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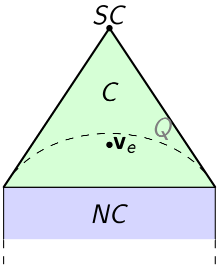

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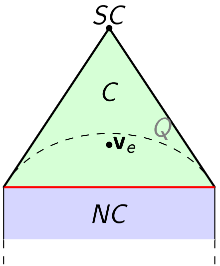

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computes tight Bell inequality
(separating hyperplane)

Contextuality as a resource

Contextuality and advantage in quantum computation

- Measurement-based quantum computation (MBQC)

  '*Contextuality in measurement-based quantum computation*'
  Raussendorf, Physical Review A, 2013.

- Magic state distillation

  '*Contextuality supplies the 'magic' for quantum computation*'
  Howard, Wallman, Veitch, Emerson, Nature, 2014.

- Shallow circuits

  '*Quantum advantage with shallow circuits*'
  Bravyi, Gossett, Koenig, Science, 2018.

  - Contextuality analysis: Aasnæss, Forthcoming, 2019.

Overview: Contextuality as a resource

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Example

'*Popescu-Rohrlich correlations as a unit of nonlocality*'
Barrett, Pironio, Physical Review Letters, 2005.

- PR boxes simulate all 2-outcome bipartite boxes
- A tripartite quantum box that cannot be simulated from PR boxes

Structure of resources

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1. **Resource theories** (coming from Physics):
   Algebraic theory of '**free operations**' that do not introduce more of the resource in question.

   Resource B can be obtained from resource A if it can be built from A using free operations.

   '*Contextual fraction as a measure of contextuality*', Abramsky, B, Mansfield, PRL, 2017.
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2. **Simulations or reducibility** (coming from Computer Science):
   Notion of **simulation** between behaviours of systems.

   One resource can be reduced to another if it can be simulated by it.
   Cf. (in)computability, degrees of unsolvability, complexity classes.

   '*Categories of empirical models*', Karvonen, QPL 2018.

Free operations

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Free operations

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- What operations can we perform (*non-contextually*) on them?

Free operations

- **Zero model** z: unique empirical model on the empty measurement scenario

$$\langle \emptyset, \Delta_0 = \{\emptyset\}, () \rangle \ .$$

- **Singleton model** u: unique empirical model on the 1-outcome 1-measurement scenario

$$\langle \mathbf{1} = \{\star\}, \Delta_1 = \{\emptyset, \mathbf{1}\}, (O_\star = \mathbf{1}) \rangle \ .$$

- **Probabilistic mixing**: Given empirical models e and d in $\langle X, \Sigma, O \rangle$ and $\lambda \in [0, 1]$, the model $e +_\lambda d : \langle X, \Sigma, O \rangle$ is given by the mixture $\lambda e + (1 - \lambda) d$.

Free operations

- **Tensor**: Let $e : \langle X, \Sigma, O \rangle$ and $d : \langle Y, \Delta, P \rangle$. Then

$$e \otimes d : \langle X \sqcup Y, \Sigma * \Delta, [O, P] \rangle$$

where $\Sigma * \Theta := \{\sigma \cup \tau | \sigma \in \Sigma, \tau \in \Delta\}$. *Runs e and d independently and in parallel.*

- **Coarse-graining**: Given $e : \langle X, \Sigma, O \rangle$ and a family of functions $h = (h_x : O_x \longrightarrow O'_x)_{x \in X}$, get a coarse-grained model

$$e/h : \langle X, \Sigma, O' \rangle$$

.

- **Measurement translation**: Given $e : \langle X, \Sigma, O \rangle$ and a simplicial map $f : \Sigma' \longrightarrow \Sigma$, the model $f^* e : \langle X', \Sigma', O \rangle$ is defined by pulling e back along the map f.

New free operation

- **Conditioning on a measurement**: Given $e : \langle X, \Sigma, O \rangle$, $x \in X$ and a family of measurements $(y_o)_{o \in O_x}$ with $y_o \in \mathsf{Vert}(\mathsf{lk}_x \Sigma)$. Consider a new measurement $x?(y_o)_{o \in O_x}$, abbreviated $x?y$. Get

$$e[x?y] : \langle X \cup \{x?y\}, \Sigma[x?y], O[x?y \mapsto O_{x?y}] \rangle$$

that results from adding $x?y$ to e.

If Σ is a simplicial complex and a $\sigma \in \Sigma$ is a face, the **link** of σ in Σ is the subcomplex of Σ whose faces are

$$\mathsf{lk}_\sigma \Sigma := \{\tau \in \Sigma \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Sigma\} \ .$$

What contexts are still available once the measurements in σ have been performed.

Free operations

Free operations generate terms typed by measurement scenarios:

$$
\begin{aligned}
\mathsf{Terms} \ni t \quad ::= \;& v \in \mathsf{Var} \;\mid\; \mathsf{z} \;\mid\; \mathsf{u} \;\mid\; f^* t \;\mid\; t/h \\
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Can d be transformed to e?

Formally: is there a typed term $v : \langle Y, \Delta, P \rangle \vdash t : \langle X, \Sigma, O \rangle$ such that $t[d/v] = e$?

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Contextual fraction and quantum advantages

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- Measure of contextuality $\rightsquigarrow$ **quantify such advantages**.

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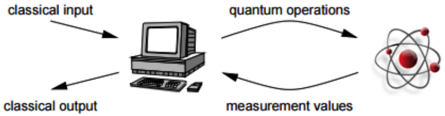
- additional power to compute non-linear functions resides in resource empirical models.



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E.g. Raussendorf (2013) ℓ2-MBQC

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  (m input bits, l output bits, n parties)

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- Probabilistic version: non-linear function computed with sufficently large probability of success implies contextuality.

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- Then,

$$\boxed{1 - \bar{p}_S \quad \geq \quad \mathsf{NCF}(e)\, \nu(f)}$$

Questions...

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