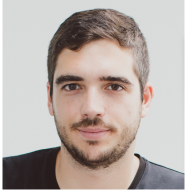
Partial Boolean algebras and the logical exclusivity principle



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Quantum physics and logic

Traditional quantum logic

Birkhoff & von Neumann (1936), '*The logic of quantum mechanics*'.

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Kochen (2015), '*A reconstruction of quantum mechanics*'.

- Kochen develops a large part of foundations of quantum theory in this framework.

Overview

- Partial Boolean algebras
- Free extensions of comeasurability
- Contextuality
- Exclusivity principles
- Tensor products

Partial Boolean algebras

Boolean algebras

Boolean algebra $\langle A, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- constants $0, 1 \in A$
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E.g.: $\mathrm{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.
Conjunction, i.e. product of projectors, becomes partial, defined only on **commuting** projectors.

The category **pBA**

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- We give a direct construction of colimits.
- More generally, we show how to freely generate from a given partial Boolean algebra a new one satisfying prescribed additional commeasurability relations.

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- *There is a* **pBA***-morphism $\eta : A \longrightarrow A[\circledcirc]$ satisfying $a \circledcirc b \implies \eta(a) \odot_{A[\circledcirc]} \eta(b)$.*
- *For every partial Boolean algebra B and* **pBA***-morphism $h : A \longrightarrow B$ satisfying $a \circledcirc b \implies h(a) \odot_B h(b)$, there is a unique homomorphism $\hat{h} : A[\circledcirc] \longrightarrow B$ such that*

$$\begin{array}{ccc} A & \xrightarrow{\eta} & A[\circledcirc] \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle \hat{h}} \\ & & B \end{array}$$

Free extensions of comeasurability

The result is proved constructively, by giving an inductive system of proof rules for commeasurability and equivalence relations over a set of syntactic terms generated from A.

- Generators $G := \{\imath(a) \mid a \in A\}$.
- Pre-terms P: closure of G under Boolean operations and constants.

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- $A[\circledcirc] = T/\equiv$, with obvious definitions for $\odot$ and operations.

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$$\frac{t(\vec{x}) \equiv_{\mathsf{Bool}} u(\vec{x}),\ \bigwedge_{i,j} v_i \odot v_j}{t(\vec{v}) \equiv u(\vec{v})} \qquad \frac{t \equiv t',\ u \equiv u',\ t \odot u}{t \wedge u \equiv t' \wedge u',\ t \vee u \equiv t' \vee u'} \qquad \frac{t \equiv u}{\neg t \equiv \neg u}$$

Coequalisers and colimits

A variation of this construction is also useful, where instead of just forcing commeasurability, one forces equality by the additional rule

$$\frac{a \odot a'}{\imath(a) \equiv \imath(a')}$$

This builds a pBA $A[\odot, \equiv]$.

Theorem

Let $h : A \longrightarrow B$ be a **pBA***-morphism such that $a \odot a' \implies h(a) = h(a')$. Then there is a unique* **pBA***-morphism $\hat{h} : A[\odot, \equiv] \longrightarrow B$ such that $h = \hat{h} \circ \eta$.*

This can be used to give an explicit construction of coequalisers, and hence general colimits, in **pBA**.

Contextuality

Kochen–Specker contextuality property

The original KS formulation of contextuality was:

> There is no embedding of the partial Boolean algebra of projectors $\mathsf{P}(\mathcal{H})$ into a (non-trivial) Boolean algebra when $\dim \mathcal{H} \geq 3$.

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Note the analogy with strong vs. logical contextuality.

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- hence there is a mediating morphism $A \longrightarrow B$!

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As such, it is complete and cocomplete, but it also admits the one-element algebra **1**, in which $0 = 1$. Note that **1** does **not** have a homomorphism to **2**.

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Theorem

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property, i.e. it has no morphism to $\mathbf{2}$.*
2. *The colimit in* **BA** *of the diagram $\mathcal{C}(A)$ of boolean subalgebras of A in* **BA** *is $\mathbf{1}$.*

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3. *$A[A^2] = \mathbf{1}$.*

Contextuality in partial Boolean algebras

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But where do states come in?

States

Definition

A **state** or **probability valuation** on a partial Boolean algebra A is a map $\nu : A \longrightarrow [0,1]$ such that:

1. $\nu(0) = 0$;
2. $\nu(\neg x) = 1 - \nu(x)$;
3. for all $x, y \in A$ with $x \odot y$, $\nu(x \vee y) + \nu(x \wedge y) = \nu(x) + \nu(y)$.

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Proposition

States can be characterised as the maps $\nu : A \longrightarrow [0,1]$ such that, for every Boolean subalgebra B of A, the restriction of ν to B is a finitely additive probability measure on B.

We can define a state $\nu : A \to [0,1]$ to be **probabilically non-contextual** if ν extends to $A[A^2]$; that is, there is a state $\hat{\nu} : A[A^2] \to [0,1]$ such that $\nu = \hat{\nu} \circ \eta$.

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By the universal property of $A[A^2]$, this is equivalent to asking that there is some Boolean algebra B, morphism $h : A \to B$, and state $\hat{\nu} : B \to [0,1]$ such that $\nu = \hat{\nu} \circ \eta$.

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Note that if A is K-S, $A[A^2] = \mathbf{1}$, and there is no state on $\mathbf{1}$.

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- A lot of effort has gone into trying to characterise the set of quantum behaviours by imposing additional, physically motivated conditions, leading to various approximations from above to this quantum set.
- We consider two **exclusivity principles**:
  - one acts at the 'logical' level, i.e. the level of events or elements of a partial Boolean algebra
  - the other acts at the 'probabilistic' level, applying to states of a partial Boolean algebra.

Exclusive events

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For $a, b \in A$, we write $a \leq b$ to mean $a \odot b$ and $a \wedge b = a$.

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Two elements $a, b \in A$ are said to be **exclusive**, written $a \perp b$, if there is a $c \in A$ such that $a \odot c$ with $a \leq c$ and $b \odot c$ with $b \leq \neg c$.

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- Note that $a \perp b$ is a weaker requirement than $a \wedge b = 0$.
- The two would be equivalent in a Boolean algebra.
- But in a general partial Boolean algebra, there might be exclusive events that are not commeasurable (and for which, therefore, the $\wedge$ operation is not defined).

LEP and PEP

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A state $\nu : A \longrightarrow [0,1]$ on A is said to satisfy the **probabilistic exclusivity principle (PEP)** if for any set $S \subseteq A$ of pairwise exclusive elements, i.e. such that $\forall a, b \in S.\ (a = b \vee a \perp b)$, then $\sum_{a \in S} \nu(a) \leq 1$.

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- But we can construct a new pba whose states yield states of A that satisfy PEP.

Theorem

A state $\nu : A \longrightarrow [0, 1]$ *satisfies PEP if there is a state* $\hat{\nu}$ *of* $A[\perp]$ *such that*

$$\begin{array}{ccc} A & \xrightarrow{\eta} & A[\perp] \\ & \searrow^{\nu} & \downarrow \hat{\nu} \\ & & [0,1] \end{array}$$

A reflective adjunction for logical exclusivity

- It's not clear whether $A[\perp]$ necessarily satisfies LEP.
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Theorem

The category **epBA** *is a reflective subcategory of* **pBA**, *i.e. the inclusion functor* $I : \mathbf{epBA} \longrightarrow \mathbf{pBA}$ *has a left adjoint* $X : \mathbf{pBA} \longrightarrow \mathbf{epBA}$.

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Theorem

Concretely, to any partial Boolean algebra A, we can associate a partial Boolean algebra $X(A) = A[\bot]^*$ satisfying LEP such that:

- *there is a homomorphism $\eta : A \longrightarrow A[\bot]^*$;*
- *for any homomorphism $h : A \longrightarrow B$ where B is a partial Boolean algebra B satisfying LEP, there is a unique homomorphism $\hat{h} : A[\bot]^* \longrightarrow B$ such that:*

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Proof. Adapt our earlier construction, adding the following rule to the inductive system:

$$\frac{u \wedge t \equiv u, \; v \wedge \neg t \equiv v}{u \odot v}$$

Tensor products of partial Boolean algebras

A (first) tensor product by generators and relations

Heunen & van den Berg show that **pBA** has a monoidal structure:

$$A \otimes B := \mathrm{colim}\, \{ C + D \mid C \in \mathcal{C}(A), D \in \mathcal{C}(B) \}$$

where $C + D$ is the coproduct of Boolean algebras.

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We can use our construction to give an explicit generators-and-relations description.

Proposition

Let A and B be partial Boolean algebras. Then

$$A \otimes B \;\cong\; (A \oplus B)[\odot]$$

where $\odot$ is the relation on the carrier set of $A \oplus B$ given by $\imath(a) \odot \jmath(b)$ for all $a \in A$ and $b \in B$.

A more expressive tensor product

- There functor $\mathrm{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA} :: \mathcal{H} \longmapsto \mathrm{P}(\mathcal{H})$ is lax monoidal.
- Embedding $\mathrm{P}(\mathcal{H}) \otimes \mathrm{P}(\mathcal{K}) \longrightarrow \mathrm{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the obvious embeddings $\mathrm{P}(\mathcal{H}) \longrightarrow \mathrm{P}(\mathcal{H} \otimes \mathcal{K}) :: p \longmapsto p \otimes 1$ and $\mathrm{P}(\mathcal{K}) \longrightarrow \mathrm{P}(\mathcal{H} \otimes \mathcal{K}) :: q \longmapsto 1 \otimes q$

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- Nevertheless, this result is suggestive.
  It poses the challenge of finding a stronger notion of tensor product.

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- Consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$.
- to show that they are **orthogonal**, we have a disjunctive requirement: $p_1 \perp q_1$ **or** $p_2 \perp q_2$.
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  Indeed, the idea that propositions can be defined on quantum systems even though subexpressions are not is emphasized by Kochen.

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This amounts to composing with the reflection to **epBA**; $\boxtimes := X \circ \otimes$. Explicitly, we define the logical exclusivity tensor product by

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- This is sound for the Hilbert space model. More precisely, P is still a lax monoidal functor wrt this tensor product.
- It remains to be seen how close it gets us to the full Hilbert space tensor product.

A limitative result

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- Can extending commeasurability by a relation $\circledcirc$ induce the K-S property in $A[\circledcirc]$ when it did not hold in A?

Theorem (K-S faithfulness of extensions)

Let A be a partial Boolean algebra, and $\circledcirc \subseteq A^2$ a relation on A. Then A is K-S if and only if $A[\circledcirc]$ is K-S.

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Corollary

If A and B are not K-S, then neither is $A \otimes B[\perp]^k$.

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Theorem (K-S faithfulness of extensions)

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So, we need a stronger tensor product to track this emergent complexity in the quantum case.

Questions...

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