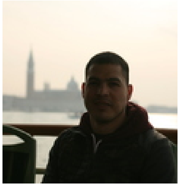
The quantum monad on relational structures: towards quantum finite model theory?


Samson Abramsky


Rui Soares Barbosa


Nadish de Silva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INL INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

UNIVERSITY OF CAMBRIDGE

Universidad Nacional Autónoma de México

Comonad meet up
16th July 2020

- ‘*The quantum monad on relational structures*’
Abramsky, B, de Silva, Zapata, MFCS 2017, `arXiv:1705.07310 [cs.LO]`.

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- ‘*The pebbling comonad in finite model theory*’
  Abramsky, Dawar, Wang, LICS 2017, `arXiv:1704.05124 [cs.LO]`.

- ‘*Relating structure and power: Comonadic semantics for computational resources*’
  Abramsky, Shah, CSL 2018, `arXiv:1806.09031 [cs.LO]`.

- ‘*Game Comonads & Generalised Quantifiers*’
  Ó Conghaile, Dawar, `arXiv:2006.16039 [cs.LO]`.

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- $\rightsquigarrow$ quantum finite model theory?

Introduction

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- use **quantum resources** for information-processing tasks
- delineate the scope of **quantum advantage**

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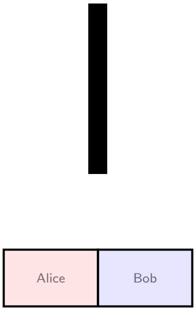
- A setting in which this has been explored is **non-local games**

Non-local games

Alice and Bob cooperate in solving a task set by Verifier

They may share prior information,



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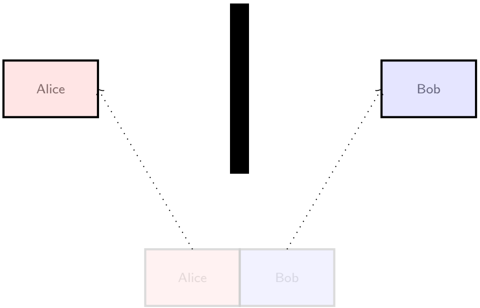

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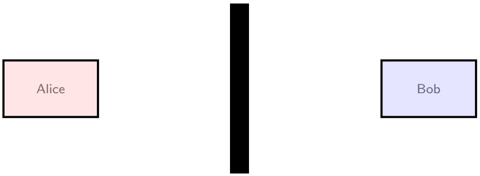

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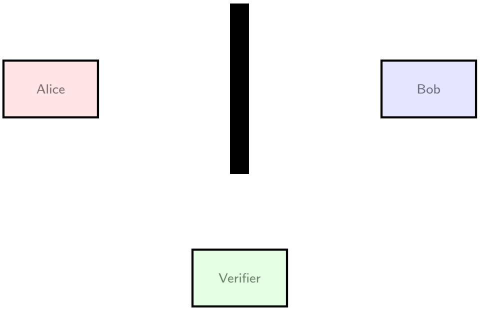

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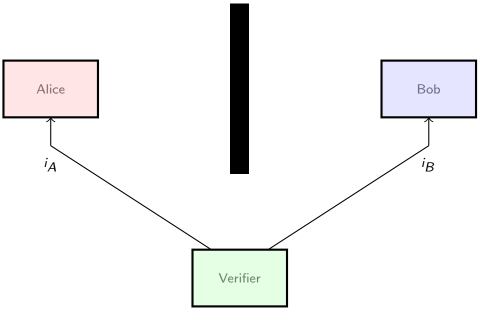

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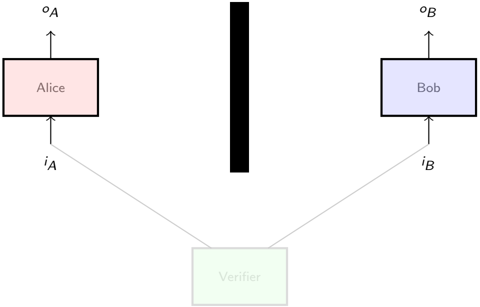

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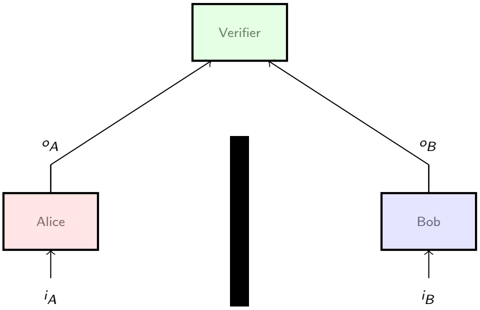

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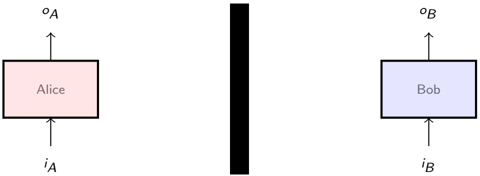

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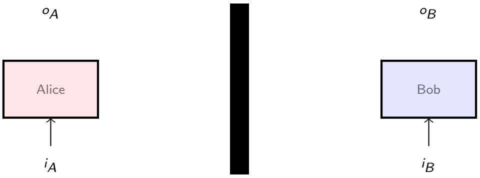

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A **perfect strategy** is one that wins with probability 1.

E.g.: Binary constraint systems

Magic square:

- Fill with 0s and 1s
- rows and first two columns: even parity
- last column: odd parity

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A	B	C
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System of linear equations over $\mathbb{Z}_2$:

$$\begin{aligned} A \oplus B \oplus C &= 0 \\ D \oplus E \oplus F &= 0 \\ G \oplus H \oplus I &= 0 \end{aligned} \qquad \begin{aligned} A \oplus D \oplus G &= 0 \\ B \oplus E \oplus H &= 0 \\ C \oplus F \oplus I &= 1 \end{aligned}$$

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Clearly, this is not satisfiable in $\mathbb{Z}_2$.

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The system has a **quantum solution** but no classical solution!

Examples of non-local games

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Many of these works have some aspects in common.

We aim to flesh this out by subsuming them under a common framework.

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Finite relational structures and homomorphisms

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Many relevant questions in these areas can be phrased in terms of (existence, number of, ...) homomorphisms between finite relational structures.

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- We formulate the task of constructing a homomorphism between relational structures as a **non-local game**.
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What could it mean to quantise these fundamental structures?

- We formulate the task of constructing a homomorphism between relational structures as a **non-local game**.
- **Uniformly** obtain quantum analogues for a range of classical notions from CS, logic, ...
- We then show that the use of quantum resources for these tasks is captured in a high-level way as **quantum homomorphisms**,
- which can be described through a **monadic** interface.

Outline

- Introduce **homomorphism game** for relational structures
- Arrive at the notion of **quantum homomorphism**, removing the two-player aspect

  (generalises Cleve & Mittal and Mančinska & Roberson)
- **Quantum monad**: capture quantum homomorphisms as classical homomorphisms to a *quantised* version of a relational structure

  (inspired on Mančinska & Roberson for graphs)
- Establish connection between non-locality and **state-independent strong contextuality**
- Towards quantum finite model theory and descriptive complexity. . . ?

Homomorphism game
for relational structures

Relational structures and homomorphisms

Relational vocabulary σ:

- relational symbols $R_1, \ldots, R_p$
- R_l has an arity $k_l \in \mathbb{N}$, for each $l \in [p] := \{1, \ldots, p\}$.

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A σ-**structure** is $\mathcal{A} = (A; R_1^{\mathcal{A}}, \ldots R_p^{\mathcal{A}})$, where:

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(For simplicity, from now on consider a single relational symbol R of arity k)

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What about quantum resources?

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Perfect strategy conditions:

$$\text{(QS1)} \qquad \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes I)\psi = 0 \qquad \text{if } \mathbf{y} \notin R^{\mathcal{B}}$$
$$\text{(QS2)} \qquad \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}} \otimes \mathcal{F}_{x,y})\psi = 0 \qquad \text{if } x = \mathbf{x}_i \text{ and } y \neq \mathbf{y}_i$$

From quantum perfect strategies to quantum homomorphisms

Simplifying quantum strategies

Theorem[1] The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathsf{x}}\}, \{\mathcal{F}_x\})$ with the following properties:

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N.B. In passing to the special form, the dimension is **reduced**: the process by which we obtain projective measurements is not at all akin to dilation.

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These $P_{x,y}$ are enough to determine the strategy!

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so that we have a PVM $P_{\mathbf{x}} = \{P_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y} \in B^k}$ where $P_{\mathbf{x},\mathbf{y}} := P_{\mathbf{x}_1,\mathbf{y}_1} \cdots P_{\mathbf{x}_k,\mathbf{y}_k}$.

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Theorem For finite structures $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent:

1. The ($\mathcal{A}$,$\mathcal{B}$)-homomorphism game has a quantum perfect strategy.
2. There is a quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$. ($\mathcal{A} \xrightarrow{q} \mathcal{B}$)

Quantum homomorphisms and the quantum monad

Quantum homomorphisms as Kleisli maps

For σ-structure $\mathcal{A}$ and $d \in \mathbb{N}$, define a σ-structure $\mathcal{Q}_d\mathcal{A}$ such that there is a 1-1 correspondence:[2]

$$\mathcal{A} \xrightarrow{q}_d \mathcal{B} \quad \cong \quad \mathcal{A} \longrightarrow \mathcal{Q}_d\mathcal{B}$$

- quantum homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ of dimension d
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[2] Mančinska & Roberson: analogous construction for (their) graph homomorphisms.

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Universe of structure $\mathcal{Q}_d\mathcal{A}$: d-dimesional projector-valued distributions on A,
i.e. set of functions $p : A \longrightarrow \mathrm{Proj}(d)$ with finite support and $\sum_{x \in A} p(x) = I$.

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Quantum homomorphisms as Kleisli maps

$\mathcal{Q}_d$ is a functor and moreover part of a **graded monad** on the category $R(\sigma)$.

Monads play a major rôle in programming language theory, providing a uniform way of encapsulating various notions of computation:

- partiality
- exceptions
- non-determinism
- probabilistic
- state updates
- input/output
- ...

Monads

Functor $T : \mathfrak{C} \longrightarrow \mathfrak{C}$ such that a T-program, a computation producing values of type B from values of type A with T-effects, is seen as a map $A \longrightarrow T\,B$ in the category $\mathfrak{C}$.

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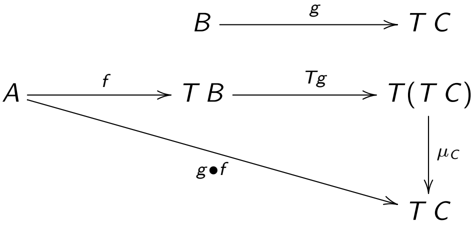

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The quantum monad is graded by dimension

$$\mu_{\mathcal{A}}^{d,d'} : \mathcal{Q}_d(\mathcal{Q}_{d'}\,\mathcal{A}) \longrightarrow \mathcal{Q}_{dd'}\,\mathcal{A}$$

$\rightsquigarrow$ composition of quantum homomorphisms keeps track of the dimension of the resources

$$\mathcal{B} \xrightarrow{\quad f \quad} \mathcal{Q}_{d'}\,\mathcal{C}$$

$$\mathcal{A} \xrightarrow{\quad f \quad} \mathcal{Q}_d\,\mathcal{B}$$

Composition of quantum homomorphisms

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The quantum monad is graded by dimension

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$\rightsquigarrow$ composition of quantum homomorphisms keeps track of the dimension of the resources

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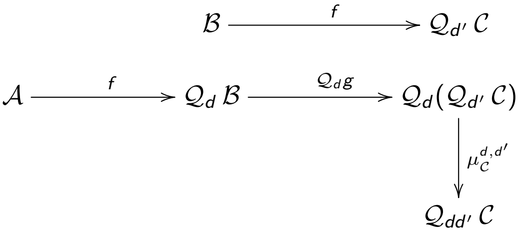

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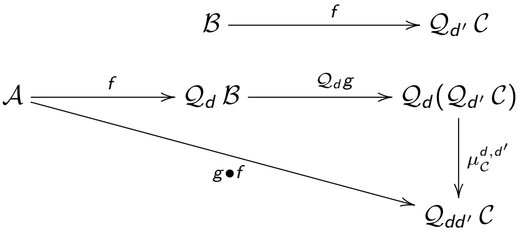

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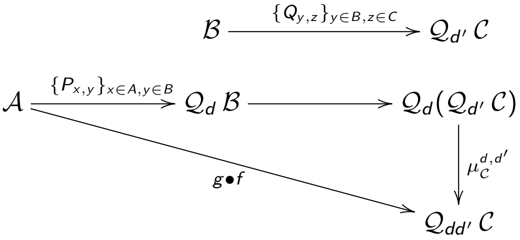

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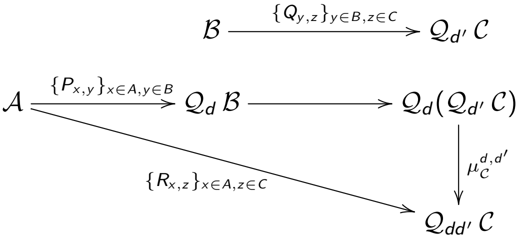

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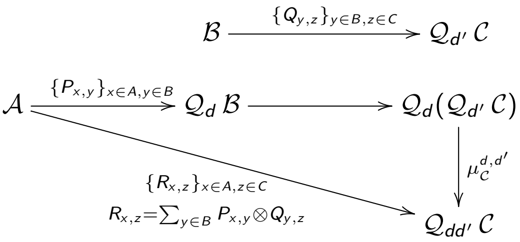

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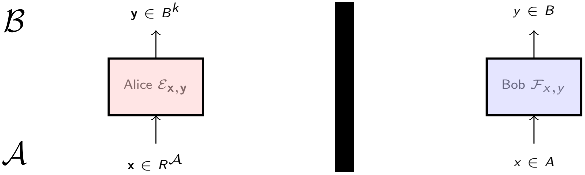

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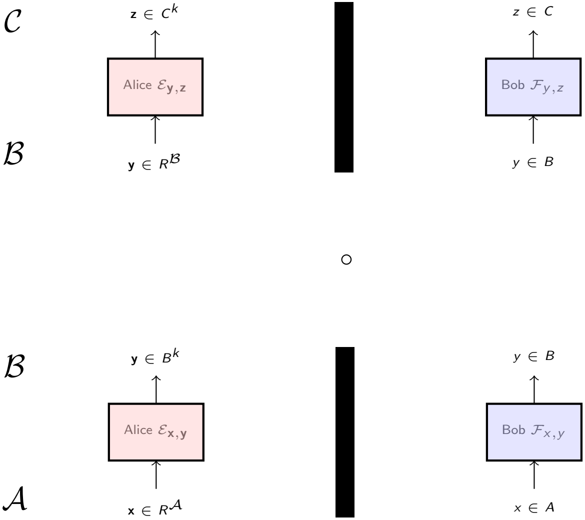

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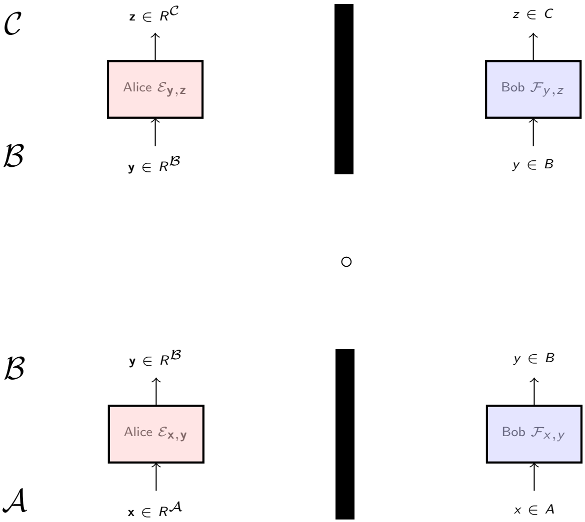

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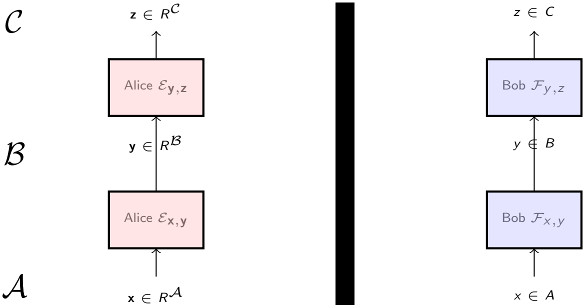

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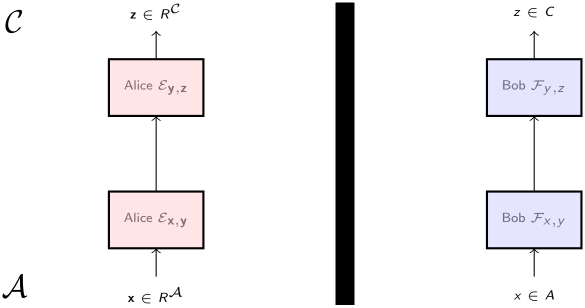

Contextuality and non-locality

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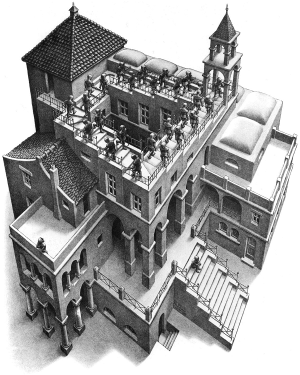

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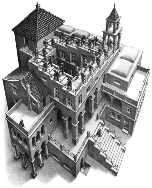
Non-locality is a particular case of contextuality for Bell scenarios

. . . but we show that certain contextuality proofs can be underwritten by non-locality arguments.



Contextuality

Measurement scenario $(X, \mathcal{M}, O)$:

- X is a finite set of measurements
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Strong contextuality: there is no global assignment $g : X \longrightarrow O$ such that

$$\forall C \in \mathcal{M}. \quad e_C(g|_C) = 1.$$

That is, no global assignment consistent with the model in the sense of yielding **possible** outcomes in all measurement contexts.

E.g.: GHZ, Kochen–Specker, (post-quantum) PR box

Strong contextuality

Strong Contextuality:
no consistent global assignment.

A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_0	b_0	✓	×	×	✓
a_0	b_1	✓	×	×	✓
a_1	b_0	✓	×	×	✓
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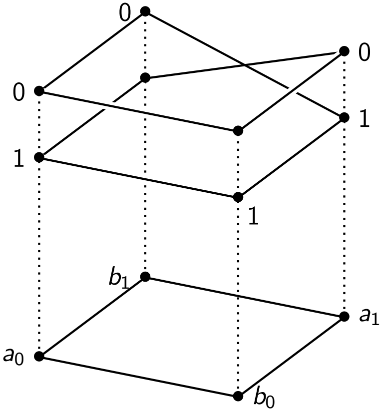

Strong contextuality and constraint satisfaction

The support of e can be described as a CSP $\mathcal{K}_e$

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- consistent global assignements for e

Hence, e is strongly contextual iff $\mathcal{K}_e$ has no (classical) solution.

Quantum correspondence

Quantum witness for e:

- state φ
- PVM $P_x = \{P_{x,o}\}_{o \in O}$ for each $x \in X$
- $[P_{x,o}, P_{x',o'}] = \mathbf{0}$ whenever $x, x' \in C \in M$
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General way of turning state-independent contextuality proofs into Bell non-locality arguments (generalising Heywood & Redhead's construction).

Outlook

Some open questions

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- Monad of distributions valued in any partial Boolean algebra $\rightsquigarrow$ logical exclusivity

Outlook: Towards quantum finite model theory

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- $\rightsquigarrow$ equivalences coarser than isomorphism:

$$\mathcal{A} \equiv^{\mathcal{L}} \mathcal{B} \quad := \quad \forall \phi \in \mathcal{L}.\, \mathcal{A} \models \phi \Leftrightarrow \mathcal{B} \models \phi$$

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- computational complexity $\leftrightarrow$ expressive power of logics

E.g.	PH $\leftrightarrow$ SO	(second-order logic)
	NP $\leftrightarrow$ $\exists$SO	(existential second-order logic)
	$\mathsf{AC}^0 \leftrightarrow \mathsf{FO}(+, \times)$	(first-order logic with $+$ and $\times$)

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Thank you!

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