Partial Boolean algebras: The logic of contextuality

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https://qpl2021.eu/

QUANTUM PHYSICS AND LOGIC

is an annual conference that brings together researchers working on mathematical foundations of quantum physics, quantum computing, and related areas, with a focus on structural perspectives and the use of logical tools, ordered algebraic and category-theoretic structures, formal languages, semantical methods, and other computer science techniques applied to the study of physical behaviour in general. Work that applies structures and methods inspired by quantum theory to other fields (including computer science) is also welcome.

Important dates	
Paper submission deadline	February 12th, 2021
Author notification	March 31st, 2021
Early registration deadline	May 14th, 2021
Final papers ready	May 28th, 2021
Conference	June 7th to 11th, 2021

Preamble

Quantum foundations

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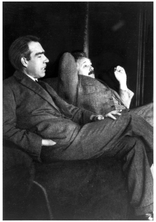

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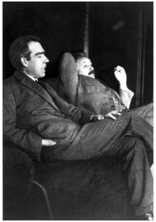

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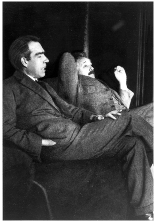

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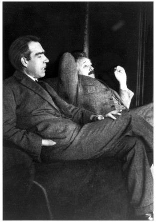

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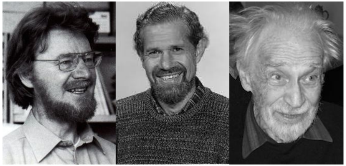

- Bell–Kochen–Specker (60s):
  Non-locality and contextuality as fundamental **empirical** phenomena rather than shortcomings of the formalism.

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- A central question is to characterise **quantum advantage**
- Focus on **non-classical** aspects of quantum theory

$\boxed{\text{Not a bug but a feature!}}$

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- Non-locality (Bell's theorem) is a special case.
- Related to many instances of quantum advantage in computation and informatics.
- Empirical predictions of quantum mechanics are incompatible with **all** observables being assigned values simultaneously.
- More abstractly: data that are **locally consistent** but **globally inconsistent**.

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- This is work in progress. Many open questions.

- Paper in CSL 2021: `arXiv:2011.03064 [quant-ph]`

- This talk: focus on logical aspects, ignore e.g. probabilistic.
  - Contextuality in logical form
  - Towards tracking the quantum tensor product
  - Logical exclusivity principle
  - Free extension of commeasurability

Logic and quantum mechanics

From classical to quantum

John von Neumann (1932), '*Mathematische Grundlagen der Quantenmechanik*'.

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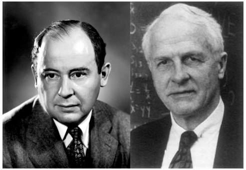
- Quantum properties or propositions are **projectors**:

$$p : \mathcal{H} \to \mathcal{H} \qquad \text{s.t.} \qquad p = p^\dagger = p^2$$

which correspond to closed subspaces of $\mathcal{H}$.

Quantum physics and logic



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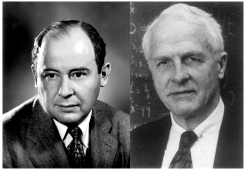

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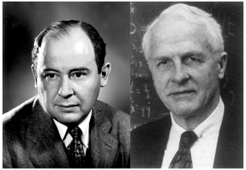

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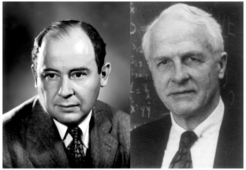

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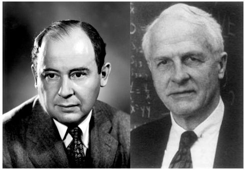

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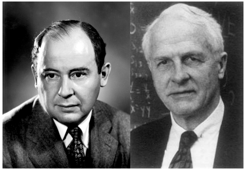

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- Only commuting measurements can be performed together.
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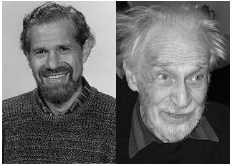
Quantum physics and logic



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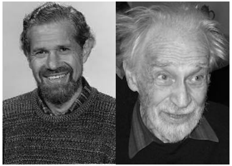

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- Represent incompatibility by **partiality**

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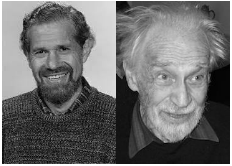

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Kochen (2015), '*A reconstruction of quantum mechanics*'.

- Kochen develops a large part of foundations of quantum theory in this framework.

Partial Boolean algebras

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Boolean algebra $\langle A, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
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E.g.: $\mathrm{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.
Conjunction, i.e. product of projectors, becomes partial, defined only on **commuting** projectors.

The category **pBA**

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- Coequalisers, and general colimits: shown to exist via the Adjoint Functor Theorem.

The category **pBA**

classifying toposes (6.56). One's first reaction on seeing this theorem is to admire its elegance and generality; the second reaction (which comes quite a long time later) is to realize its fundamental uselessness—a quality which, by the way, it shares with the General Adjoint Functor Theorem. For the

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- We give a direct construction of colimits.
- More generally, we show how to freely generate from a given partial Boolean algebra a new one satisfying prescribed additional commeasurability relations.

Free extensions of commeasurability

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- *There is a* **pBA**-*morphism* $\eta : A \longrightarrow A[\circledcirc]$ *satisfying* $a \circledcirc b \implies \eta(a) \odot_{A[\circledcirc]} \eta(b)$.
- *For every partial Boolean algebra B and* **pBA**-*morphism* $h : A \longrightarrow B$ *satisfying* $a \circledcirc b \implies h(a) \odot_B h(b)$, *there is a unique homomorphism* $\hat{h} : A[\circledcirc] \longrightarrow B$ *such that*

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "h"'] & A[\circledcirc] \arrow[d, "\hat{h}"] \\ & B \end{tikzcd}$$

Free extensions of commeasurability

The result is proved constructively, by giving an inductive system of proof rules for commeasurability and equivalence relations over a set of syntactic terms generated from A.

- Generators $G := \{\imath(a) \mid a \in A\}$.
- Pre-terms P: closure of G under Boolean operations and constants.

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- $T := \{t \in P \mid t\downarrow\}$.
- $A[\circledcirc] = T/\equiv$, with obvious definitions for $\odot$ and operations.

The inductive construction

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$$\frac{a \in A}{\imath(a)\downarrow} \qquad \frac{a \odot_A b}{\imath(a) \odot \imath(b)} \qquad \frac{a \circledcirc b}{\imath(a) \odot \imath(b)}$$

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$$\frac{}{0\downarrow,\ 1\downarrow} \qquad \frac{t \odot u}{t \wedge u\downarrow,\ t \vee u\downarrow} \qquad \frac{t\downarrow}{\neg t\downarrow}$$

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$$\frac{a \in A}{\imath(a)\downarrow} \qquad \frac{a \odot_A b}{\imath(a) \odot \imath(b)} \qquad \frac{a \circledcirc b}{\imath(a) \odot \imath(b)}$$

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Coequalisers and colimits

A variation of this construction is also useful, where instead of just forcing commeasurability, one forces equality by the additional rule

$$\frac{a \circledcirc a'}{\imath(a) \equiv \imath(a')}$$

This builds a pBA $A[\circledcirc, \equiv]$.

Theorem

Let $h : A \longrightarrow B$ be a **pBA***-morphism such that $a \circledcirc a' \implies h(a) = h(a')$. Then there is a unique* **pBA***-morphism $\hat{h} : A[\circledcirc, \equiv] \longrightarrow B$ such that $h = \hat{h} \circ \eta$.*

This is used to give an explicit construction of coequalisers, and hence general colimits, in **pBA**.

Contextuality

Kochen–Specker contextuality property

The original KS formulation of contextuality was:

> There is no embedding of the partial Boolean algebra of projectors $\mathsf{P}(\mathcal{H})$ into a (non-trivial) Boolean algebra when $\dim \mathcal{H} \geq 3$.

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Thus the **strongest** contextuality property is:

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This is what Kochen and Specker prove for $\mathrm{P}(\mathcal{H})$ with $\dim \mathcal{H} \geq 3$.

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- **BA** is a full subcategory of **pBA**.
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But note that **BA** is an equational variety of algebras over **Set**.

As such, it is complete and cocomplete, but it also admits the one-element algebra **1**, in which $0 = 1$. Note that **1** does **not** have a homomorphism to **2**.

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Theorem

Let A be a partial Boolean algebra. The following are equivalent:

1. *A has the K-S property, i.e. it has no morphism to $\mathbf{2}$.*
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3. *$A[A^2] = \mathbf{1}$.*

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- A satisfies $\varphi(\vec{a})$ if $t \equiv 1$ in $A[\emptyset]$.

Theorem

The following are equivalent:

1. *A has the K-S property.*
2. *There is a $\varphi(\vec{x}) \equiv_{\mathsf{Bool}} 0$ and assignment $\vec{x} \mapsto \vec{a}$ s.t. A satisfies $\varphi(\vec{a})$.*

Tensor products and partial Boolean algebras

A (first) tensor product by generators and relations

Heunen & van den Berg show that **pBA** has a monoidal structure:

$$A \otimes B := \operatorname{colim} \{ C + D \mid C \in \mathcal{C}(A), D \in \mathcal{C}(B) \}$$

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We can use our construction to give an explicit generators-and-relations description.

Proposition

Let A and B be partial Boolean algebras. Then

$$A \otimes B \;\cong\; (A \oplus B)[\odot]$$

where $\odot$ is the relation on the carrier set of $A \oplus B$ given by $\imath(a) \odot \jmath(b)$ for all $a \in A$ and $b \in B$.

A more expressive tensor product

- The functor $\mathrm{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA} :: \mathcal{H} \longmapsto \mathrm{P}(\mathcal{H})$ is lax monoidal.
- Embedding $\mathrm{P}(\mathcal{H}) \otimes \mathrm{P}(\mathcal{K}) \longrightarrow \mathrm{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the obvious embeddings $\mathrm{P}(\mathcal{H}) \longrightarrow \mathrm{P}(\mathcal{H} \otimes \mathcal{K}) :: p \longmapsto p \otimes 1$ and $\mathrm{P}(\mathcal{K}) \longrightarrow \mathrm{P}(\mathcal{H} \otimes \mathcal{K}) :: q \longmapsto 1 \otimes q$

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  - The images of $\mathsf{P}(\mathcal{H})$ and $\mathsf{P}(\mathcal{K})$ generate $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$, for any finite-dimensional $\mathcal{H}$ and $\mathcal{K}$.
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- Nevertheless, this result is suggestive.
  It poses the challenge of finding a stronger notion of tensor product.

A more expressive tensor product (ctd)

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- Consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$.
- to show that they are **orthogonal**, we have a disjunctive requirement: $p_1 \perp q_1$ **or** $p_2 \perp q_2$.
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Indeed, the idea that propositions can be defined on quantum systems even though subexpressions are not is emphasized by Kochen.

Logical exclusivity principle

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Two elements $a, b \in A$ are said to be **exclusive**, written $a \perp b$, if there is a $c \in A$ such that $a \odot c$ with $a \leq c$ and $b \odot c$ with $b \leq \neg c$.

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Definition

A is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are logically exclusive are also commeasurable, i.e. if $\perp \subseteq \odot$.

Logical exclusivity and transitivity

Definition

A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, if $a \leq b$ and $b \leq c$, then $a \leq c$, i.e. $\leq$ is (globally) a partial order on A.

Proposition

A partial Boolean algebra satisfies LEP if and only if it is transitive.

A reflective adjunction for logical exclusivity

- It's not clear whether $A[\perp]$ necessarily satisfies LEP.
- While the principle holds for all its elements in the image of $\eta : A \to A[\perp]$, it may fail to hold for other elements in $A[\perp]$.

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Theorem

The category **epBA** *of partial Boolean algebras satisfying LEP is a reflective subcategory of* **pBA**, *i.e. the inclusion functor* $I : \mathbf{epBA} \longrightarrow \mathbf{pBA}$ *has a left adjoint* $X : \mathbf{pBA} \longrightarrow \mathbf{epBA}$.

A reflective adjunction for logical exclusivity

Theorem

Concretely, to any partial Boolean algebra A, we can associate a partial Boolean algebra $X(A) = A[\perp]^*$ satisfying LEP such that:

- *there is a homomorphism $\eta : A \longrightarrow A[\perp]^*$;*
- *for any homomorphism $h : A \longrightarrow B$ where B is a partial Boolean algebra B satisfying LEP, there is a unique homomorphism $\hat{h} : A[\perp]^* \longrightarrow B$ such that:*

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "h"'] & A[\perp]^* \arrow[d, "\hat{h}"] \\ & B \end{tikzcd}$$

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Proof. Adapt our earlier construction, adding the following rule to the inductive system:

$$\frac{u \wedge t \equiv u, \;\; v \wedge \neg t \equiv v}{u \odot v}$$

Towards a more expressive tensor

Logical exclusivity tensor product

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This amounts to composing with the reflection to **epBA**; $\boxtimes := X \circ \otimes$. Explicitly, we define the logical exclusivity tensor product by

$$A \boxtimes B = (A \otimes B)[\perp]^* = (A \oplus B)[\oplus][\perp]^*.$$

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- This is sound for the Hilbert space model. More precisely, P is still a lax monoidal functor wrt this tensor product.
- How close does it get to the full Hilbert space tensor product?

A limitative result

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- Can extending commeasurability by a relation $\circledcirc$ induce the K-S property in $A[\circledcirc]$ when it did not hold in A?

Theorem (K-S faithfulness of extensions)

Let A be a partial Boolean algebra.
For any relation $\circledcirc$ on A, A is K-S if and only if $A[\circledcirc]$ is K-S.
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So, we need a stronger tensor product to track this emergent complexity in the quantum case.

Questions...

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