Free transformations in the resource theory of contextuality


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This talk

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Quantum Physics

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Closing Bell: Boxing black box simulations in the resource theory of contextuality

Rui Soares Barbosa, Martti Karvonen, Shane Mansfield

This chapter contains an exposition of the sheaf-theoretic framework for contextuality emphasising resource-theoretic aspects, as well as some original results on this topic. In particular, we consider functions that transform empirical models on a scenario S to empirical models on another scenario T, and characterise those that are induced by classical procedures between S and T corresponding to 'free' operations in the (non-adaptive) resource theory of contextuality. We proceed by expressing such functions as empirical models themselves, on a new scenario built from S and T. Our characterisation then boils down to the non-contextuality of these models. We also show that this construction on scenarios provides a closed structure in the category of measurement scenarios.

Comments: 36 pages. To appear as part of a volume dedicated to Samson Abramsky in Springer's Outstanding Contributions to Logic series
Subjects: **Quantum Physics (quant-ph)**; Logic in Computer Science (cs.LO); Category Theory (math.CT)
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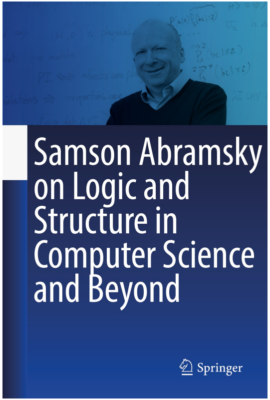

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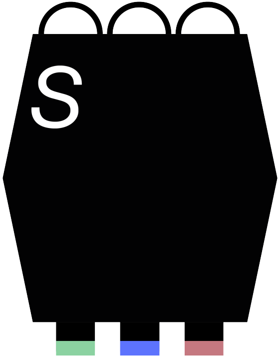
  - F realisable by classical procedure $S \longrightarrow T$ **iff** e_F is noncontextual (and satisfies a certain predicate).
  - $[-, -]$ provides a **closed structure** on (a variant of) the category of measurement scenarios.

Contextuality

Type or interface: measurement scenario



- Interaction with system: perform measurements and observe respective outcomes

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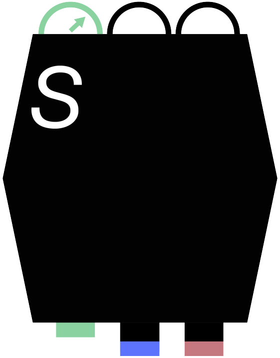

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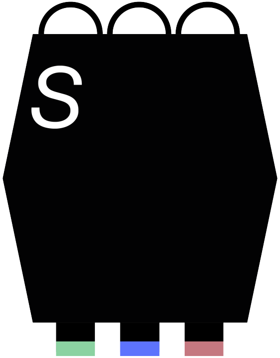

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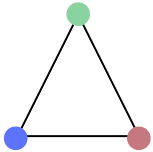

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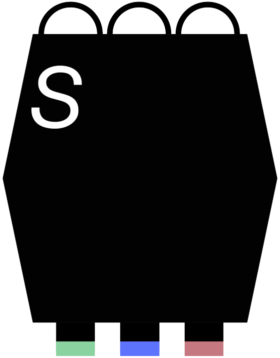

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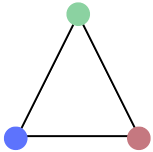

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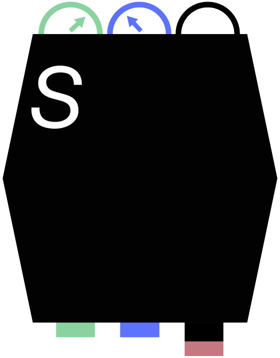

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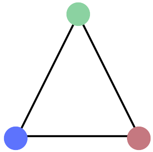

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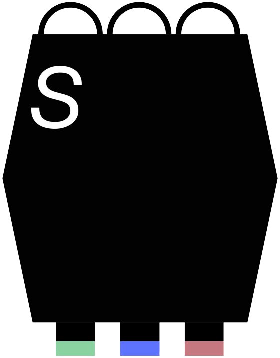

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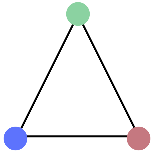

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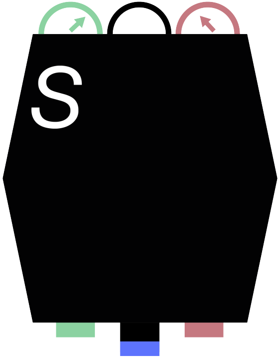

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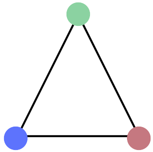

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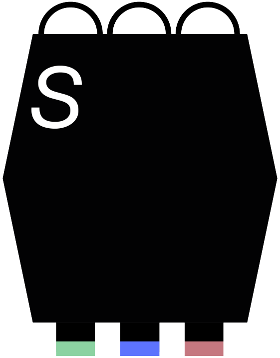

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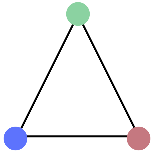

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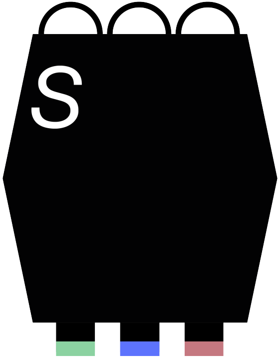

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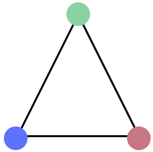

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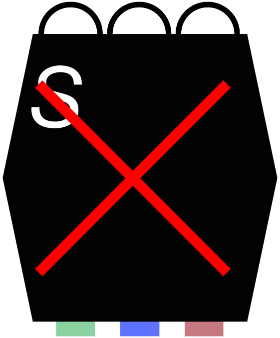

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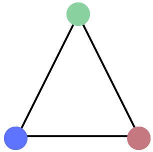

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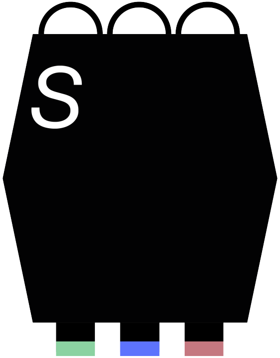

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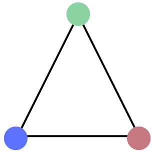

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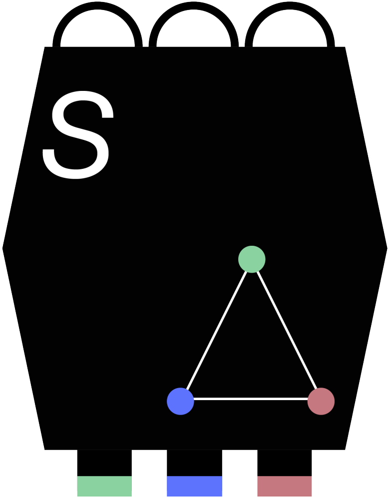

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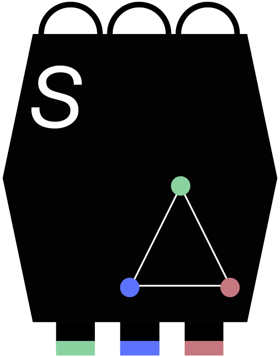

Measurement scenario $S = \langle X_S, \Sigma_S, O_S \rangle$:

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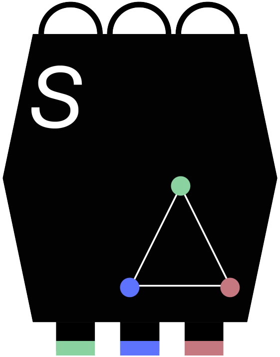

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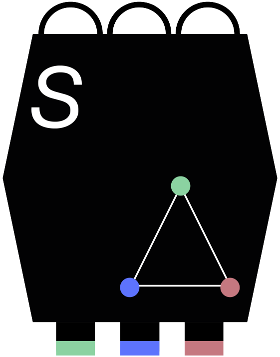

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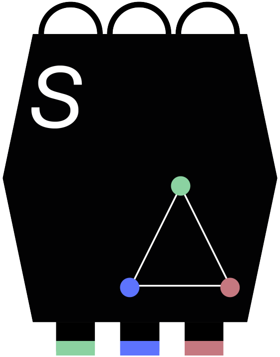

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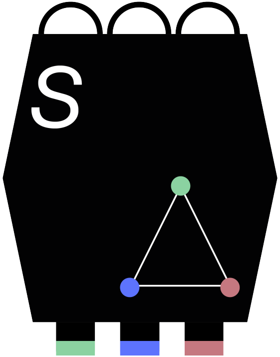

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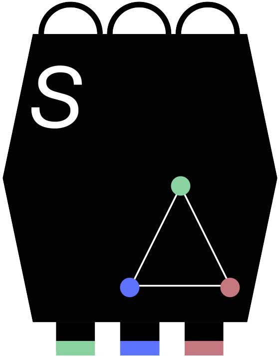

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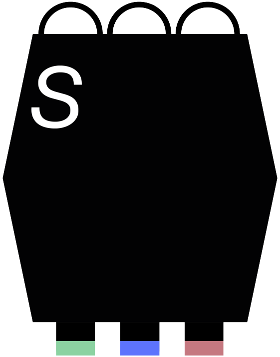
Behaviour: empirical model



- Behaviour of system is described by measurement statistics

		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y				
y	z				
x	z				

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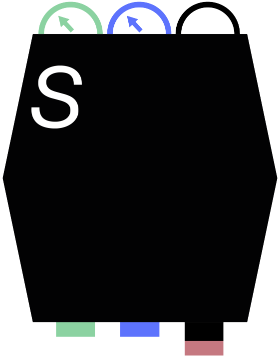

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y	z				
x	z				

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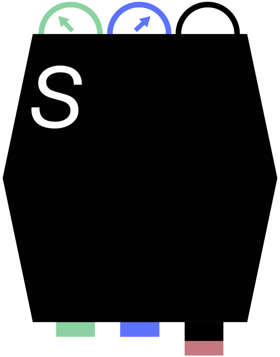

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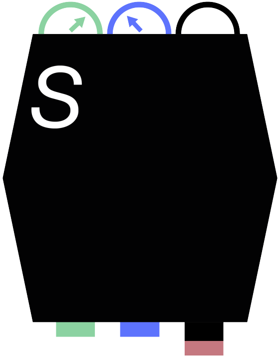

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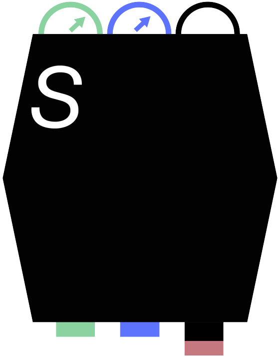

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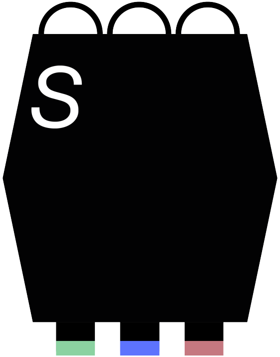

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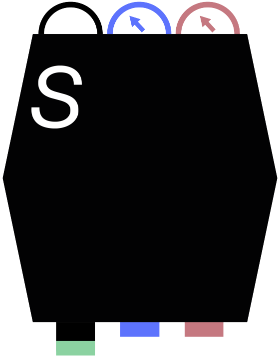

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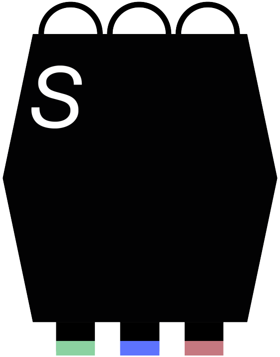
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$			
x	z				

Behaviour: empirical model



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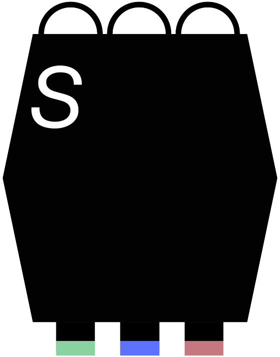
		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

Behaviour: empirical model



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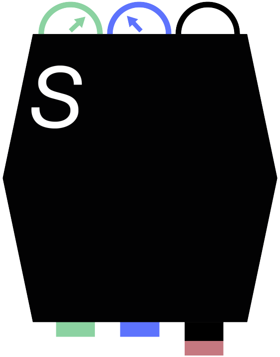
		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

Behaviour: empirical model



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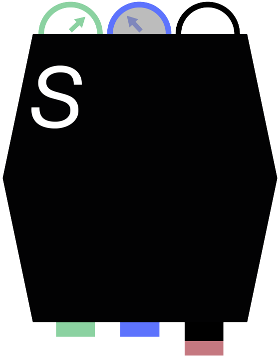
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$P(\mathsf{x},\mathsf{y} \mapsto a, b)$$

Behaviour: empirical model



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		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
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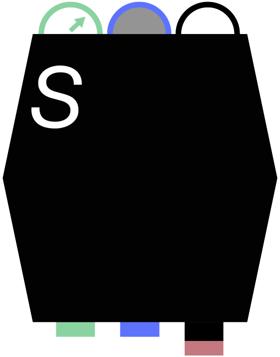


- Behaviour of system is described by measurement statistics

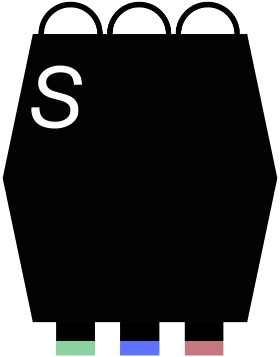
		(0, 0)	(0, 1)	(1, 0)	(1, 1)
x	y	3/8	1/8	1/8	3/8
y	z	3/8	1/8	1/8	3/8
x	z	1/8	3/8	3/8	1/8

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathbf{x}, \mathbf{y} \mapsto a, b)$$

Behaviour: empirical model



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		$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

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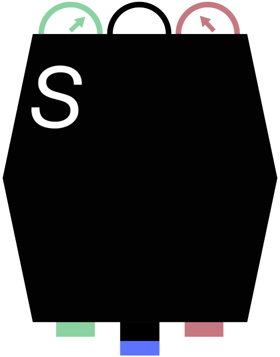


- Behaviour of system is described by measurement statistics

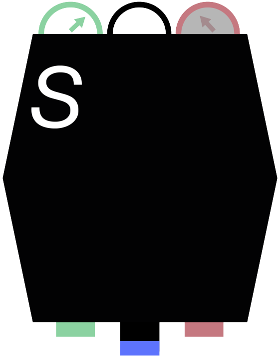
		(0, 0)	(0, 1)	(1, 0)	(1, 1)
x	y	3/8	1/8	1/8	3/8
y	z	3/8	1/8	1/8	3/8
x	z	1/8	3/8	3/8	1/8

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathsf{x}, \mathsf{y} \mapsto a, b) \qquad\qquad P(\mathsf{x}, \mathsf{z} \mapsto a, c)$$

Behaviour: empirical model



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		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

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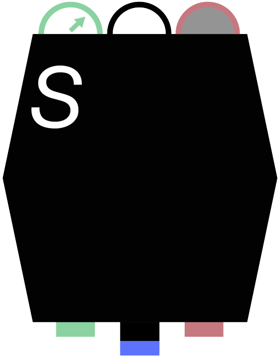

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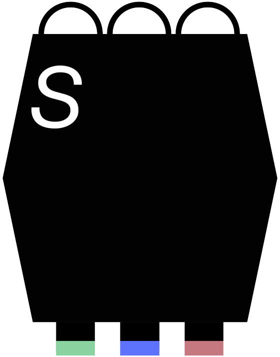
		(0, 0)	(0, 1)	(1, 0)	(1, 1)
x	y	3/8	1/8	1/8	3/8
y	z	3/8	1/8	1/8	3/8
x	z	1/8	3/8	3/8	1/8

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathbf{x}, \mathbf{y} \mapsto a, b) \qquad \sum_c P(\mathbf{x}, \mathbf{z} \mapsto a, c)$$

Behaviour: empirical model



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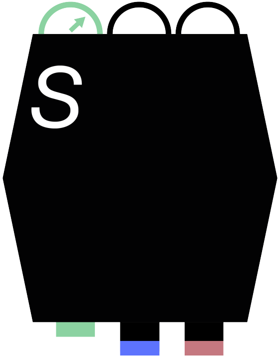
		(0,0)	(0,1)	(1,0)	(1,1)
x	y	3/8	1/8	1/8	3/8
y	z	3/8	1/8	1/8	3/8
x	z	1/8	3/8	3/8	1/8

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathsf{x},\mathsf{y} \mapsto a,b) \quad = \quad \sum_c P(\mathsf{x},\mathsf{z} \mapsto a,c)$$

Behaviour: empirical model



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		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
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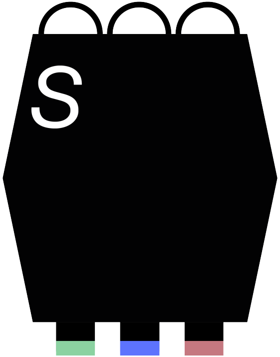
- Marginal distributions agree

$$\sum_b P(\mathsf{x},\mathsf{y} \mapsto a, b) \qquad \sum_c P(\mathsf{x},\mathsf{z} \mapsto a, c)$$

$$=$$

$$P(\mathsf{x} \mapsto a)$$

Behaviour: empirical model



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$$=$$

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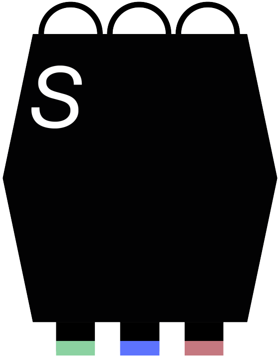
Behaviour: empirical model

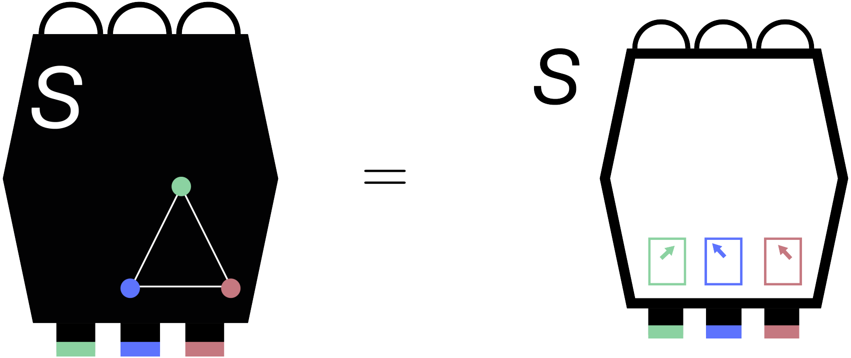


Empirical model $e : S$ is a family $\{e_\sigma\}_{\sigma \in \Sigma_S}$ where:

- e_σ is a probability distribution on the set of joint outcomes $\mathbf{O}_{S,\sigma} := \prod_{x \in \sigma} O_{S,x}$
- These satisfy **no-disturbance**:
  if $\tau \subset \sigma$, then $e_\sigma|_\tau = e_\tau$.

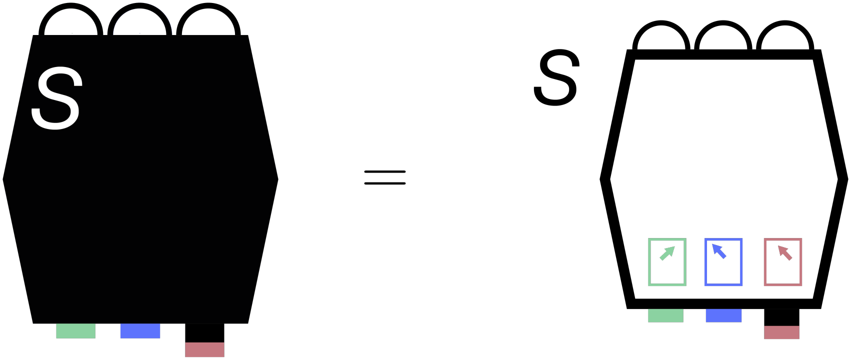
Contextuality

Deterministic model



Contextuality

Deterministic model



Contextuality

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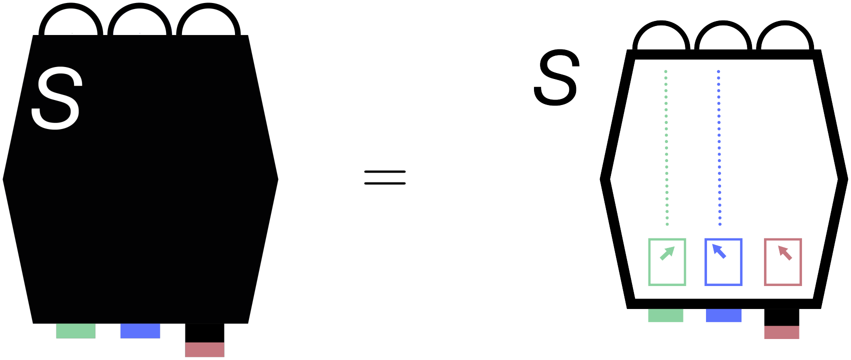

Contextuality

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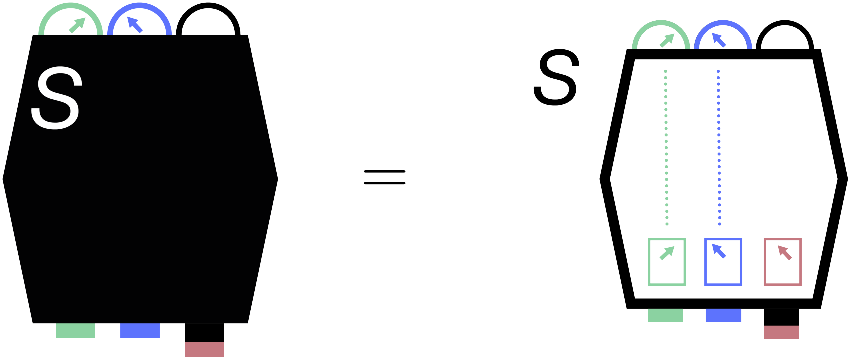
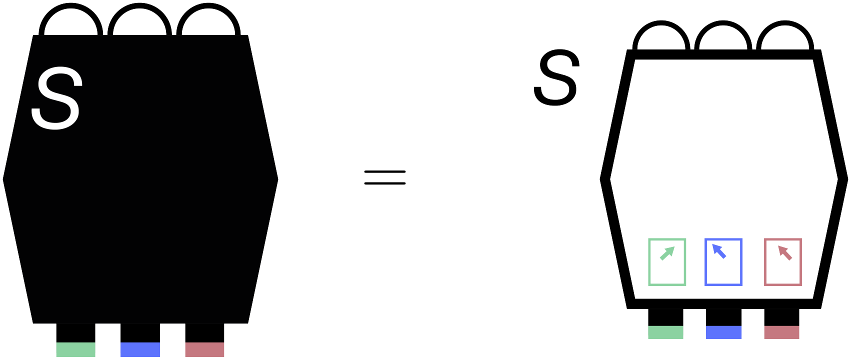


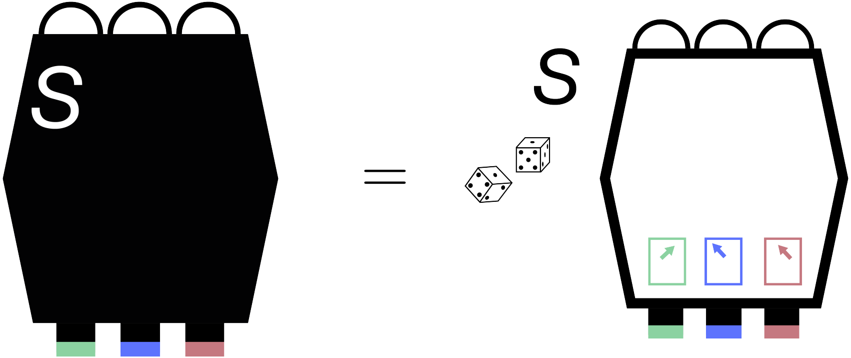
Contextuality

Deterministic model



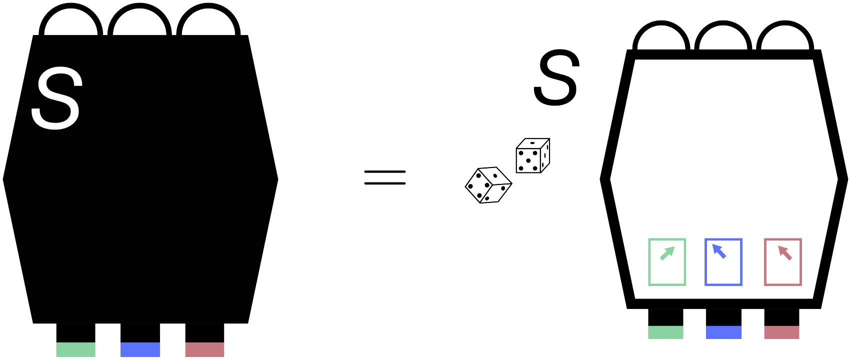
Contextuality

Non-contextual model



Contextuality

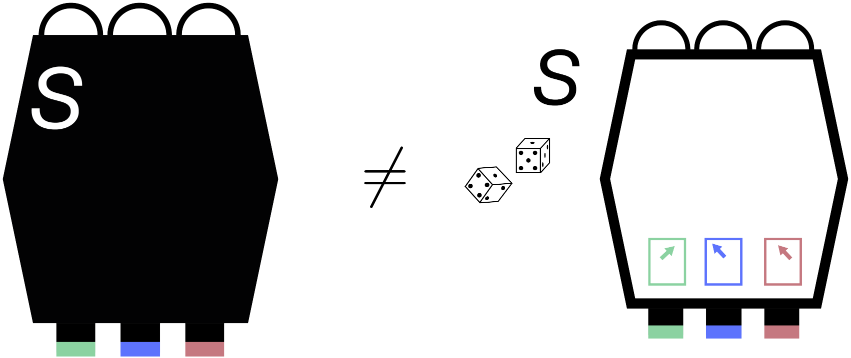
Non-contextual model



$\exists$ probability distribution d on $\mathbf{O}_{S,X_S} = \prod_{x \in X_S} O_{S,x}$ such that $d|_\sigma = e_\sigma$ for all $\sigma \in \Sigma_S$.

Contextuality

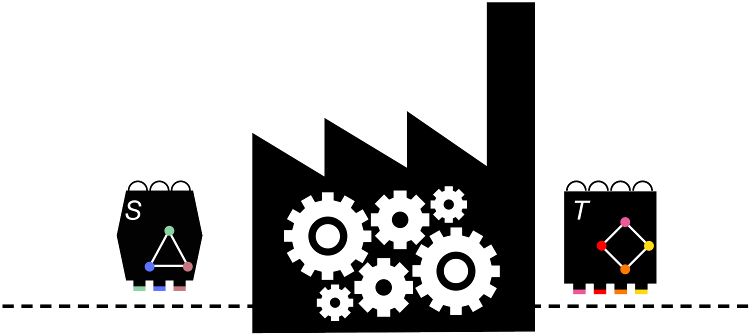
Contextual model



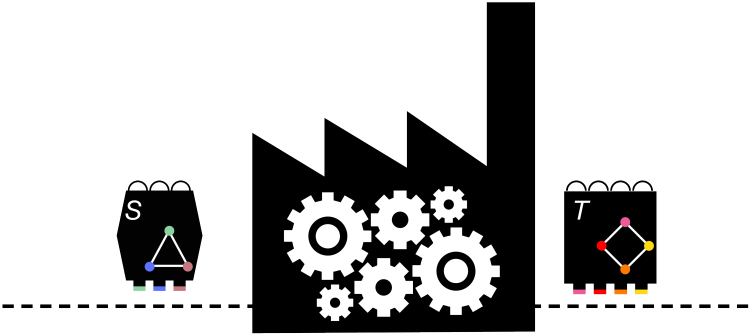
$\nexists$ probability distribution d on $\mathbf{O}_{S,X_S} = \prod_{x \in X_S} O_{S,x}$ such that $d|_\sigma = e_\sigma$ for all $\sigma \in \Sigma_S$.

Resource theory of contextuality

Resource theories

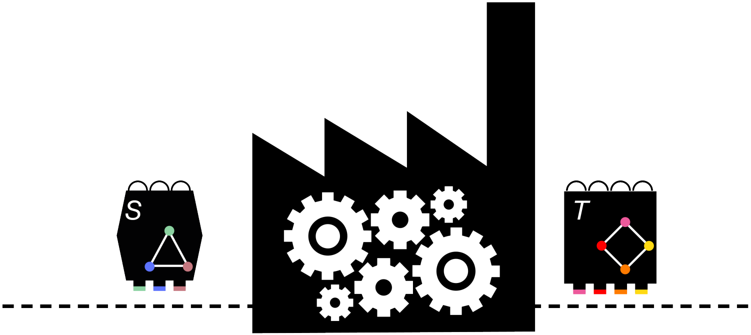


Resource theories



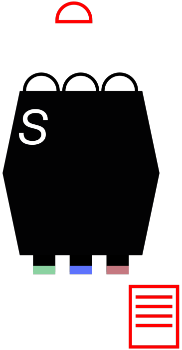
- Consider 'free' (i.e. classical) operations:

Resource theories



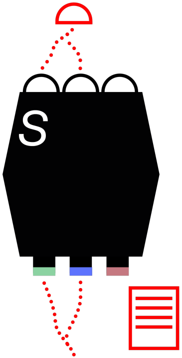
- Consider 'free' (i.e. classical) operations:
  (classical) procedures that use a box of type S to simulate a box of type T

Experiments and procedures



- An O-valued S-**experiment** is a protocol for an interaction with the box S producing a value in O:
  - which measurements to perform;
  - how to interpret their joint outcome into an outcome in O.

Experiments and procedures



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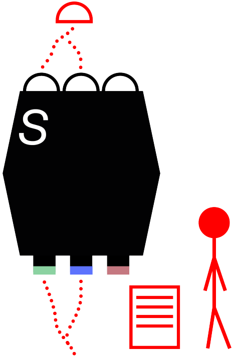

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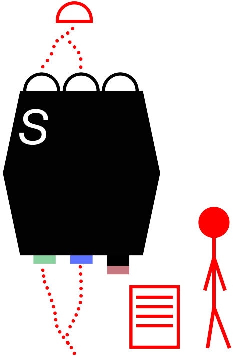

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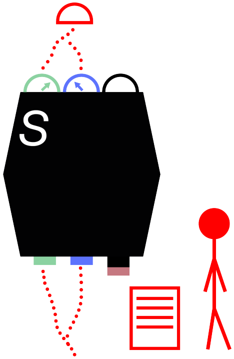

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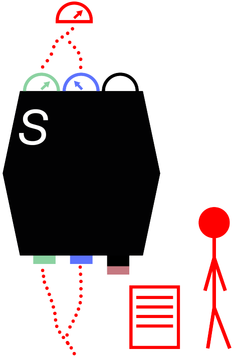

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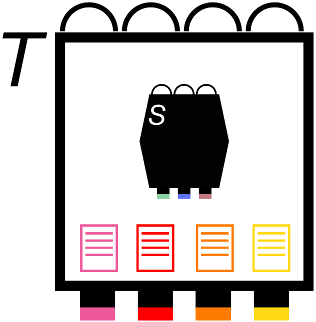
Experiments and procedures



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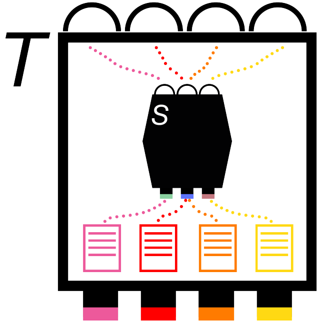
- A **deterministic procedure** $S \longrightarrow T$ specifies an S-experiment ($O_{T,x}$-valued) for each measurement x of T.

Experiments and procedures



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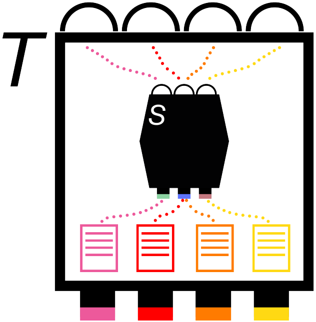
Experiments and procedures



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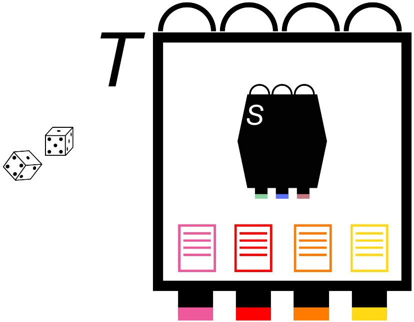
Experiments and procedures



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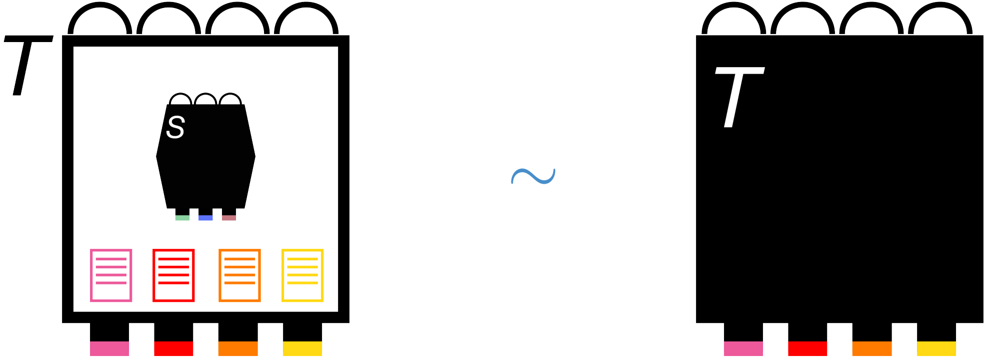
- A **classical procedure** is a probabilistic mixture of deterministic procedures.

Classical procedures and simulations



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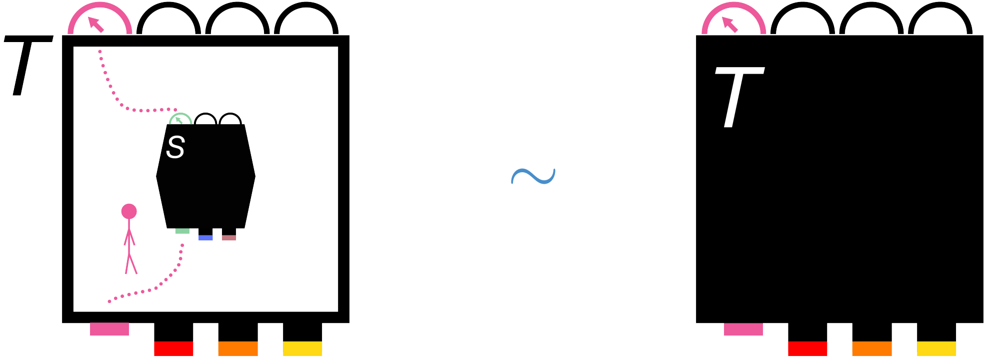
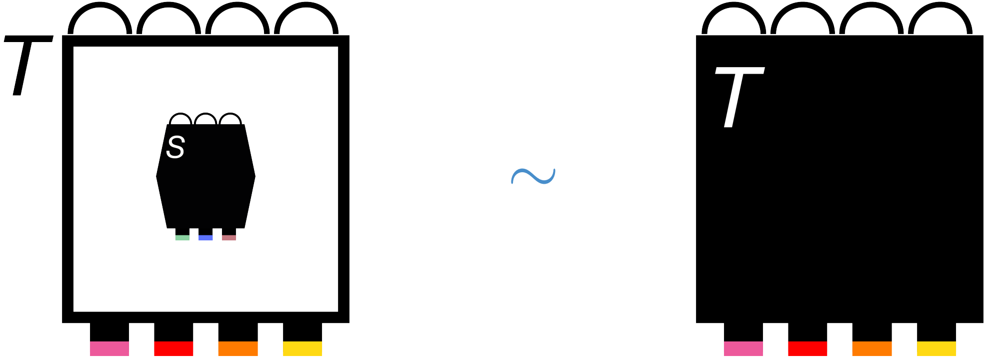


Classical procedures and simulations



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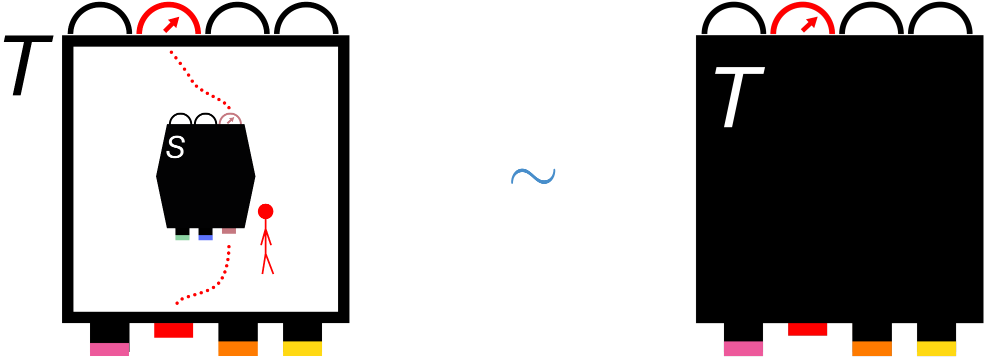

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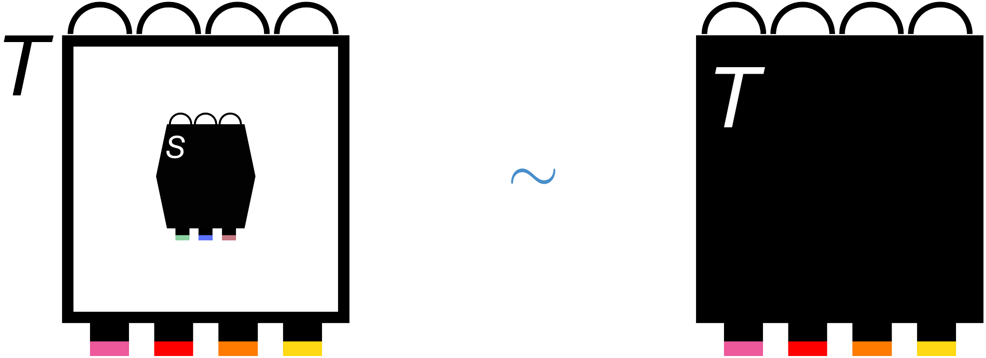

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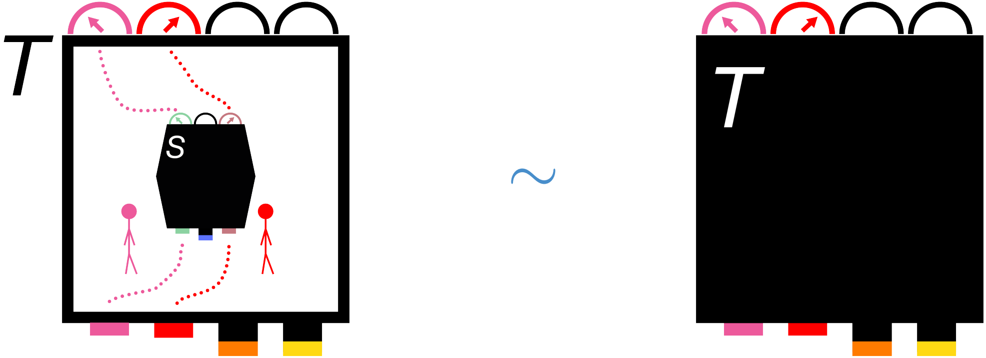

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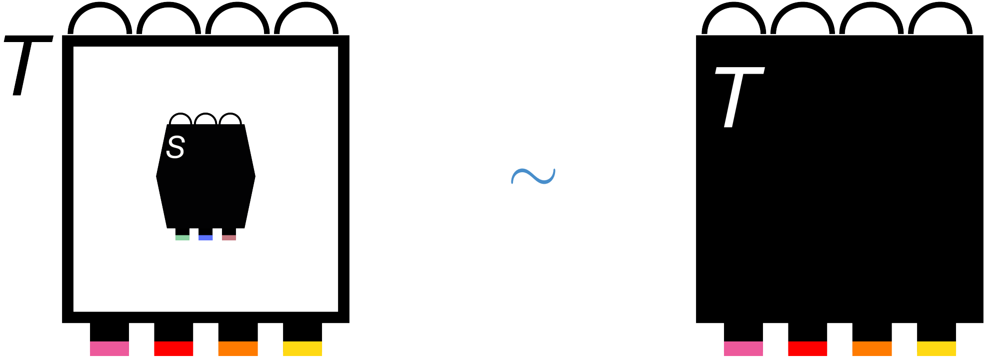

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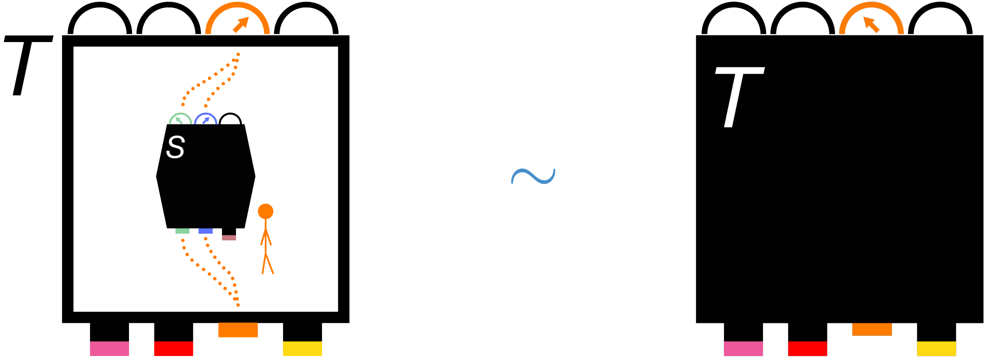

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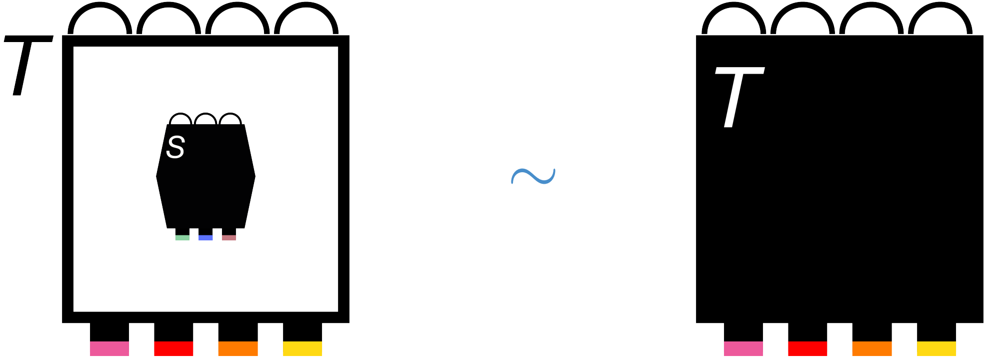

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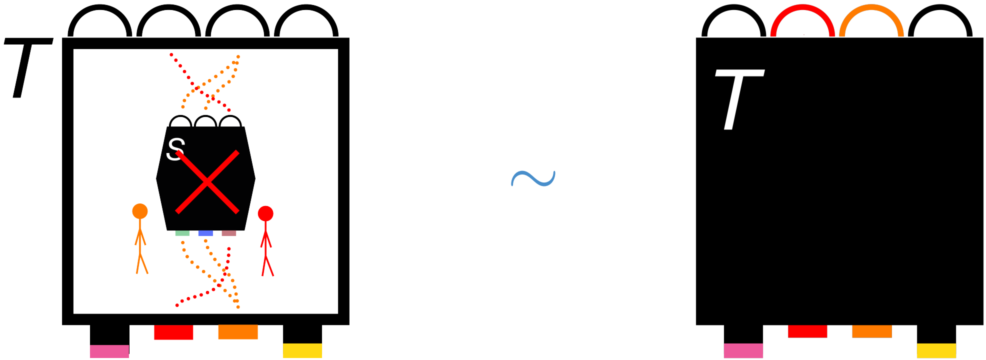

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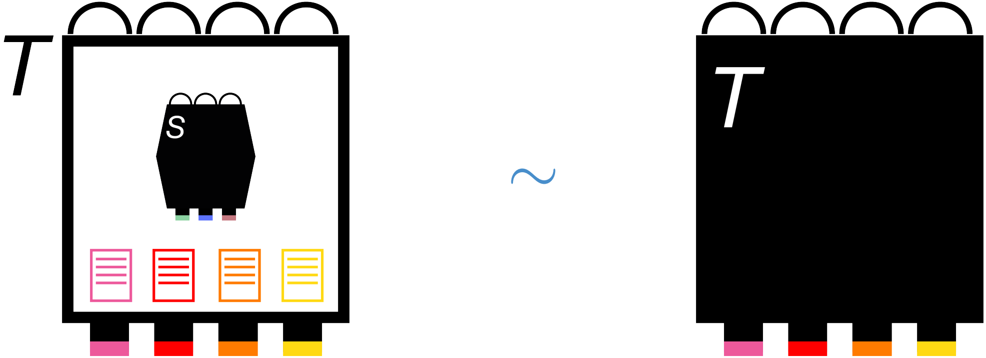
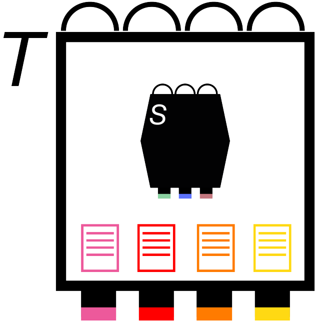


Classical procedures



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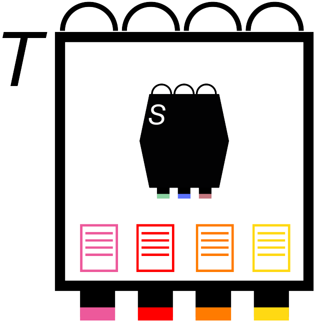

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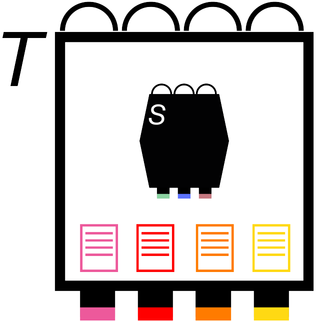

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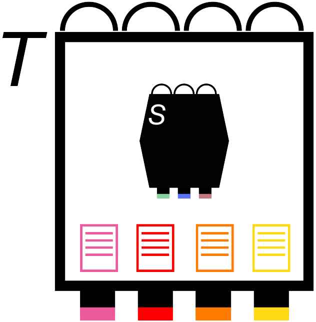

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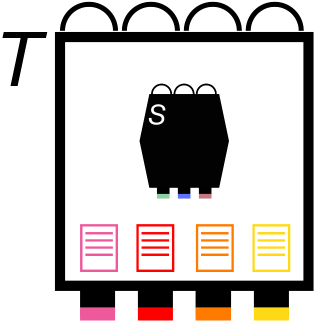

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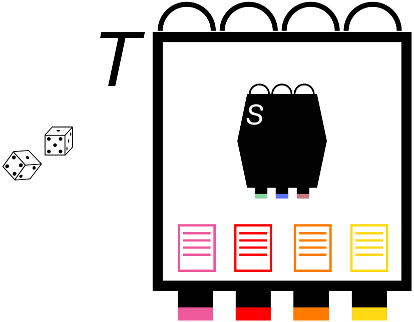

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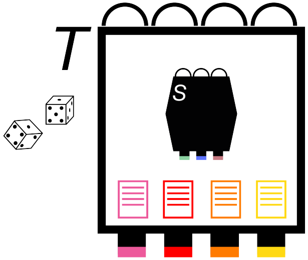
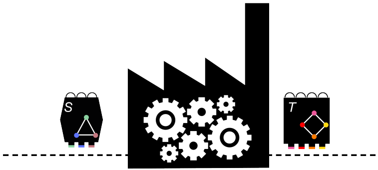
Probabilistic procedure $f : S \longrightarrow T$ is $f = \sum_i r_i f_i$ where $r_i \geq 0$, $\sum_i r_i = 1$, and $f_i : S \longrightarrow T$ deterministic procedures.

Classical simulations

- A classical procedure induces a (convex-preserving) map between empirical models:

$$f : S \longrightarrow T \qquad \leadsto \qquad \mathbf{Emp}(f) : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$$





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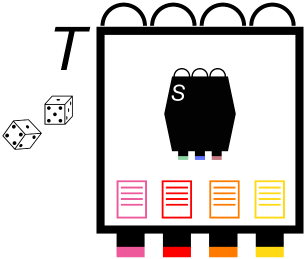
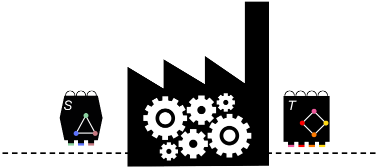




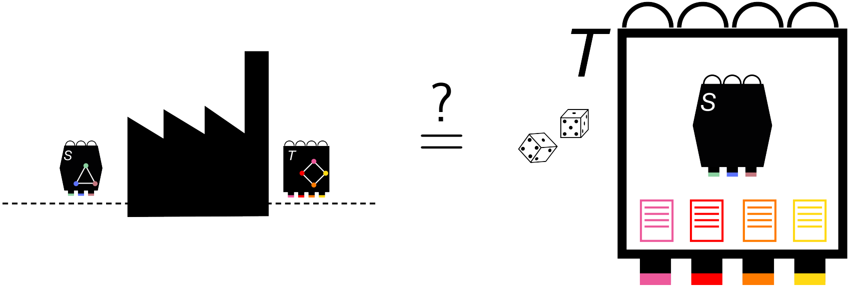
- Which black-box transformations arise in this fashion?

Characterising free transformations

Main question and sketch of the answer

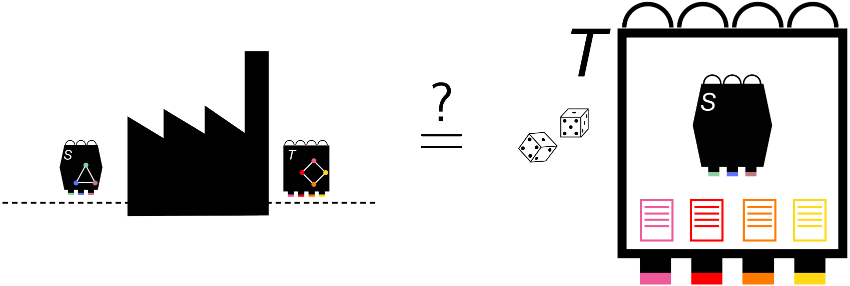
Main question

Given $F : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$, can it be realised by a classical procedure?
I.e. is there a procedure $f : S \longrightarrow T$ s.t. $F = \mathbf{Emp}(f)$?



Relativising contextuality

Given $F : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$, can it be realised by an experimental procedure? I.e. is there a procedure $f : S \longrightarrow T$ s.t. $F = \mathbf{Emp}(f)$?



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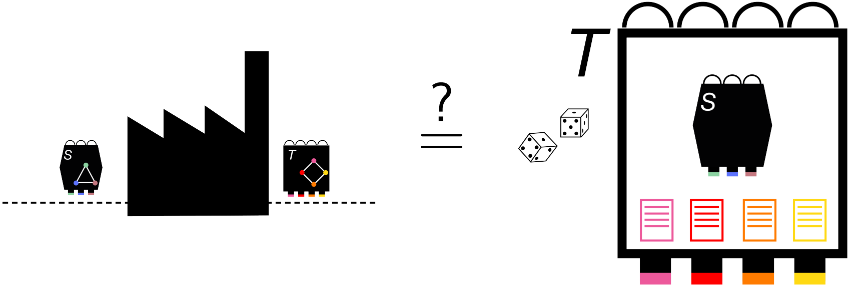

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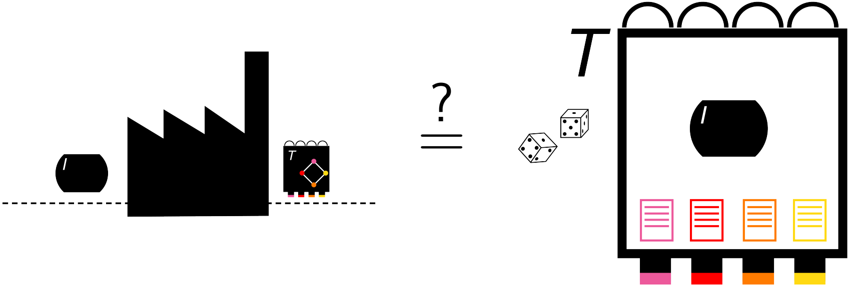

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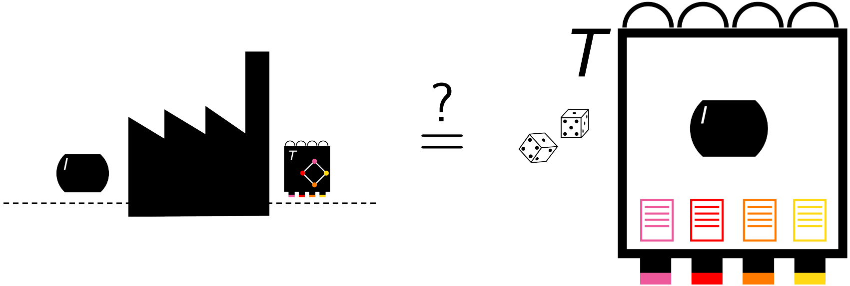

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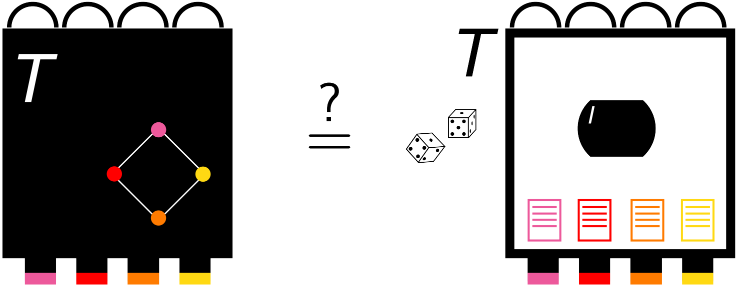
Given an empirical model $e \in \mathbf{Emp}(T)$, can it be realised by an classical procedure? I.e. is there a procedure $f : I \longrightarrow T$ s.t. $F = \mathbf{Emp}(f)$?



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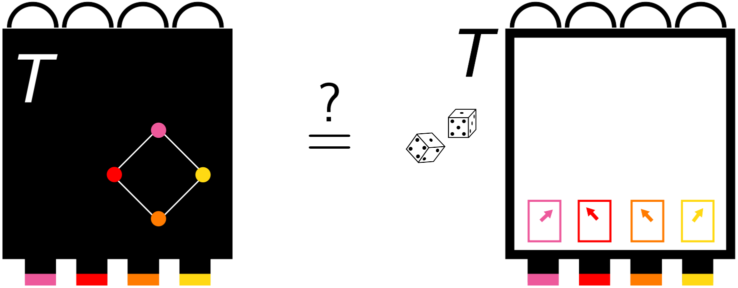

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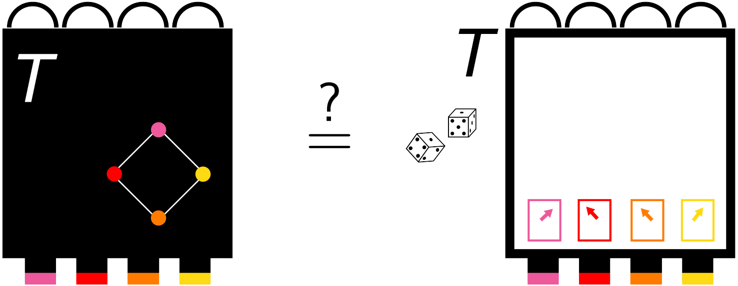
(Non-contextual models are those which can be simulated from nothing.)



From objects to morphisms . . .

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↑ is special case of

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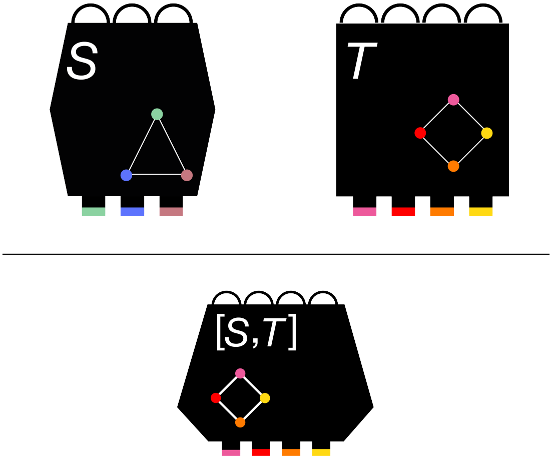
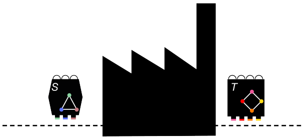
Answering the question by internalisation



From two scenarios S and T, we build a new scenario $[S, T]$.

Answering the question by internalisation



A convex preserving $F : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$

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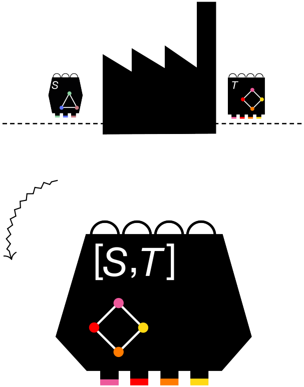

A convex preserving F : $\mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$ induces a canonical model $e_F : [S, T]$.

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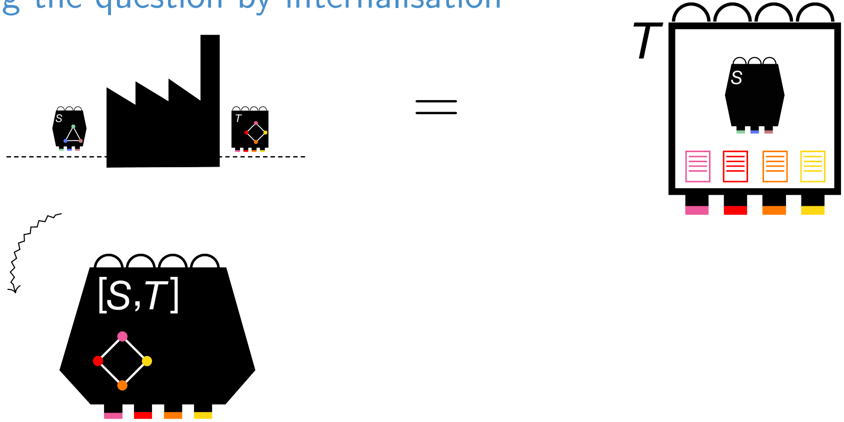

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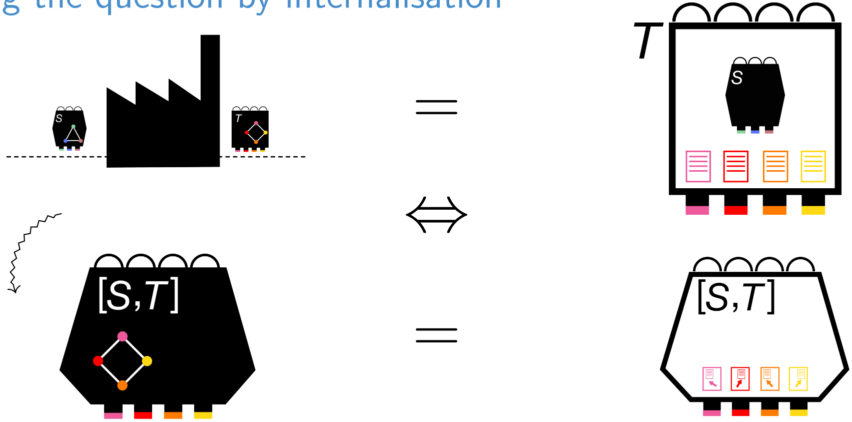

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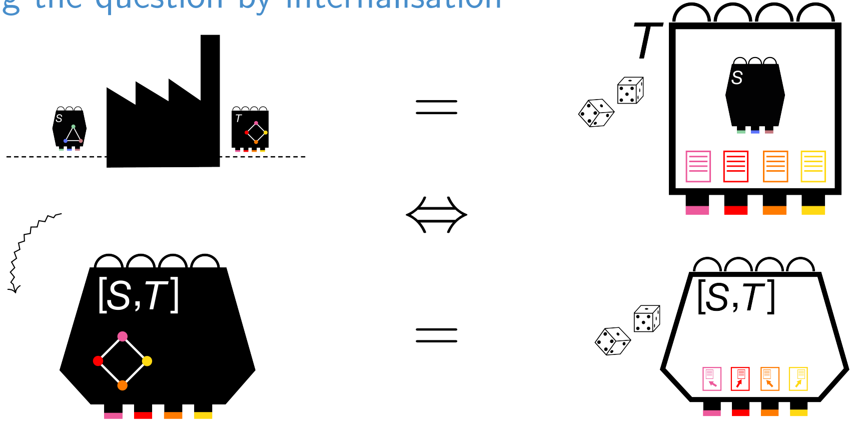

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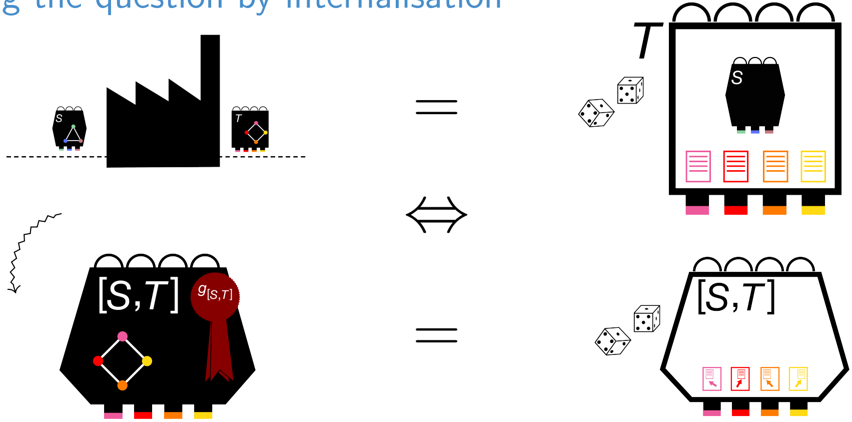


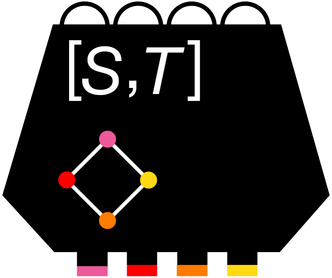
A convex preserving $F : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$ induces a canonical model $e_F : [S, T]$.

F is realised by a deterministic procedure iff e_F is deterministic and satisfies $g_{[S,T]}$.

F is realised by a classical procedure iff e_F is non-contextual and satisfies $g_{[S,T]}$.

Further details

The hom scenario [S,T]



- **Measurements** are those of T.

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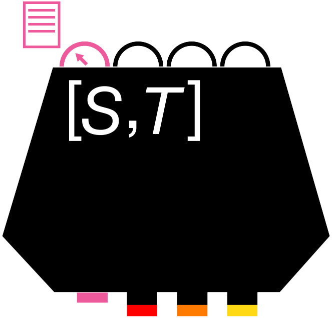

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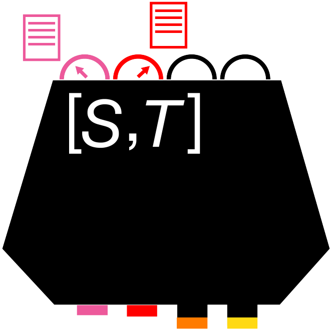

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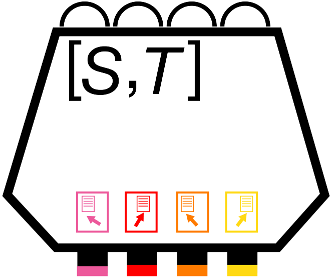

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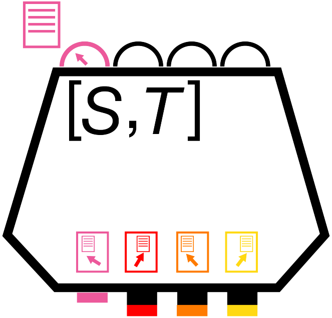

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Evaluation map

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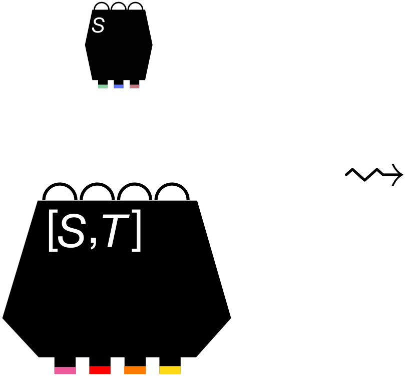

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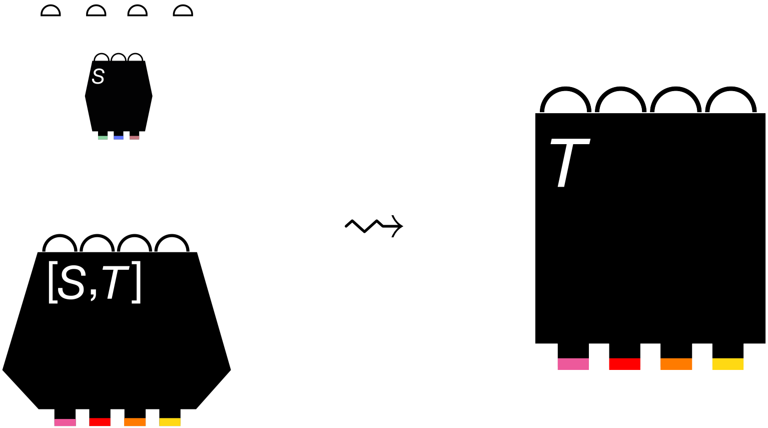

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Answering the question I

Facts:

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Therefore, a convex-preserving function $\mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$ is determined by its action on deterministic models, $\mathbf{Det}(S)$.

Answering the question II: for experiments

An S-experiment valued in $\{1, \ldots, n\}$ is a classical procedure $S \longrightarrow \mathbf{n}$.

- $\mathbf{n}$ is the scenario with a single measurement with outcomes in $\{1, \ldots, n\}$.
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Fact:

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Similarly, $\sum r_i f_i$ is induced by an experiment if each U_{f_i} is a compatible set of measurements.

Answering the question III: internalisation

A convex-preserving map $F : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$ is determined by its action on $\mathbf{Det}(S)$.

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Given a compatible set of measurements on T, we then get a mixture of deterministic functions from $\mathbf{Det}(S)$ to joint outcomes of these measurements.

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Lemma

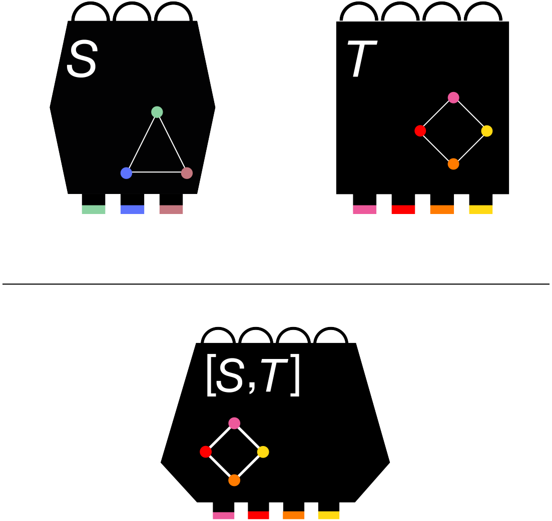
A convex-preserving function $F : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$ *induces a canonical no-signalling empirical model* $e_F : [S, T]$*.*

Main results

Theorem

F is induced by a classical procedure iff e_F is non-contextual and satisfies $g_{[S,T]}$.

Caveat: adding predicates



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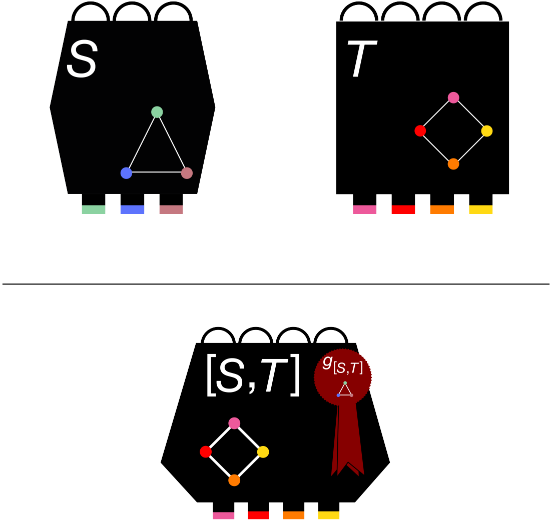

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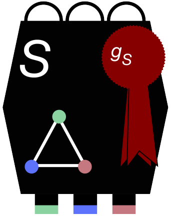
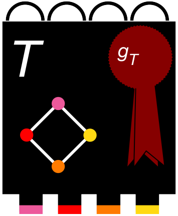


$g_S : S \longrightarrow \mathbf{2}$



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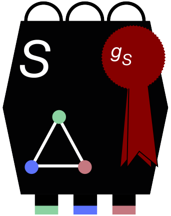

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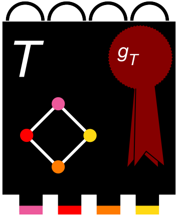

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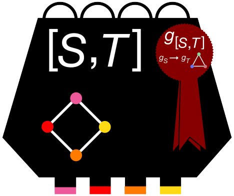

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F is induced by a classical procedure iff e_F is non-contextual and satisfies $g_{[S,T]}$.

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- A morphism $f : \langle S, g \rangle \longrightarrow \langle T, h \rangle$ is given by a procedure $f : S \longrightarrow T$ such that
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Theorem

$[-,-]$ *(appropriately modified) makes this category into a closed category.*

Closed structure

Getting closure

$$[S, T] \ \text{“}\otimes\text{”} \ S \longrightarrow T$$

Getting closure

$$
\begin{array}{c}
[S, T] \;\; \text{“}\otimes\text{”} \;\; S \longrightarrow T \\
\Big\downarrow {\scriptstyle S \cong [I, S]} \\
[S, T] \;\; \text{“}\otimes\text{”} \;\; [I, S] \longrightarrow [I, T]
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$\rightsquigarrow$ generalise

$$[S, T] \text{ ``}\otimes\text{''} [R, S] \longrightarrow [R, T]$$

Getting closure

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$$[S,T] \;\text{“}\otimes\text{”}\; [I,S] \longrightarrow [I,T]$$

$$\Big\downarrow{\scriptstyle \text{generalise}}$$

$$[S,T] \;\text{“}\otimes\text{”}\; [R,S] \longrightarrow [R,T]$$

$$\Big\downarrow{\scriptstyle \text{curry}}$$

$$\boxed{\mathsf{L}^R_{S,T} : [S,T] \longrightarrow [[R,S],[R,T]]}$$

Getting closure

Closed category

$$[-,-] : \mathsf{Scen}^{\mathsf{op}} \times \mathsf{Scen} \longrightarrow \mathsf{Scen}$$

- $i_S : S \xrightarrow{\cong} [I, S]$ natural in S
- $j_S : I \longrightarrow [S, S]$ extranatural in S (identity transformations)
- $\mathsf{L}^R_{S,T} : [S, T] \longrightarrow [[R, S], [R, T]]$ natural in S, T, extranatural in R (curried composition)
- $+$ reasonable coherence axioms

Outlook

Further questions

- External characterisation of adaptive procedures?
  Note that $[S, T]$ can be defined in the adaptive case, but there is no obvious way of building a canonical adaptive empirical model out of a convex-preserving function $\mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$.

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- Doing the same possibilistically?

- Does the set of all predicates on S generalise partial Boolean algebras to arbitrary measurement compatibility structures?

- Examining the closed structure?
  Note that it's not monoidal wrt. the usual monoidal structure, but seems closed wrt a 'directed' tensor product.

Questions...

?