Contextuality in logical form

Duality for transitive partial CABAs


Samson Abramsky


Rui Soares Barbosa

s.abramsky@ucl.ac.uk

rui.soaresbarbosa@inl.int

UCL

INL INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

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Overview

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- Tarski duality between **CABA** and **Set**
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  - In logic, between syntax and semantics

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- Tarski duality between **CABA** and **Set**
  - Simplest of dualities relating algebra and topology
  - In logic, between syntax and semantics
- partial Boolean algebras
  - Introduced by Kochen and Specker (1965)
  - algebraic-logic setting for contextual systems
  - original formulation of KS theorem

Motivation

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Boolean algebras	Stone spaces
finite Boolean algebras	finite sets
complete atomic Boolean algebras	sets

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- Given a space X,
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Here, I mean *commutativity* in a loose, informal sense.
For lattices, this would be *distributivity* (think: idempotents of a ring).

From classical to quantum

John von Neumann (1932), ‘*Mathematische Grundlagen der Quantenmechanik*’.

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Classical mechanics

- Described by **commutative** C^*-algebras or von Neumann algebras.
- By Gel'fand duality, these are algebras of continuous (or measurable) functions on topological spaces, the state spaces.

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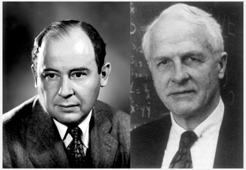
- Measurements are self-adjoint operators.
- Quantum properties or propositions are **projectors** (dichotomic measurements):

$$p : \mathcal{H} \longrightarrow \mathcal{H} \qquad \text{s.t.} \qquad p = p^\dagger = p^2$$

which correspond to closed subspaces of $\mathcal{H}$.



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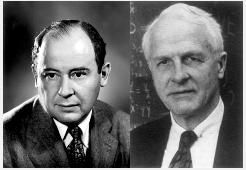

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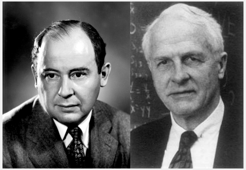

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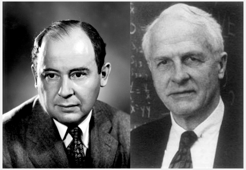
Quantum physics and logic



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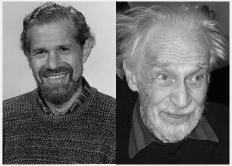
- Distributivity fails: $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$.
- Only commuting measurements can be performed together.
  So, what is the operational meaning of $p \wedge q$, when p and q **do not commute**?



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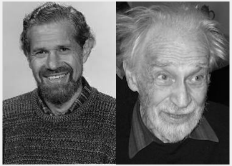

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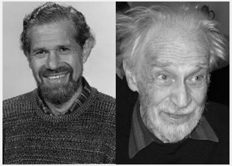
- Only admit physically meaningful operations.
- Represent incompatibility by **partiality**.



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Kochen (2015), '*A reconstruction of quantum mechanics*'.

- Kochen develops a large part of foundations of quantum theory in this framework.

Boolean algebras

Boolean algebra $\langle A, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- constants $0, 1 \in A$
- a unary operation $\neg : A \longrightarrow A$
- binary operations $\vee, \wedge : A^2 \longrightarrow A$

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satisfying the usual axioms: $\langle A, \vee, 0 \rangle$ and $\langle A, \wedge, 1 \rangle$ are commutative monoids,
$\vee$ and $\wedge$ distribute over each other,
$a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

E.g.: $\langle \mathcal{P}(X), \varnothing, X, \cup, \cap \rangle$, in particular $\mathbf{2} = \{0, 1\} \cong \mathcal{P}(\{\star\})$.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
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Morphisms of pBAs are maps preserving commeasurability, and the operations wherever defined. This gives the category **pBA**.

Contextuality, or the Kochen–Specker theorem

Kochen & Specker (1965).

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- Spectrum of a pBA cannot have *points*…

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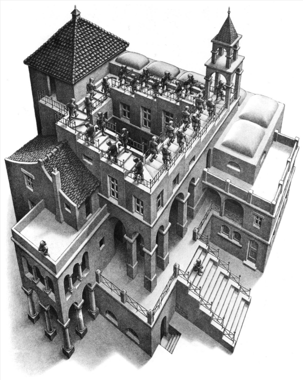

M. C. Escher, *Ascending and Descending*

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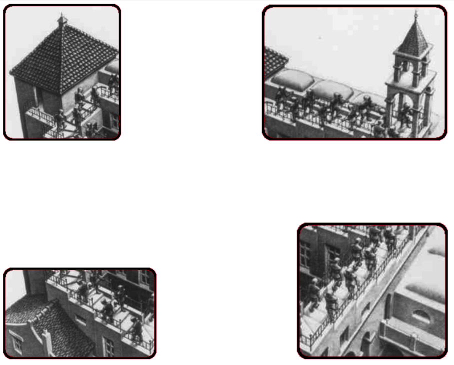

Local consistency

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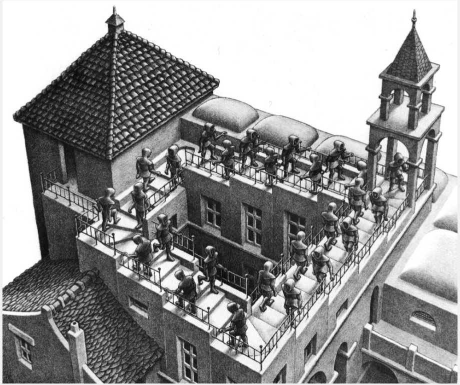

Local consistency *but* **Global inconsistency**

No-go theorems for noncommutative dualities

- Reyes (2012)
  - Any extension of Zariski spectrum to a functor $\mathbf{Rng}^{\mathrm{op}} \longrightarrow \mathbf{Top}$ trivialises on $\mathbb{M}_n(\mathbb{C})$ ($n \geq 3$).
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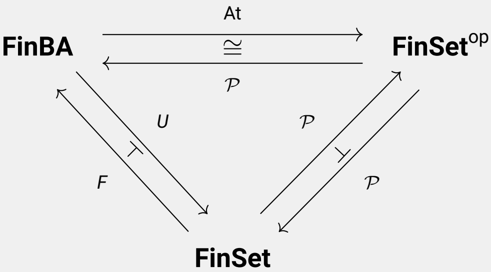
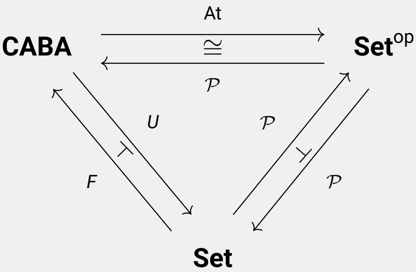
'What is proved by impossibility proofs is lack of imagination.' – John S. Bell

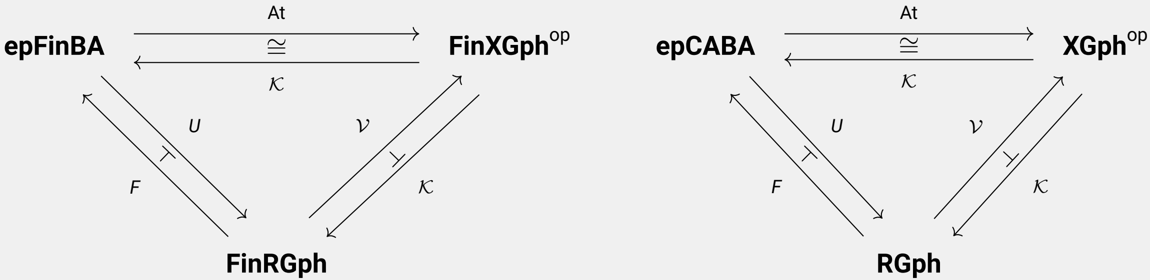
Results

Tarski duality





Partial Tarski duality



Recap: Tarski duality

Partial order

Let A be a Boolean algebra.

Definition
For $a, b \in A$, we write $a \leq b$ when one (hence all) of the following equivalent conditions hold:

- $a \wedge b = a$
- $a \vee b = b$
- $a \wedge \neg b = 0$
- $\neg a \vee b = 1$

$\leq$ is a partial order.

It determines A as a Boolean algebra: e.g. $\vee$ (resp. $\wedge$) is supremum (resp. infimum) wrt $\leq$.

CABAs

Definition (Complete Boolean algebra)

A Boolean algebra A is said to be **complete** if any subset of elements $S \subseteq A$ has a supremum $\bigvee S$ in A (and consequently an infimum $\bigwedge S$, too). It thus has additional operations

$$\bigwedge, \bigvee : \mathcal{P}(A) \longrightarrow A \,.$$

Definition (Atomic Boolean algebra)

An **atom** of a Boolean algebra is a minimal non-zero element, i.e. an element $x \neq 0$ such that $a \leq x$ implies $a = 0$ or $a = x$.

A Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

A **CABA** is a complete, atomic Boolean algebra.

Example

Any finite Boolean algebra is trivially a CABA.

The powerset $\mathcal{P}(X)$ of an arbitrary set X is a CABA.

- completeness: closed under arbitrary unions
- atoms: singletons $\{x\}$ for $x \in X$

This is in fact the 'only' (up to iso) example.

Proposition

In a CABA, every element is the join of the atoms below it:

$$a = \bigvee U_a \quad \textit{where } U_a := \{x \in A \mid x \textit{ is an atom and } x \leq a\}\,.$$

Proof.

Suppose $a \not\leq \bigvee U_a$, i.e. $a \wedge \neg \bigvee U_a \neq 0$. Atomicity implies there's an atom $x \leq a \wedge \neg \bigvee U_a$. On the one hand, $x \leq \neg \bigvee U_a$. On the other, $x \leq a$, i.e. $x \in U_a$, hence $x \leq \bigvee U_a$. Hence $x = 0$. ↯ □

Tarski duality

$$\mathbf{CABA} \underset{\mathrm{At}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathbf{Set}^{\mathrm{op}} \qquad \mathbf{CABA} \cong \mathbf{Set}^{\mathrm{op}}$$

Tarski duality

$$\mathbf{CABA} \underset{\mathrm{At}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathbf{Set}^{\mathrm{op}} \qquad \cong$$

$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{CABA}$ is the contravariant powerset functor:

- on objects: a set X is mapped to its powerset $\mathcal{P}X$ (a CABA).
- on morphisms: a function $f : X \longrightarrow Y$ yields a complete Boolean algebra homomorphism

$$\begin{aligned} \mathcal{P}(f) : \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X) \\ (T \subseteq Y) &\longmapsto f^{-1}(T) = \{x \in X \mid f(x) \in T\} \end{aligned}$$

Tarski duality

$$\mathbf{CABA} \underset{\mathrm{At}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathbf{Set}^{\mathrm{op}} \qquad (\cong)$$

$\mathrm{At} : \mathbf{CABA}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is defined as follows:

- on objects: a CABA A is mapped to its set of atoms.
- on morphisms: a complete Boolean homomorphism $h : A \longrightarrow B$ yields a function

  $$\mathrm{At}(h) : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$$

  mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

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Tarski duality

Lemma

Let $h : A \longrightarrow B$ *in* **CABA**. *For all* $y \in \mathrm{At}(A)$, *there is a unique* $x \in \mathrm{At}(A)$ *with* $y \leq h(x)$.

Proof.

Facts about atoms in any BA:

- If $x \neq x'$ are atoms, then $x \wedge_A x' = 0$.
- If x is an atom and $x \leq \bigvee S$, there is $a \in S$ with $x \leq a$.

Existence

A complete atomic implies $1_A = \bigvee \mathrm{At}(A)$. Hence,

$$1_B = h(1_A) = h(\bigvee \mathrm{At}(A)) = \bigvee \{h(x) \mid x \in \mathrm{At}(A)\}$$

Since $y \leq 1_B$, we conclude $y \leq h(x)$ for some $x \in \mathrm{At}(A)$.

Uniqueness

If $y \leq h(x)$ and $y \leq h(x')$, then $y \leq h(x) \wedge_B h(x') = h(x \wedge x')$, hence $x = x'$. □

Tarski duality

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- Given a CABA A, the isomorphism $A \cong \mathcal{P}(\mathrm{At}(A))$ maps $a \in A$ to the set of elements

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- Given a set X, the bijection $X \cong \mathrm{At}(\mathcal{P}(X))$ maps $x \in X$ to the singleton $\{x\}$, which is an atom of $\mathcal{P}(X)$.

  A possible world is identified with its characteristic property (which fully determines it).

Duality for partial CABAs

Logical exclusivity principle

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For $a, b \in A$, we write $a \leq b$ to mean $a \odot b$ and $a \wedge b = a$.

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Definition

A is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are logically exclusive are also commeasurable, i.e. if $\perp \subseteq \odot$.

Logical exclusivity principle

Note that $\leq$ is always reflexive and antisymmetric.

Definition

A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, $a \leq b$ and $b \leq c$, then $a \leq c$, i.e. $\leq$ is (globally) a partial order on A.

Proposition

A partial Boolean algebra satisfies LEP if and only if it is transitive.

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We restrict atention to partial Boolean algebras satisfying LEP in this talk.

Theorem

The category **epBA** *of partial Boolean algebras satisfying LEP is a reflective subcategory of* **pBA***, i.e. the inclusion functor* $I : \mathbf{epBA} \longrightarrow \mathbf{pBA}$ *has a left adjoint* $X : \mathbf{pBA} \longrightarrow \mathbf{epBA}$.

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \bigodot \longrightarrow A$$

satisfying the following property: any set $S \in \bigodot$ is contained in a set $T \in \bigodot$ which forms a complete Boolean algebra under the restriction of the operations.

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A **partial CABA** is a complete, atomic partial Boolean algebra.

Graph

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Given a vertex $x \in X$ and sets of vertices $S, T \subset X$, we write:

- $x \# S$ when for all $y \in S$, $x \# y$;
- $S \# T$ when for all $x \in S$ and $y \in T$, $x \# y$;
- $x^{\#} := \{y \in X \mid y\#x\}$ for the neighbourhood of the vertex x;
- $S^{\#} := \bigcap x \in Sx^{\#} = \{y \in X \mid y \# S\}$ for the common neighbourhood of the set S.

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A **clique** is a set of pairwise-adjacent vertices, i.e. a set $K \subset X$ with $x \# K \setminus \{x\}$ for all $x \in K$.

A graph $(X, \#)$ has **finite clique cardinal** if all cliques are finite sets.

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

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Recall that in a CABA, any element is uniquely written as a join of atoms, viz. $a = \bigvee U_a$ with

$$U_a := \{x \in \mathrm{At}(A) \mid x \leq a\}$$

In a pBA, U_a may not be pairwise commeasurable, hence their join need not even be defined.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

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Proof.

Let $a \in A$ and K be a clique of $\mathrm{At}(A)$ maximal in U_a.

Being a clique in $\mathrm{At}(A)$, $K \in \bigodot$ and thus $\bigvee K$ is defined.

Since $K \subset U_a$, all $k \in K$ satisfy $k \leq a$ and in particular $k \odot a$. Hence, $K \cup \{a\} \in \bigodot$, implying that it is contained in a complete Boolean subalgebra. Consequently, $\bigvee K \leq a$.

Now, suppose $a \not\leq \bigvee K$, i.e. $a \wedge \neg \bigvee K \neq 0$. Then atomicity implies there is an atom $x \leq a \wedge \neg \bigvee K$. By transitivity, $x \leq a$ and $x \leq \neg k$ (hence $x \perp k$) for all $k \in K$. This makes $K \cup \{x\}$ a clique of atoms contained in U_a, contradicting maximality of K. □

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Proposition

Let K and L be cliques in $\mathrm{At}(A)$. *Then* $\bigvee K \leq \bigvee L$ *iff* $L^{\#} \subseteq K^{\#}$ *iff* $K \subseteq L^{\#\#}$.

Corollary

$\bigvee K = \bigvee L$ *iff* $K^{\#} = L^{\#}$.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \ :\Leftrightarrow \ K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

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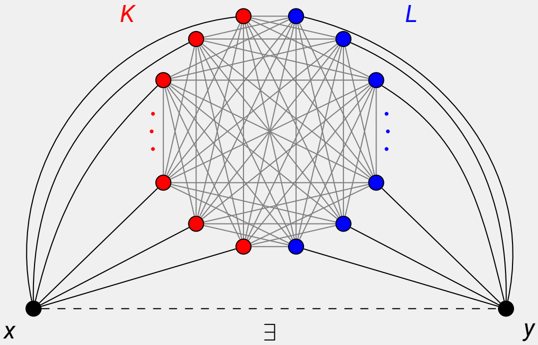
Which conditions on a graph $(X, \#)$ allow for such reconstruction?

Complete exclusivity graphs

Definition

A **complete exclusivity graph** is a graph $(X, \#)$ such that for K, L cliques and $x, y \in X$:

1. If $K \sqcup L$ is a maximal clique, then $K^{\#} \# L^{\#}$, i.e. $x \# K$ and $y \# L$ implies $x \# y$.
2. $x^{\#} \subseteq y^{\#}$ implies $x = y$.



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A helpful intuition is to see these as generalising sets with a $\neq$ relation (the complete graph).

- A graph is symmetric and irreflexive.
- To be an inequivalence relation, we need cotransitivity: $x \# z$ implies $x \# y$ or $x \# z$.

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- A graph is symmetric and irreflexive.
- To be an inequivalence relation, we need cotransitivity: $x \# z$ implies $x \# y$ or $x \# z$.
- Condition 1. is a weaker version of cotransitivity.
- Condition 2. eliminates redundant elements: cotransitive + 2. implies $\neq$.

Graph of atoms is complete exclusivity graph

Proposition

Let A be a partial Boolean algebra. Then $\mathrm{At}(A)$ *is a complete exclusivity graph.*

Proof.

Let $K, L \subset X$ such that $K \sqcup L$ is a maximal clique, and let x, y be atoms of A.
$c := \bigvee K = \neg \bigvee L$.
$x \# K$ means $x \leq \neg \bigvee K = \neg c$ and $x \# L$ means $y \leq \neg \bigvee L = c$.
By transitivity, we conclude that $x \odot y$, □

Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \; R \; y$, $x' \; R \; y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

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3. trivialises.

Given $h : A \longrightarrow B$ define $y \; R \; x$ iff $y \leq h(x)$.

Morphisms of CE graphs and pCABA homomorphisms

Proposition

Let A and B be transitive partial CABAs. Given $h : A \longrightarrow B$ a partial complete Boolean algebra homomorphism, the relation $R_h : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$ given by

$$xR_hy \qquad \textit{iff} \qquad x \leq h(y)$$

is a morphism of complete exclusivity graphs. Moreover, the assignment $h \mapsto R_h$ is functorial.

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Proposition

For any A and B be transitive partial CABAs, $\mathbf{epCABA}(A, B) \cong \mathbf{XGph}(\mathrm{At}(B), \mathrm{At}(A))$.

Global points

Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

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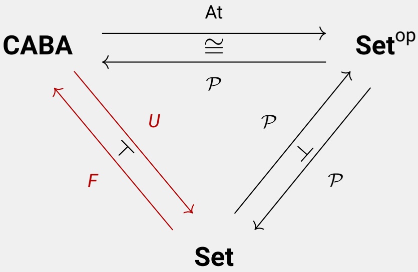
i.e. a subset of atoms of A satisfying:

1. it is an independent (or stable) set
2. it is a maximal clique transversal, i.e. it has a vertex in each maximal clique

Free-forgetful adjunction for CABAs



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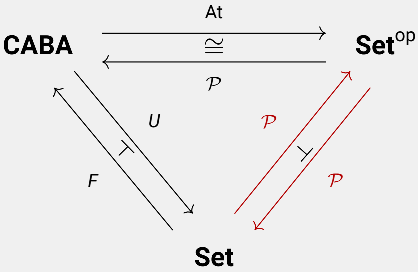
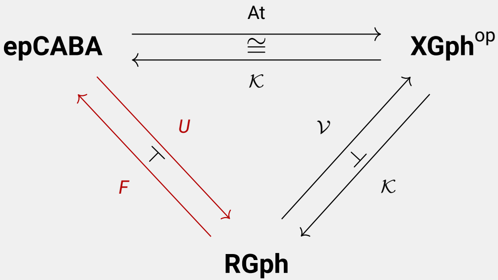


- Under the duality, it corresponds to the contravariant powerset self-adjunction.
- It gives the construction of the free CABA as a double powerset.

Free-forgetful adjunction for partial CABAs



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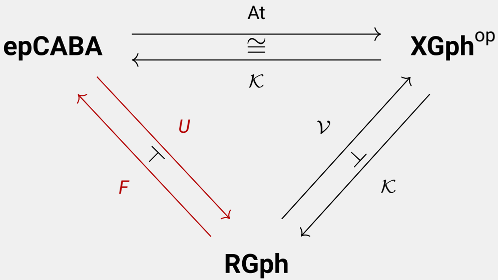
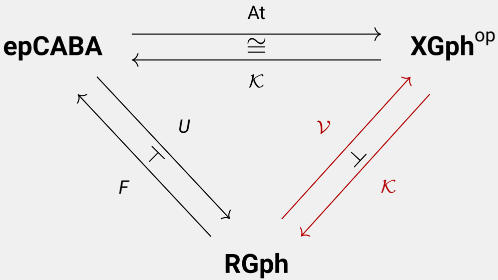


- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$

Free-forgetful adjunction for partial CABAs



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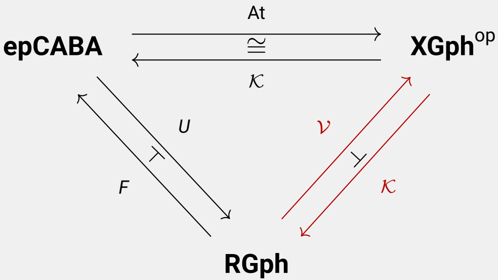

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$$\langle C, \gamma \rangle \# \langle D, \delta \rangle \qquad \text{iff} \qquad \exists x \in C \cap D.\ \ \gamma(x) \neq \delta(x)\ .$$

Outlook

Reconstruction via double-neighbourhood-closed sets

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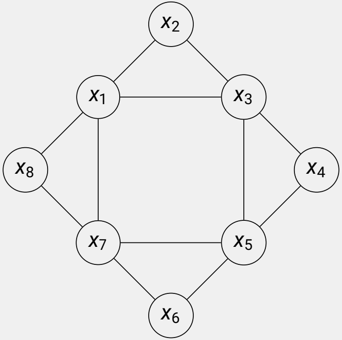

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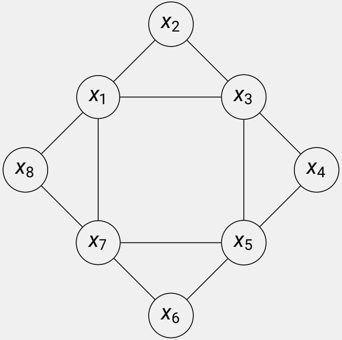

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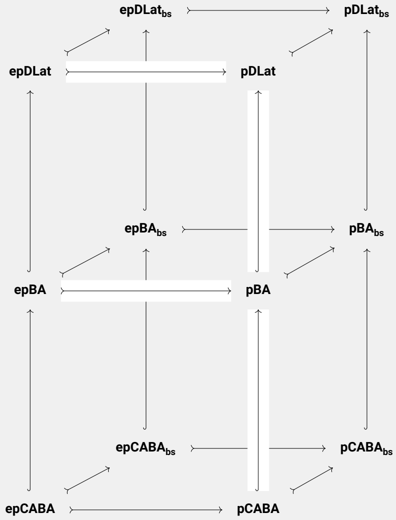
Can we characterise which $^{\#\#}$-closed sets arise from cliques?

The spatial landscape of partial Boolean algebra



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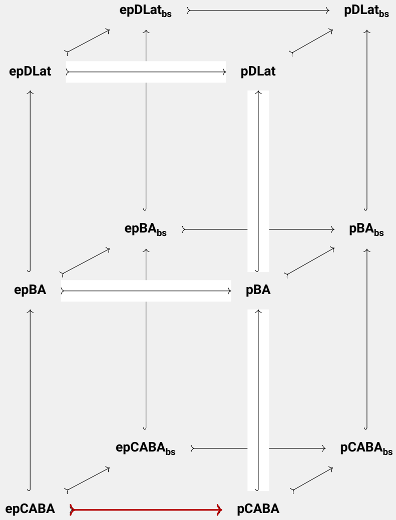

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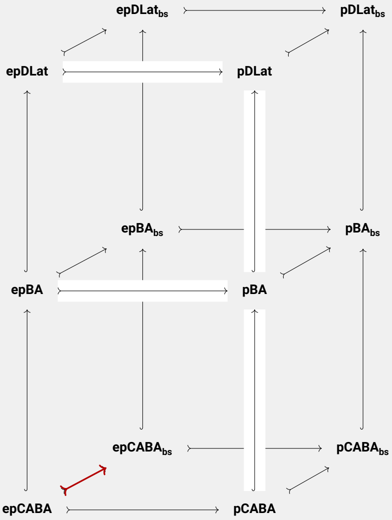

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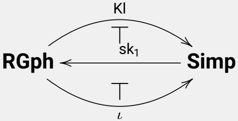

$\rightsquigarrow$Czelakowski's *pBAs in a broader sense*

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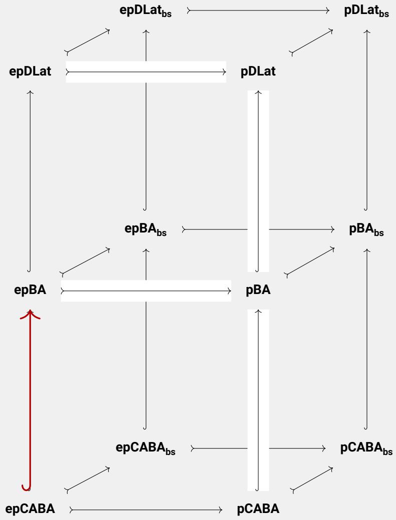

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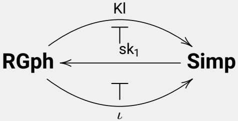

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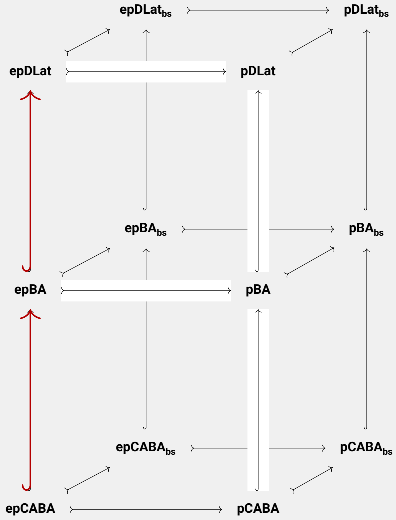

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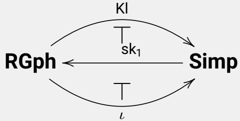

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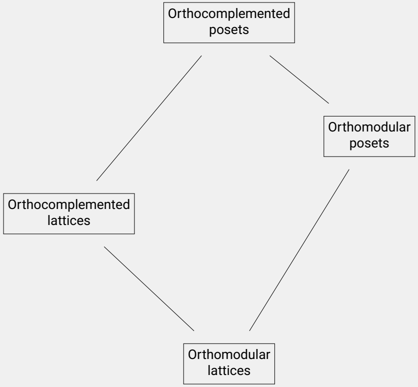
- Dropping completeness and atomicity (e.g. P(A) for vN algebra A with factor not of type I)

$\rightsquigarrow$ analogues of Stone, Priestley, ...
Stone's motto: *'always topologise'* – but how?

The wider spatial landscape of ‘quantum’ logics



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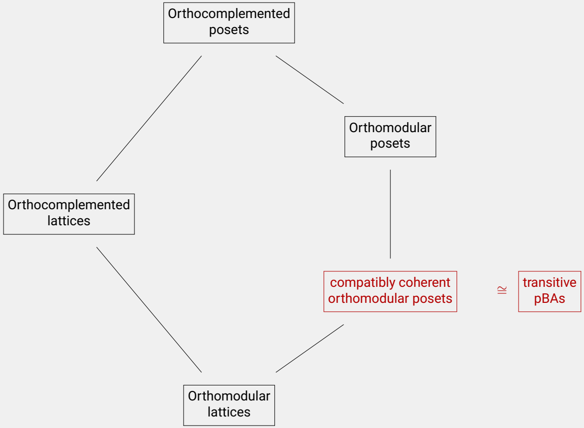

(Gudder, 1972)

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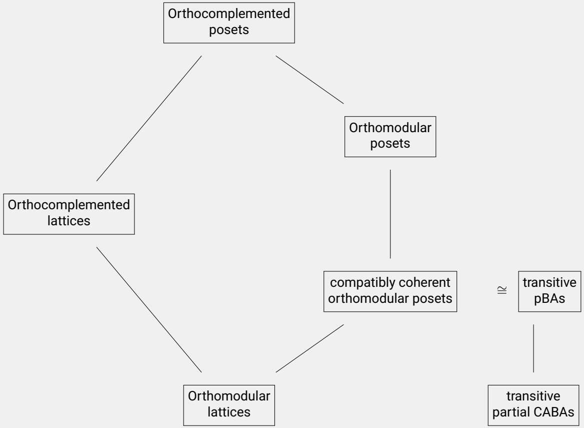

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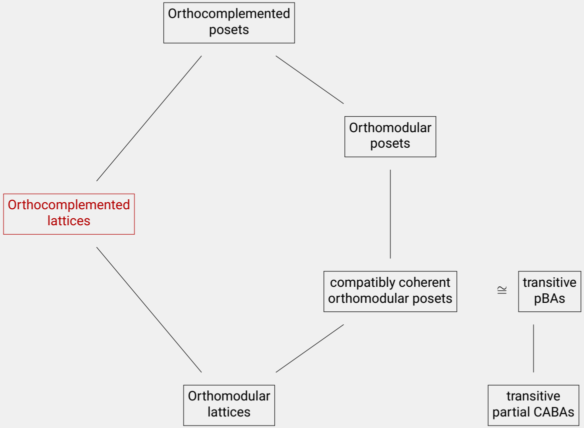

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OLs ⇜⇝ Minimal quantum logic
(Dishkant, Goldblatt, Dalla Chiara, 1970s)

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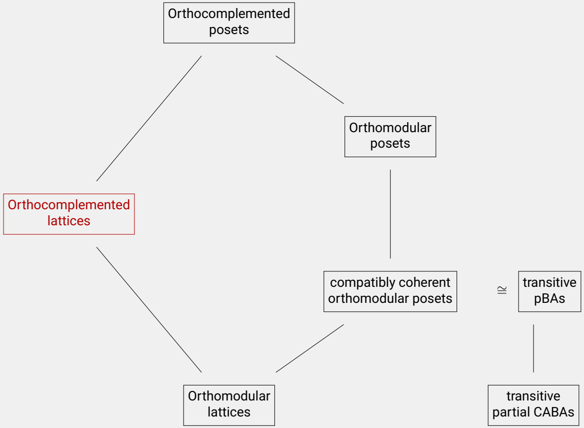

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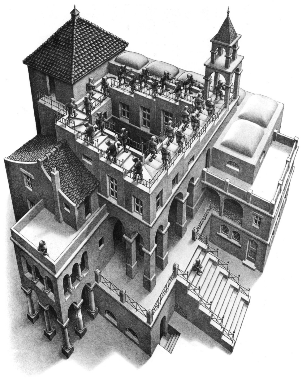
Stone representation for OLs
(Goldblatt, 1975)

- related to our construction
- all graphs, all nhood-regular sets
- nothing on morphisms

Towards noncommutative dualities?

- Can one find a more encompassing duality theory for ‘noncommutative’ or ‘quantum’ structures by viewing them through multiple partial classical snapshots?



Questions...

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