Partial algebraic structures and the logic of quantum computation

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INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

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CMAT – Centre of Mathematics, Universidade do Minho
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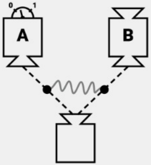
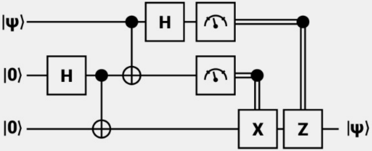
Introduction

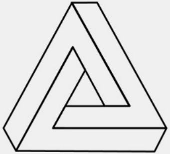
Motivation



Quantum Foundations



Quantum Computer Science



Mathematics of Quantum Structures

- What **phenomena distinguish** quantum mechanics from classical physical theories?

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- What is the **informatic advantage** afforded by quantum resources?
- What **structures** are useful to reason about quantum systems?
    - capturing the essence of their non-classicality
    - in a compositional fashion

From quantum foundations to quantum technologies

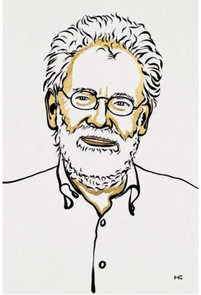
The Nobel Prize in Physics 2022

Ill. Niklas Elmehed © Nobel Prize Outreach

Alain Aspect

Ill. Niklas Elmehed © Nobel Prize Outreach

John F. Clauser



Ill. Niklas Elmehed © Nobel Prize Outreach

Anton Zeilinger

‘for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science’

Contextuality and quantum advantage

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It's been established as a useful resource conferring **advantage** in quantum computation:

- Measurement-based quantum computation (MBQC)

  '*Contextuality in measurement-based quantum computation*'
  Raussendorf, Physical Review A, 2013.

  '*Contextual fraction as a measure of contextuality*'
  Abramsky, B, Mansfield, Physical Review Letters, 2017.

- Magic state distillation

  '*Contextuality supplies the 'magic' for quantum computation*'
  Howard, Wallman, Veitch, Emerson, Nature, 2014.

- Shallow circuits

  '*Quantum advantage with shallow circuits*'
  Bravyi, Gossett, Koenig, Science, 2018.

  '*A generalised construction of quantum advantage with shallow circuits*'
  Aasnæss, DPhil thesis, 2022.

The essence of contextuality

- Not all properties may be observed simultaneously.
- Sets of jointly observable properties provide **partial, classical snapshots**.

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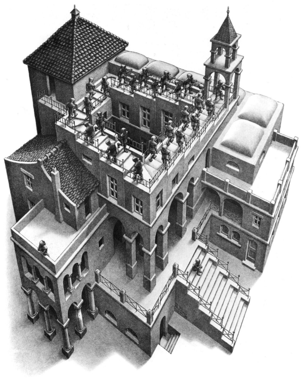

M. C. Escher, *Ascending and Descending*

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Local consistency

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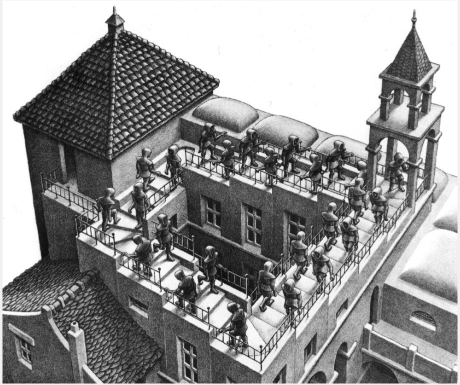


Local consistency *but* **Global inconsistency**

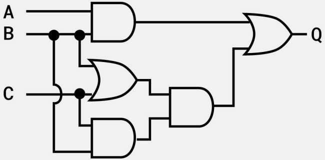
Algebra, Logic, Computation

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

Algebra

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Logic



Computation

Boolean algebra

Boolean algebra

- Algebraic structure satisfying certain equational axioms
- Partial order with certain properties
- Semantics for classical propositional calculus
- Foundation of digital circuits

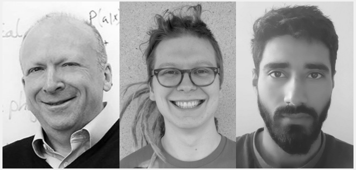
This talk

Recent work with Samson Abramsky on algebraic-logical view of contextuality, revisiting Kochen & Specker's partial Boolean algebras.

'*The logic of contextuality*'
Abramsky & B, CSL 2021.

'*Contextuality in logical form: Duality for transitive partial CABAs*'
Abramsky & B, TACL 2022, QPL 2023.

Joint work in progress with
Samson Abramsky, Martti Karvonen, Raman Choudhary, . . .



The logic of quantum theory

From states to properties

I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all, Hilbert space (as far as quantum mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of 'conserving the validity of all formal rules' [...]. Now we begin to believe that it is not the *vectors* which matter, but the lattice of all linear (closed) subspaces. Because: 1) The vectors ought to represent the physical *states*, but they do it redundantly, up to a complex factor, only 2) and besides, the states are merely a derived notion, the primitive (phenomenologically given) notion being the qualities which correspond to the *linear closed subspaces* [von Neumann (1935) as quoted in Birkhoff (1966)]

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John von Neumann (1932), '*Mathematische Grundlagen der Quantenmechanik*'.

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Classical mechanics

- Described by **commutative** C^*-algebras or von Neumann algebras.
- By Gel'fand duality, these are algebras of continuous (or measurable) functions on topological spaces, the state spaces.

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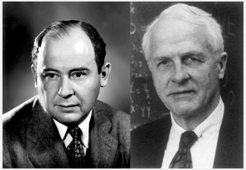
- Measurements are self-adjoint operators.
- Quantum properties or propositions are **projectors** (dichotomic measurements):

$$p : \mathcal{H} \longrightarrow \mathcal{H} \qquad \text{s.t.} \qquad p = p^\dagger = p^2$$

which correspond to closed subspaces of $\mathcal{H}$.

Quantum physics and logic



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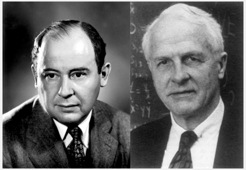

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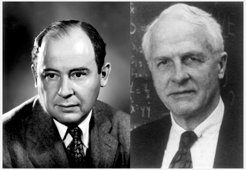

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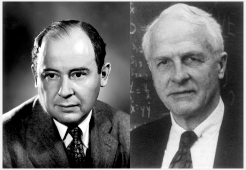

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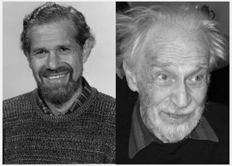
- Distributivity fails: $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$.
- Taking the *phenomenological* requirement seriously:
  in QM, only **commuting** measurements can be performed together.

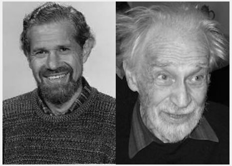
  So, what is the operational meaning of $p \wedge q$, when p and q **do not commute**?

Quantum physics and logic



An alternative approach

Kochen & Specker (1965), '*The problem of hidden variables in quantum mechanics*'.



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- Only admit physically meaningful operations,
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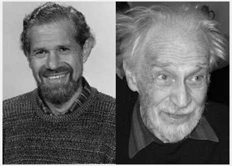

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Kochen (2015), ‘*A reconstruction of quantum mechanics*’.

- Kochen develops a large part of foundations of quantum theory in this framework.

Classical snapshots

- If A and B commute, then their product AB commutes with both.

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- When $A, B, C = AB$ are jointly measured on **any** quantum state, the observed outcomes a, b, c satisfy $c = ab$.

- More generally, for $A_1, \ldots, A_n$ pairwise commuting and any Borel $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, then $f(A_1, \ldots, A_n)$ commutes with all A_i and eigenvalues satisty the same functional relation.

Partial Boolean algebras

Boolean algebras

Boolean algebra $\langle A, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
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satisfying the usual axioms: $\langle A, \vee, 0 \rangle$ and $\langle A, \wedge, 1 \rangle$ are commutative monoids,
$\vee$ and $\wedge$ distribute over each other,
$a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

E.g.: $\langle \mathcal{P}(X), \varnothing, X, \cup, \cap \rangle$, in particular $\mathbf{2} = \{0, 1\} \cong \mathcal{P}(\{\star\})$.

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Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

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- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
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E.g.: $\mathrm{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.
Conjunction, i.e. meet of projectors, becomes partial, defined only on **commuting** projectors.

Partial Boolean algebras

A more concrete formulation of the defining axioms is:

- operations preserve commeasurability: for each n-ary operation f,

$$\frac{a_1 \odot c, \ \ldots, \ a_n \odot c}{f(a_1, \ldots, a_n) \odot c}$$

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- for any triple a, b, c of pairwise-commeasurable elements, the axioms of Boolean algebra are satisfied, e.g.

$$\frac{a \odot b}{a \wedge b = b \wedge a} \qquad \frac{a \odot b, \; a \odot c, \; b \odot c}{a \wedge (b \vee c) \; = \; (a \wedge b) \vee (a \wedge c)}$$

The category **pBA**

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Heunen & van der Berg (2012), '*Non-commutativity as a colimit*'.

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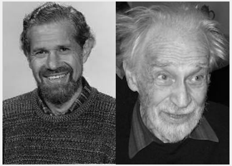
Abramsky & B (2021), '*The logic of contextality*'.

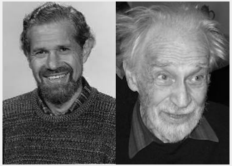
- We give a direct construction of colimits.
- More generally, we show how to freely generate from a given partial Boolean algebra A a new one satisfying prescribed additional commeasurability relations $\circ$, denoted $A[\circledcirc]$.

Contextuality, or the Kochen–Specker theorem



Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathrm{P}(\mathcal{H})$ its pBA of projectors.



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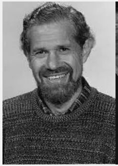
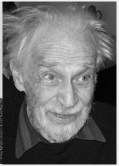
$$\boxed{\text{There is } \textbf{no} \text{ pBA homomorphism } \mathsf{P}(\mathcal{H}) \longrightarrow \mathbf{2}.}$$

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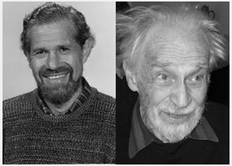

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- No assignment of truth values to all propositions which respects logical operations on jointly testable propositions.

An apparent contradiction

- **BA** is a full subcategory of **pBA**.

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Kochen–Specker and conditions of ‘impossible’ experience

We could say that such a diagram is “implicitly contradictory”: in trying to combine all the information in a colimit, we obtain the manifestly contradictory **1**.

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Let A be a partial Boolean algebra. The following are equivalent:

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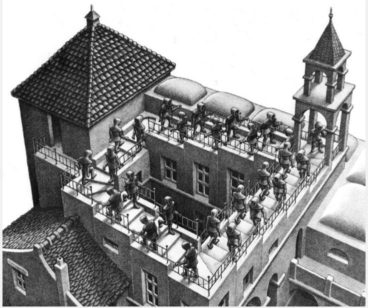
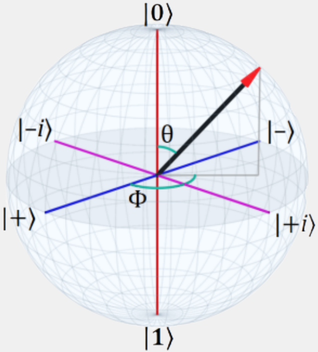


At the borders of paradox:
the contradiction is never directly observed!

Pauli measurements

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



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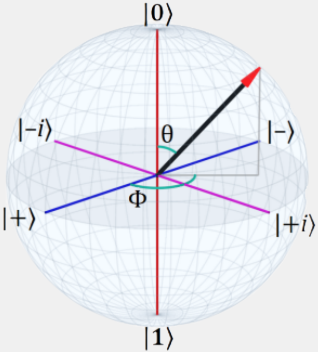

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$$\lambda\, P_1 \otimes \cdots \otimes P_n$$

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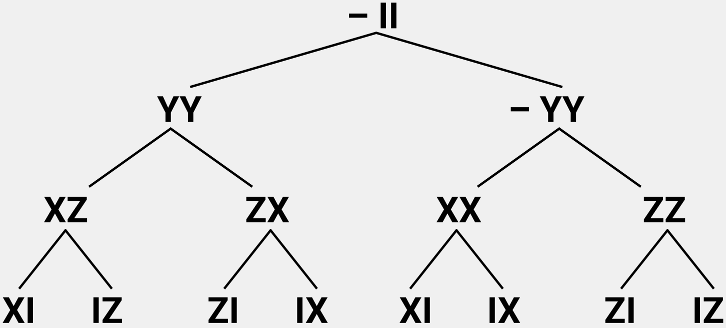

Compound systems

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DISCUSSION OF PROBABILITY RELATIONS BETWEEN SEPARATED SYSTEMS

BY E. SCHRÖDINGER

[Communicated by Mr M. BORN]

[*Received* 14 August, *read* 28 October 1935]

1. When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives (or ψ-functions) have become entangled. To disentangle them we must

Question

$\boxed{\text{How do properties of systems compose?}}$

A [first] tensor product by generators and relations

Heunen & van den Berg show that **pBA** has a monoidal structure:

$$A \otimes B := \text{colim}\,\{C + D \mid C \in \mathcal{C}(A), D \in \mathcal{C}(B)\}$$

where $C + D$ is the coproduct of Boolean algebras.

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We can use our construction to give an explicit generators-and-relations description.

Proposition

Let A and B be partial Boolean algebras. Then

$$A \otimes B \;\cong\; (A \oplus B)[\odot]$$

where $\odot$ is the relation on the carrier set of $A \oplus B$ given by $\imath(a) \odot \jmath(b)$ for all $a \in A$ and $b \in B$.

Tracking the quantum mechanical tensor product?

- There is an embedding $\mathsf{P}(\mathcal{H}) \otimes \mathsf{P}(\mathcal{K}) \longrightarrow \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the obvious embeddings

$$\mathsf{P}(\mathcal{H}) \longrightarrow \mathsf{P}(\mathcal{H} \otimes \mathcal{K}) :: p \longmapsto p \otimes 1$$
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  - Nevertheless, this result is suggestive.
    It poses the challenge of finding a stronger notion of tensor product.

Mysteries of partiality

A slight detour: free partial Boolean algebra

Free partial Boolean algebra on a reflexive graph $(X, \frown)$
(a 'graphical' measurement scenario).

- Generators $G := \{\imath(x) \mid x \in X\}$.
- Pre-terms P: closure of G under Boolean operations and constants.

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- $F(X) = T/\equiv$, with obvious definitions for $\odot$ and operations.

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Mysteries of partiality

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- But the pBA (internally) generated by a subset of a pBA *A* may be infinite

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- So, for $X \subseteq A$, the map $F(X, \odot_X) \longrightarrow \langle X \rangle_A$ need not be surjective!
- How come? The reason is that **new compatibilities** arise!

not just $$\frac{t \odot u, \; t \odot v, \; u \odot v}{(t \wedge u) \odot v}$$

A more expressive tensor product

- Consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$.
- to show that they are **orthogonal**, we have a disjunctive requirement: $p_1 \perp q_1$ **or** $p_2 \perp q_2$.
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  Indeed, the idea that propositions can be defined on quantum systems even though subexpressions are not is emphasized by Kochen.

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- This is sound for the Hilbert space model.
- It remains to be seen how close it gets us to the full Hilbert space tensor product.

A limitative result

- Can extending commeasurability by a relation $\circledcirc$ induce the K-S property in $A[\circledcirc]$ when it did not hold in A?

Theorem (K-S faithfulness of extensions)

Let A be a partial Boolean algebra, and $\circledcirc \subseteq A^2$ a relation on A. Then A is K-S if and only if $A[\circledcirc]$ is K-S.

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In particular, $\mathrm{P}(\mathbb{C}^2) \boxtimes \mathrm{P}(\mathbb{C}^2)$ does not have the K-S property.

We need an even stronger tensor product to track the emergent complexity in the quantum case!

A simpler problem

Restrict the problem

Forget some structure:

- Parity or XOR/NOT logic
- i.e. $(\neg, \oplus)$-fragment
- this is the 'linear (or actually *affine*) part' of Boolean algebra

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Consider the Pauli operators

- $P \in (\mathbb{C}^2)^{\otimes n}$
- s.t. $P = \lambda(P_1 \otimes \cdots \otimes P_n)$,
  with $P_i \in \{X, Y, Z, \mathbf{1}\}$, $\lambda \in \{\pm 1, \pm i\}$

Boolean affine space

Boolean affine space $\langle A, 0, \oplus, \neg \rangle$:

- a set A
- constant $0 \in A$
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Note that $\neg a = a \oplus 1$, so we could define this with 1.

Partial Boolean affine space

Partial Boolean affine space $\langle A, \odot, 0, \oplus, \neg \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
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E.g.: $\mathrm{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.

But also: (projectors associated with) n-Pauli operators, $\mathcal{P}_n \preceq \mathrm{P}((\mathbb{C}^2)^{\otimes n})$

Recovering the Paulis

$$\frac{t \odot u,\ t \odot v,\ u \odot v}{(t \oplus u) \odot v}$$

Crucially, self-adjoint Paulis either **commute** or **anticommute**

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This fully characterises commeasurability of '$\oplus$'s of Paulis, without needing to inspect the concrete Paulis. That is, whether $\phi(\vec{a})$ is commeasurable with b does not depend on the concrete **a** and b but only on the commeasurability structure of $\{a_1, \ldots, a_n, b\}$.

This addresses the compatibility issue in reconstructing $\mathcal{P}_n$ as a partial Boolean affine space.

Questions...

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