Contextuality in logical form

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INL INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

Programming Principles, Logic, and Verification Seminar
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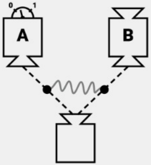
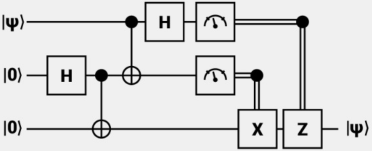
London, 22nd February 2024

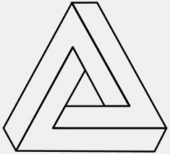
Introduction



Quantum Foundations



Quantum Computer Science



Mathematics of Quantum Structures

- What is the **informatic advantage** afforded by quantum resources?

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- What **phenomena distinguish** quantum mechanics from classical physical theories?

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- What **phenomena distinguish** quantum mechanics from classical physical theories?
- What **structures** are useful to reason about quantum systems?

From quantum foundations to quantum technologies

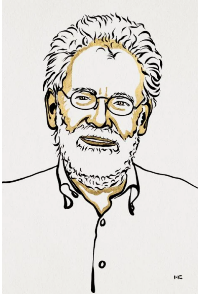
The Nobel Prize in Physics 2022

Ill. Niklas Elmehed © Nobel Prize Outreach

Alain Aspect

Ill. Niklas Elmehed © Nobel Prize Outreach

John F. Clauser



Ill. Niklas Elmehed © Nobel Prize Outreach

Anton Zeilinger

‘for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science’

Contextuality and quantum advantage

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- Measurement-based quantum computation (MBQC)

  '*Contextuality in measurement-based quantum computation*'
  Raussendorf, Physical Review A, 2013.

  '*Contextual fraction as a measure of contextuality*'
  Abramsky, B, Mansfield, Physical Review Letters, 2017.

- Magic state distillation

  '*Contextuality supplies the 'magic' for quantum computation*'
  Howard, Wallman, Veitch, Emerson, Nature, 2014.

- Shallow circuits

  '*Quantum advantage with shallow circuits*'
  Bravyi, Gossett, Koenig, Science, 2018.

  '*A generalised construction of quantum advantage with shallow circuits*'
  Aasnæss, DPhil thesis, 2022.

The essence of contextuality

- Not all properties may be observed simultaneously.
- Sets of jointly observable properties provide **partial, classical snapshots**.

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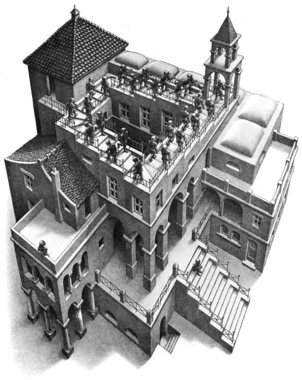

M. C. Escher, *Ascending and Descending*

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Local consistency

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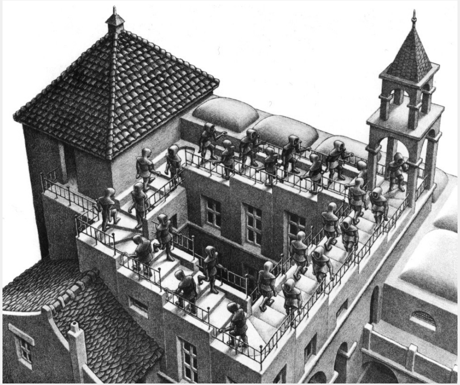

Local consistency *but* **Global inconsistency**

Overview of this talk

Generalise Tarski duality to partial Boolean algebras

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$\boxed{\text{Generalise Tarski duality to partial Boolean algebras}}$

- partial Boolean algebras (Kochen & Specker, 1965)
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  - Includes non-locality (Bell's theorem) as a special case
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  - Includes non-locality (Bell's theorem) as a special case
  - Key role in many instances of **quantum computational advantage**
- Duality between **CABA** and **Set** (Tarski, 1935)
  - Simplest of dualities relating algebra and topology
  - In logic, between syntax and semantics

The logic of quantum theory

From classical to quantum

John von Neumann (1932), '*Mathematische Grundlagen der Quantenmechanik*'.

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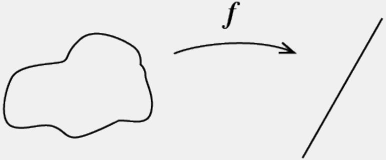

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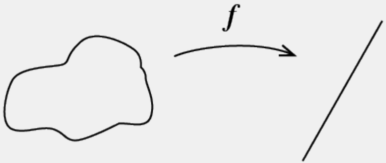

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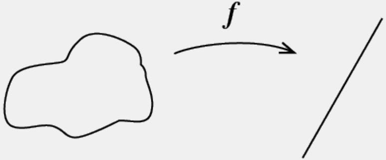

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- Measurements are self-adjoint operators.
- Quantum properties or propositions are **projectors** (dichotomic measurements):

$$p : \mathcal{H} \longrightarrow \mathcal{H} \qquad \text{s.t.} \qquad p = p^\dagger = p^2$$

which correspond to closed subspaces of $\mathcal{H}$.

From states to properties

I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all, Hilbert space (as far as quantum mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of 'conserving the validity of all formal rules' [. . .]. Now we begin to believe that it is not the *vectors* which matter, but the lattice of all linear (closed) subspaces. Because: 1) The vectors ought to represent the physical *states*, but they do it redundantly, up to a complex factor, only 2) and besides, the states are merely a derived notion, the primitive (phenomenologically given) notion being the qualities which correspond to the *linear closed subspaces* [von Neumann (1935) as quoted in Birkhoff (1966)]

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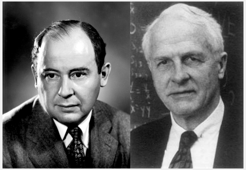
Quantum physics and logic



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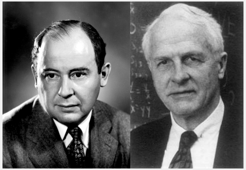

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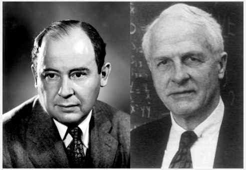

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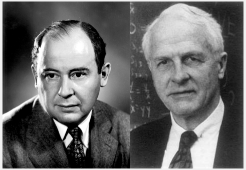

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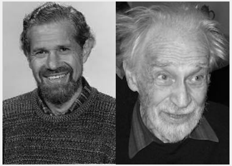
- Distributivity fails: $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$.
- Taking the *phenomenological* requirement seriously:
  in QM, only **commuting** measurements can be performed together.

  So, what is the operational meaning of $p \wedge q$, when p and q **do not commute**?



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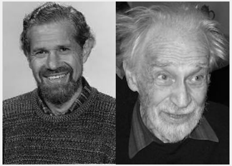

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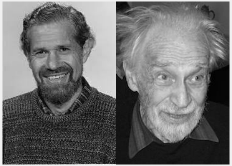

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Kochen (2015), '*A reconstruction of quantum mechanics*'.

- Kochen develops a large part of foundations of quantum theory in this framework.

Classical snapshots

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- When A, B, C with $C = AB$ are jointly measured on **any** quantum state, the observed outcomes a, b, c satisfy $c = ab$.

- More generally, for $A_1, \ldots, A_n$ pairwise commuting and any Borel $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, then $f(A_1, \ldots, A_n)$ commutes with all A_i and eigenvalues satisty the same functional relation.

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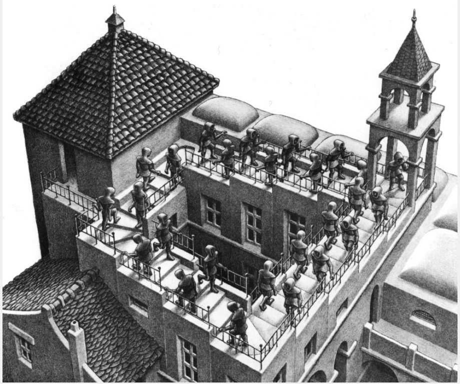

Partial Boolean algebras

Boolean algebras

Boolean algebra $\langle A, 0, 1, \neg, \vee, \wedge \rangle$:

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satisfying the usual axioms: $\langle A, \vee, 0 \rangle$ and $\langle A, \wedge, 1 \rangle$ are commutative monoids,
$\vee$ and $\wedge$ distribute over each other,
$a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

E.g.: $\langle \mathcal{P}(X), \varnothing, X, \cup, \cap \rangle$, in particular $\mathbf{2} = \{0, 1\} \cong \mathcal{P}(\{\star\})$.

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Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
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Morphisms of pBAs are maps preserving commeasurability, and the operations wherever defined. This gives the category **pBA**.

Partial Boolean algebras

A more concrete formulation of the defining axioms is:

- operations preserve commeasurability: for each n-ary operation f,

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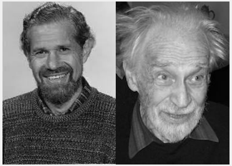
- for any triple a, b, c of pairwise-commeasurable elements, the axioms of Boolean algebra are satisfied, e.g.

$$\frac{a \odot b}{a \wedge b = b \wedge a} \qquad \frac{a \odot b, \ a \odot c, \ b \odot c}{a \wedge (b \vee c) \ = \ (a \wedge b) \vee (a \wedge c)}$$

Contextuality, or the Kochen–Specker theorem



Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathrm{P}(\mathcal{H})$ its pBA of projectors.

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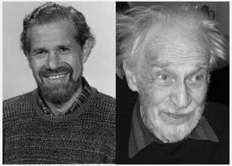

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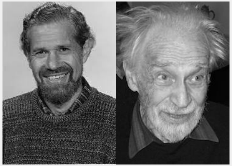

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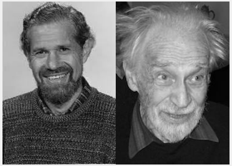

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- No assignment of truth values to all propositions which respects logical operations on jointly testable propositions.

At the borders of paradox

Let A be a partial Boolean algebra. The following are equivalent:

1. A has the K-S property, i.e. it has no morphism to **2**.
2. The colimit in **BA** of the diagram $\mathcal{C}(A)$ of boolean subalgebras of A is **1**.

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‘to be sincere contradicting oneself’
(Álvaro de Campos, *Passagem das Horas*, 1916)

But the contradiction is never directly observed!

At the borders of paradox

Let A be a partial Boolean algebra. The following are equivalent:

1. A has the K-S property, i.e. it has no morphism to $\mathbf{2}$.
2. The colimit in **BA** of the diagram $\mathcal{C}(A)$ of boolean subalgebras of A is $\mathbf{1}$.
3. There is a propositional contradiction $\varphi(\vec{x})$ and an assignment $\vec{x} \mapsto \vec{a}$ such that $A \models \varphi(\vec{a})$.

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Quantum realisation

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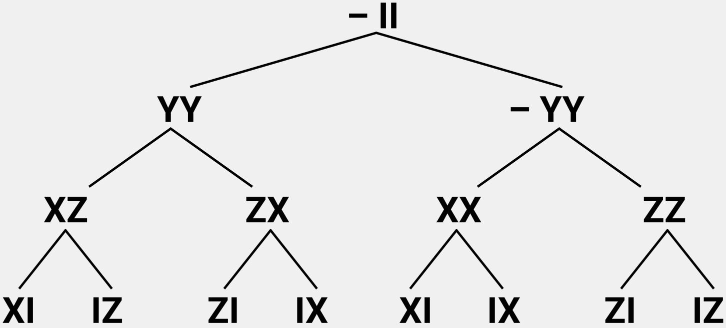

$$\langle \{0, 1\}, \oplus \rangle \quad \longleftrightarrow \quad \langle \{1, -1\}, \cdot \rangle$$

The mirror of mathematics

Dualities between algebra and topology

'Commutative algebra is like topology, only backwards.' – John Baez

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Commutative C^*-algebras Locally compact Hausdorff spaces

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finite Boolean algebras	finite sets
complete atomic Boolean algebras	sets

Commutativity

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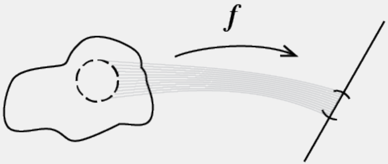
- Given a space X,
  - take the set $C(X)$ of continuous functions $X \longrightarrow \mathbb{K}$ to *scalars* $\mathbb{K}$.



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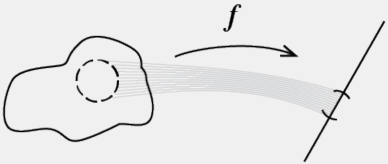

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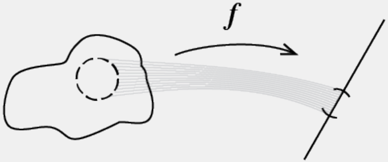

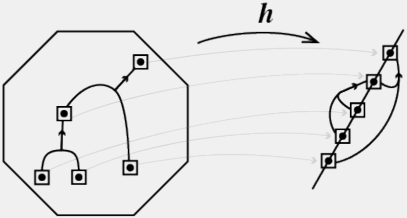

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Here, I mean **commutativity** in a loose, informal sense.
For lattices, this would be **distributivity** (think: idempotents of a ring).

No-go theorems for noncommutative dualities

- Reyes (2012)
  - Any extension of Zariski spectrum to a functor $\mathbf{Rng}^{\mathrm{op}} \longrightarrow \mathbf{Top}$ trivialises on $\mathbb{M}_n(\mathbb{C})$ ($n \geq 3$).
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    the spectrum of a pBA cannot have points!

'What is proved by impossibility proofs is lack of imagination.' – John S. Bell

Summary of results

Duality for partial CABAs: key idea

- Replace **sets** by certain **graphs**.
- Vertices are *possible worlds of maximal information*.
- Adjacency represents **exclusivity**.
- It generalises $\neq$, thus sets embed as **complete graphs**.
- These exclusivity graphs are the ‘non-commutative’ spaces in this duality.

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- These exclusivity graphs are the ‘non-commutative’ spaces in this duality.
- The partial algebra is reconstructed as equivalence classes of **cliques**, or double-neighbourhood closures of cliques.
- Morphisms of exclusivity graphs are certain **relations**, generalising **functional** ones from Tarski duality.

Tarski duality

$$
\begin{tikzcd}
\mathbf{FinBA} \arrow[rr, shift left, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{FinSet}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{P}"] \arrow[dl, shift left, "\mathcal{P}"] \\
& \mathbf{FinSet} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{P}"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
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\end{tikzcd}
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Partial Tarski duality

$$\begin{tikzcd}
\mathbf{epFinBA} \arrow[rr, shift left, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{FinXGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \\
& \mathbf{FinRGph} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv" sloped] \arrow[ur, shift left, "\mathcal{V}"] \arrow[ur, phantom, "\dashv" sloped] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \\
& \mathbf{RGph} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv" sloped] \arrow[ur, shift left, "\mathcal{V}"] \arrow[ur, phantom, "\dashv" sloped] &
\end{tikzcd}$$

Recap: Tarski duality

Partial order

Let A be a Boolean algebra.

Definition

For $a, b \in A$, we write $a \leq b$ when one (hence all) of the following equivalent conditions hold:

- $a \wedge b = a$
- $a \vee b = b$
- $a \wedge \neg b = 0$
- $\neg a \vee b = 1$

$\leq$ is a partial order.

It determines A as a Boolean algebra: e.g. $\vee$ (resp. $\wedge$) is supremum (resp. infimum) wrt $\leq$.

Definition (Complete Boolean algebra)

A Boolean algebra A is said to be **complete** if any subset of elements $S \subseteq A$ has a supremum $\bigvee S$ in A (and consequently an infimum $\bigwedge S$, too). It thus has additional operations

$$\bigwedge, \bigvee : \mathcal{P}(A) \longrightarrow A \,.$$

Definition (Atomic Boolean algebra)

An **atom** of a Boolean algebra is a minimal non-zero element, i.e. an element $x \neq 0$ such that $a \leq x$ implies $a = 0$ or $a = x$.

A Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

A **CABA** is a complete, atomic Boolean algebra.

Example

Any finite Boolean algebra is trivially a CABA.

The powerset $\mathcal{P}(X)$ of an arbitrary set X is a CABA.

- completeness: closed under arbitrary unions
- atoms: singletons $\{x\}$ for $x \in X$

This is in fact the 'only' (up to iso) example.

Proposition

In a CABA, every element is the join of the atoms below it:

$$a = \bigvee U_a \qquad \textit{where } U_a := \{x \in A \mid x \textit{ is an atom and } x \leq a\} .$$

Proof.

Suppose $a \not\leq \bigvee U_a$, i.e. $a \wedge \neg \bigvee U_a \neq 0$. Atomicity implies there's an atom $x \leq a \wedge \neg \bigvee U_a$. On the one hand, $x \leq \neg \bigvee U_a$. On the other, $x \leq a$, i.e. $x \in U_a$, hence $x \leq \bigvee U_a$. Hence $x = 0$. $\lightning$ $\square$

Tarski duality

$$\begin{tikzcd} \mathbf{CABA} \arrow[rr, bend right, "\mathrm{At}"'] \arrow[rr, phantom, "\cong"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, bend right, "\mathcal{P}"'] \end{tikzcd}$$

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$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{CABA}$ is the contravariant powerset functor:

- on objects: a set X is mapped to its powerset $\mathcal{P}X$ (a CABA).
- on morphisms: a function $f : X \longrightarrow Y$ yields a complete Boolean algebra homomorphism

$$\begin{aligned} \mathcal{P}(f) : \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X) \\ (T \subseteq Y) &\longmapsto f^{-1}(T) = \{x \in X \mid f(x) \in T\} \end{aligned}$$

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$\mathrm{At} : \mathbf{CABA}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is defined as follows:

- on objects: a CABA A is mapped to its set of atoms.
- on morphisms: a complete Boolean homomorphism $h : A \longrightarrow B$ yields a function

$$\mathrm{At}(h) : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$$

mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

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mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

Lemma

Let $h : A \longrightarrow B$ in **CABA**. *For all $y \in \mathrm{At}(A)$, there is a unique $x \in \mathrm{At}(A)$ with $y \leq h(x)$.*

Proof.

Facts about atoms in any BA:

- If $x \neq x'$ are atoms, then $x \wedge_A x' = 0$.
- If x is an atom and $x \leq \bigvee S$, there is $a \in S$ with $x \leq a$.

Existence

A complete atomic implies $1_A = \bigvee \mathrm{At}(A)$. Hence,

$$1_B = h(1_A) = h(\bigvee \mathrm{At}(A)) = \bigvee \{h(x) \mid x \in \mathrm{At}(A)\}$$

Since $y \leq 1_B$, we conclude $y \leq h(x)$ for some $x \in \mathrm{At}(A)$.

Uniqueness

If $y \leq h(x)$ and $y \leq h(x')$, then $y \leq h(x) \wedge_B h(x') = h(x \wedge x')$, hence $x = x'$. $\square$

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- Given a CABA A, the isomorphism $A \cong \mathcal{P}(\mathrm{At}(A))$ maps $a \in A$ to the set of elements

$$U_a = \{x \in \mathrm{At}(A) \mid x \leq a\}\,.$$

  A property is identified with the set of possible worlds in which it holds.

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  A property is identified with the set of possible worlds in which it holds.

- Given a set X, the bijection $X \cong \mathrm{At}(\mathcal{P}(X))$ maps $x \in X$ to the singleton $\{x\}$, which is an atom of $\mathcal{P}(X)$.

  A possible world is identified with its characteristic property (which fully determines it).

Transitive partial CABAs

Logical exclusivity principle

Let A be a partial Boolean algebra.

For $a, b \in A$, we write $a \leq b$ to mean $a \odot b$ and $a \wedge b = a$.

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Two elements $a, b \in A$ are **exclusive**, written $a \perp b$, if there is a $c \in A$ with $a \leq c$ and $b \leq \neg c$.

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- $a \perp b$ is a weaker requirement than $a \wedge b = 0$.
- The two are equivalent in a Boolean algebra.
- But in a general partial Boolean algebra, there may be exclusive events that are not commeasurable (and for which, therefore, the $\wedge$ operation is not defined).

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- But in a general partial Boolean algebra, there may be exclusive events that are not commeasurable (and for which, therefore, the $\wedge$ operation is not defined).

Definition

A is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are logically exclusive are also commeasurable, i.e. if $\perp \subseteq \odot$.

Logical exclusivity principle

Note that $\leq$ is always reflexive and antisymmetric.

Definition

A partial Boolean algebra is said to be **transitive** if $a \leq b$ and $b \leq c$ implies $a \leq c$, i.e. $\leq$ is (globally) a partial order on A.

Proposition

A partial Boolean algebra satisfies LEP if and only if it is transitive.

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We restrict atention to partial Boolean algebras satisfying LEP in this talk.

Theorem

The category **epBA** *of partial Boolean algebras satisfying LEP is a reflective subcategory of* **pBA***, i.e. the inclusion functor* $I : \mathbf{epBA} \longrightarrow \mathbf{pBA}$ *has a left adjoint* $X : \mathbf{pBA} \longrightarrow \mathbf{epBA}$.

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \bigodot \longrightarrow A$$

satisfying the following property: any set $S \in \bigodot$ is contained in a set $T \in \bigodot$ which forms a complete Boolean algebra under the restriction of the operations.

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Partial CABAs from their graphs of atoms

Graph

Definition

A **graph** $(X, \#)$ is a set equipped with a symmetric irreflexive relation.

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Given a vertex $x \in X$ and sets of vertices $S, T \subset X$, we write:

- $x \# S$ when for all $y \in S$, $x \# y$;
- $S \# T$ when for all $x \in S$ and $y \in T$, $x \# y$;
- $x^{\#} := \{y \in X \mid y \# x\}$ for the neighbourhood of the vertex x;
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A **clique** is a set of pairwise-adjacent vertices, i.e. a set $K \subset X$ with $x \# K \setminus \{x\}$ for all $x \in K$.

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

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- Can interpret these as *worlds of maximal information* and incompatibility between them.

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- If A is a Boolean algebra, then $\mathrm{At}(A)$ is the complete graph on the set of atoms ($\#$ is $\neq$).

Recall that in a CABA, any element is uniquely written as a join of atoms, viz. $a = \bigvee U_a$ with

$$U_a := \{x \in \mathrm{At}(A) \mid x \leq a\}$$

In a pBA, U_a may not be pairwise commeasurable, hence their join need not even be defined.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

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Proof.

Let $a \in A$ and K be a clique of $\mathrm{At}(A)$ maximal in U_a.

Being a clique in $\mathrm{At}(A)$, $K \in \bigodot$ and thus $\bigvee K$ is defined.

Since $K \subset U_a$, all $k \in K$ satisfy $k \leq a$ and in particular $k \odot a$. Hence, $K \cup \{a\} \in \bigodot$, implying that it is contained in a complete Boolean subalgebra. Consequently, $\bigvee K \leq a$.

Now, suppose $a \not\leq \bigvee K$, i.e. $a \wedge \neg \bigvee K \neq 0$. Then atomicity implies there is an atom $x \leq a \wedge \neg \bigvee K$. By transitivity, $x \leq a$ and $x \leq \neg k$ (hence $x \perp k$) for all $k \in K$. This makes $K \cup \{x\}$ a clique of atoms contained in U_a, contradicting maximality of K. □

Elements from atoms

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Elements from atoms

So an element a is the join of **any** clique that is maximal in U_a.

Given two maximal cliques K and L, this yields an equality

$$\bigvee K = \bigvee L$$

where the elements in $\bigvee K$ and those in $\bigvee L$ are not commeasurable.

Elements from atoms

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Proposition

Let K and L be cliques in $\mathrm{At}(A)$. Then $\bigvee K \leq \bigvee L$ *iff* $L^{\#} \subseteq K^{\#}$ *iff* $K \subseteq L^{\#\#}$.

Corollary

$\bigvee K = \bigvee L$ *iff* $K^{\#} = L^{\#}$.

Partial CABA from its graph of atoms

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elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

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$\boxed{\text{Which conditions on a graph } (X, \#) \text{ allow for such reconstruction?}}$

Exclusivity graphs

Exclusivity graphs

Definition

An **exclusivity graph** is a graph $(X, \#)$ such that for K, L cliques and $x, y \in X$:

1. If $K \sqcup L$ is a maximal clique, then $K^{\#} \# L^{\#}$, i.e. $x \# K$ and $y \# L$ implies $x \# y$.
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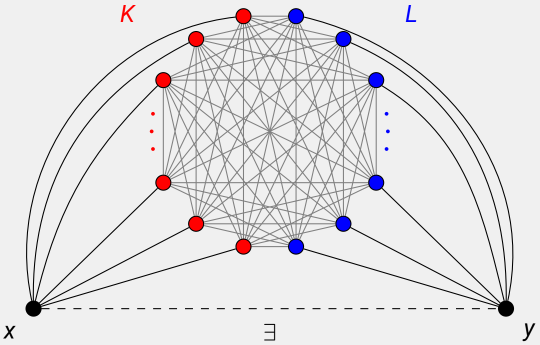

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A helpful intuition is to see these as generalising sets with a $\neq$ relation (the complete graph).

- A graph is symmetric and irreflexive.
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- To be an inequivalence relation, we need cotransitivity: $x \# z$ implies $x \# y$ or $x \# z$.
- Condition 1 is a weaker version of cotransitivity.
- Condition 2 eliminates redundant elements: cotransitive + 2 imply $\neq$.

Graph of atoms is an exclusivity graph

Proposition

Let A be a partial Boolean algebra. Then $\mathrm{At}(A)$ is a exclusivity graph.

Proof.

Let $K, L \subset X$ such that $K \sqcup L$ is a maximal clique, and let x, y be atoms of A.

Write $c := \bigvee K = \neg \bigvee L$.

$x \# K$ means $x \leq \neg \bigvee K = \neg c$ and $x \# L$ means $y \leq \neg \bigvee L = c$.

By transitivity, we conclude that $x \odot y$, hence $x \perp y$. $\square$

The ‘clique powerset’ of an exclusivity graph

Proposition

Let K, L be cliques in an exclusivity graph. The following are equivalent:

- $[K] \odot [L]$, *i.e. there exist* K', L' *with* $K' \equiv K$ *and* $L' \equiv L$ *such that* $K' \cup L'$ *is a clique.*
- *The four sets*

$$K^{\#\#} \cap L^{\#\#}, \quad K^{\#\#} \cap L^{\#}, \quad K^{\#} \cap L^{\#\#}, \quad K^{\#} \cap L^{\#},$$

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Choose maximal cliques

$$M_{11} \subset K^{\#\#} \cap L^{\#\#}, \quad M_{10} \subset K^{\#\#} \cap L^{\#}, \quad M_{01} \subset K^{\#} \cap L^{\#\#}, \quad M_{00} \subset K^{\#} \cap L^{\#},$$

and set
$$[K] \wedge [L] := [M_{11}] \qquad \text{and} \qquad [K] \vee [L] := [M_{11} \cup M_{10} \cup M_{11}] \, .$$

The ‘clique powerset’ of an exclusivity graph

Proposition

Let K, L, M be cliques in an exclusivity graph with $[K] \odot [L]$, $[K] \odot [M]$, $[L] \odot [M]$. The eight sets

$$K^{\square_1} \cap L^{\square_2} \cap M^{\square_3}\,, \qquad \square_i \in \{\#, \#\#\}$$

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Proposition

Let $\{K_i\}_{i \in I}$ be a set of cliques in an exclusivity graph whose equivalence classes are pairwise commeasurable. The sets

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Morphisms

Morphisms of exclusivity graphs

$\boxed{\text{What about morphisms?}}$

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \; R \; y$, $x' \; R \; y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
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3. trivialises.

Morphisms of exclusivity graphs and pCABA homomorphisms

Proposition

Let A and B be transitive partial CABAs. Given $h : A \longrightarrow B$ a partial complete Boolean algebra homomorphism, the relation $R_h : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$ given by

$$xR_hy \qquad \textit{iff} \qquad x \leq h(y)$$

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Proposition

For any A and B be transitive partial CABAs, $\mathbf{epCABA}(A, B) \cong \mathbf{XGph}(\mathrm{At}(B), \mathrm{At}(A))$.

Revisiting contextuality

Global points

Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

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Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

i.e. a subset of atoms of A satisfying:

1. it is an independent (or stable) set
2. it is a maximal clique transversal, i.e. it has a vertex in each maximal clique

Outlook

Reconstruction via double-neighbourhood-closed sets

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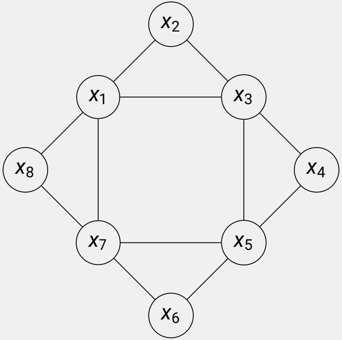

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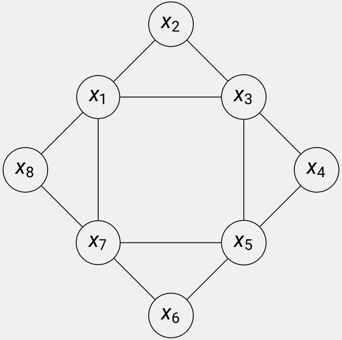

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Can we characterise which ${}^{\#\#}$-closed sets arise from cliques?

Free-forgetful adjunction for CABAs

$$\begin{tikzcd}
\mathbf{CABA} \arrow[rr, shift left=1.5, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left=1.5, "U"] \arrow[dr, phantom, "\top"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, shift left=1.5, "\mathcal{P}"] \arrow[dl, shift left=1.5, "\mathcal{P}"] \arrow[dl, phantom, "\dashv"] \\
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\end{tikzcd}$$

- Under the duality, it corresponds to the contravariant powerset self-adjunction.
- It gives the construction of the free CABA as a double powerset.

Free-forgetful adjunction for partial CABAs

$$
\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left=2, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left=2, "U"] \arrow[dr, phantom, "\top"] & & \mathbf{XGph}^{\mathsf{op}} \arrow[ll, shift left=2, "\mathcal{K}"] \arrow[dl, shift left=2, "\mathcal{K}"] \arrow[dl, phantom, "\perp"] \\
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- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$
- Under duality it corresponds to adjunction between **compatibility** and **exclusivity** graphs.

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$\boxed{\text{Can we give a concrete construction of the free CABA?}}$

- First attempt: Given $\langle P, \odot \rangle$ build a graph with vertices $\langle C, \gamma : C \longrightarrow \{0,1\} \rangle$ where C maximal compatible set, and edges $\langle C, \gamma \rangle \# \langle D, \delta \rangle$ iff $\exists x \in C \cap D.\ \ \gamma(x) \neq \delta(x)$.

The spatial landscape of partial Boolean algebra

$$
\begin{tikzcd}
& \mathbf{epDLat}_{\mathbf{bs}} \arrow[rr, tail] && \mathbf{pDLat}_{\mathbf{bs}} \\
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& \mathbf{epBA}_{\mathbf{bs}} \arrow[uu, hook] \arrow[rr, tail] && \mathbf{pBA}_{\mathbf{bs}} \arrow[uu, hook] \\
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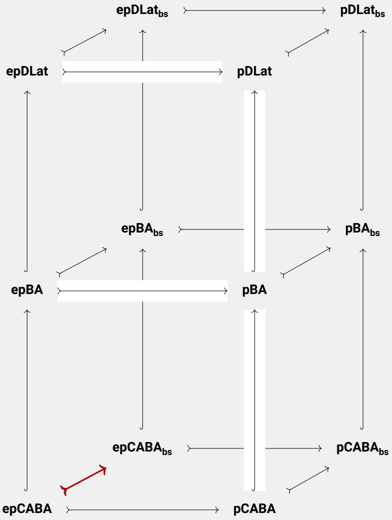
- Drop transitivity / LEP

The spatial landscape of partial Boolean algebra

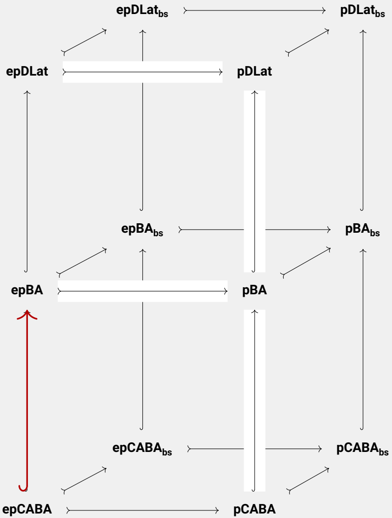


- Drop transitivity / LEP
- Relax binary to simplicial compatibility

$$\begin{tikzcd} \mathbf{RGph} \arrow[rr, bend left, "\mathsf{Kl}"] \arrow[rr, bend right, "\iota"'] & & \mathbf{Simp} \arrow[ll, "\mathsf{sk}_1"' description] \end{tikzcd}$$

$\rightsquigarrow$Czelakowski's *pBAs in a broader sense*

The spatial landscape of partial Boolean algebra



- Drop transitivity / LEP

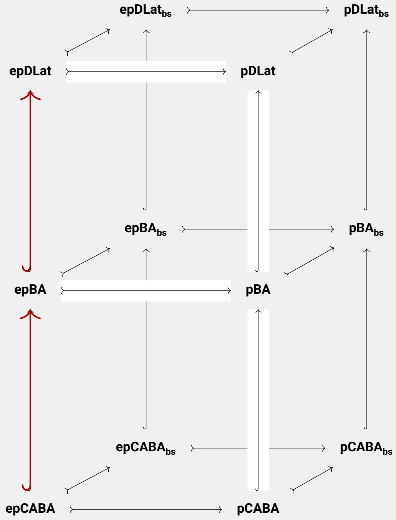
- Relax binary to simplicial compatibility

$$\begin{tikzcd} \mathbf{RGph} \arrow[r, bend left=40, "\mathrm{Kl}"] \arrow[r, bend right=40, "\iota"'] & \mathbf{Simp} \arrow[l, "\mathrm{sk}_1"'] \arrow[l, phantom, bend right=20, "\perp"] \arrow[l, phantom, bend left=20, "\perp"] \end{tikzcd}$$

$\rightsquigarrow$Czelakowski's *pBAs in a broader sense*

- Dropping completeness and atomicity
(e.g. $P(A)$ for vN algebra A with factor not of type I)

The spatial landscape of partial Boolean algebra



- Drop transitivity / LEP

- Relax binary to simplicial compatibility

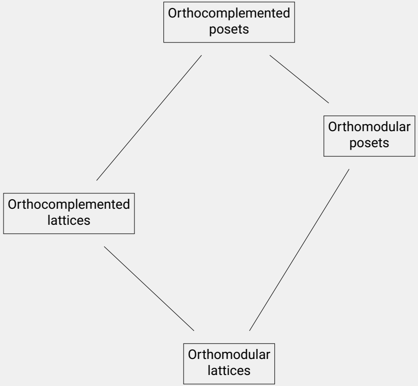
$$\begin{tikzcd} \mathbf{RGph} \arrow[r, bend left=50, "\mathsf{Kl}"] \arrow[r, bend right=50, "\iota"'] & \mathbf{Simp} \arrow[l, "\top\ \mathsf{sk}_1"'] \end{tikzcd}$$

$\rightsquigarrow$Czelakowski's *pBAs in a broader sense*

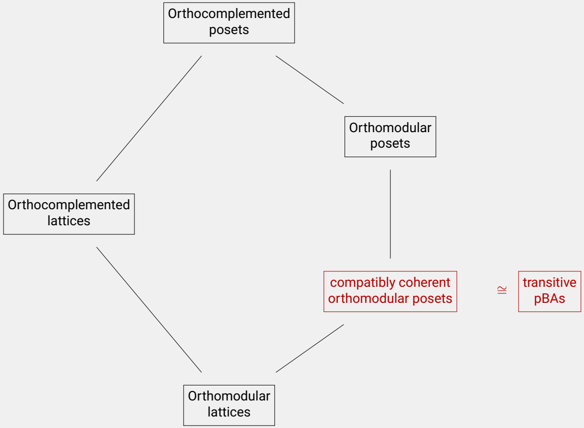
- Dropping completeness and atomicity
(e.g. P(A) for vN algebra A with factor not of type I)

$\rightsquigarrow$ analogues of Stone, Priestley, . . .
Stone's motto: *'always topologise'* – but how?

The wider spatial landscape of ‘quantum’ logics

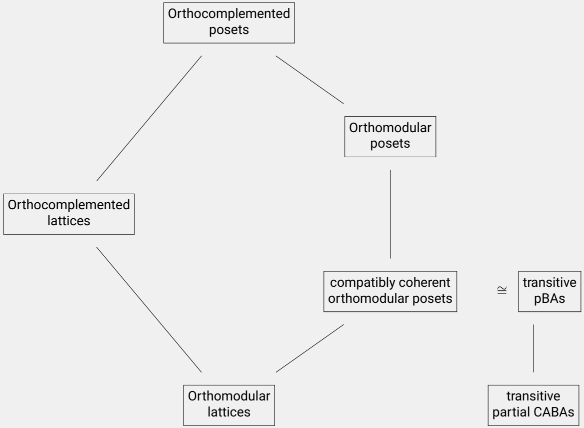


The wider spatial landscape of 'quantum' logics



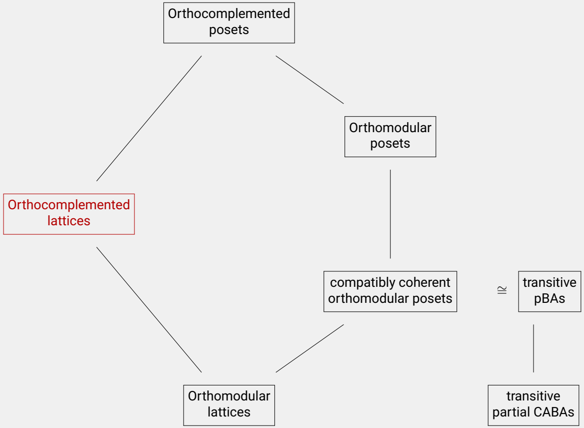
(Gudder, 1972)

The wider spatial landscape of ‘quantum’ logics



(Gudder, 1972)

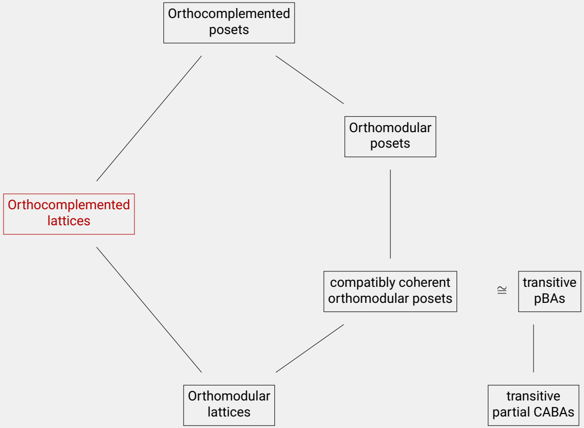
The wider spatial landscape of ‘quantum’ logics



(Gudder, 1972)

OLs $\leftrightsquigarrow$ Minimal quantum logic
(Dishkant, Goldblatt, Dalla Chiara, 1970s)

The wider spatial landscape of ‘quantum’ logics



(Gudder, 1972)

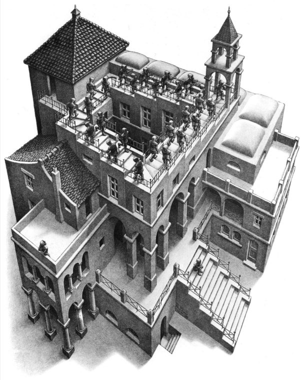
OLs $\leftrightsquigarrow$ Minimal quantum logic
(Dishkant, Goldblatt, Dalla Chiara, 1970s)

Stone representation for OLs
(Goldblatt, 1975)

- related to our construction
- all graphs, all nhood-regular sets
- nothing on morphisms

Towards noncommutative dualities?

- Can one find a more encompassing duality theory for 'noncommutative' or 'quantum' structures by viewing them through multiple partial classical snapshots?



Thank you for your attention!

Questions...

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