Combining contextuality and causality: a game semantics approach



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21st International Conference on Quantum Physics and Logic (QPL 2024)
Buenos Aires, 18th July 2024

This talk

PHILOSOPHICAL TRANSACTIONS A

royalsocietypublishing.org/journal/rsta

Research

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Cite this article: Abramsky S, Barbosa RS, Searle A. 2024 Combining contextuality and causality: a game semantics approach. *Phil. Trans. R. Soc. A* **382**: 20230002. https://doi.org/10.1098/rsta.2023.0002

Received: 10 June 2023
Accepted: 2 October 2023

One contribution of 12 to a theme issue 'Quantum contextuality, causality and freedom of choice'.

Subject Areas:
quantum physics

Keywords:
contextuality, causality, games

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We develop an approach to combining contextuality with causality, which is general enough to cover causal background structure, adaptive measurement-based quantum computation and causal networks. The key idea is to view contextuality as arising from a game played between Experimenter and Nature, allowing for causal dependencies in the actions of both the Experimenter (choice of measurements) and Nature (choice of outcomes).

This article is part of the theme issue 'Quantum contextuality, causality and freedom of choice'.

- Preprint available at arXiv:2307.04786 [quant-ph].

FoQaCiA

Funded by the European Union

NSERC CRSNG

UK Research and Innovation

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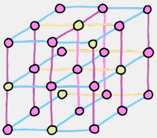

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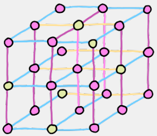

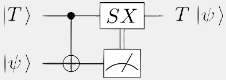

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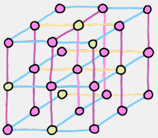

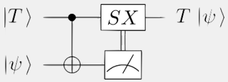

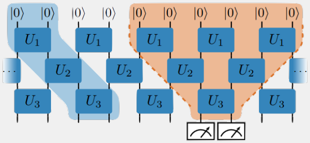

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- Shallow circuits
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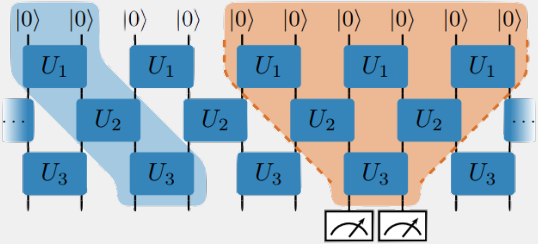

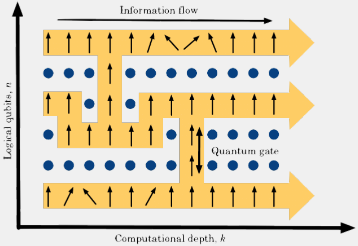

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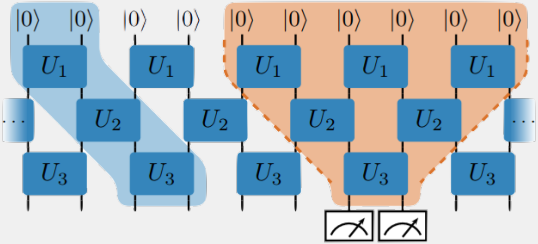

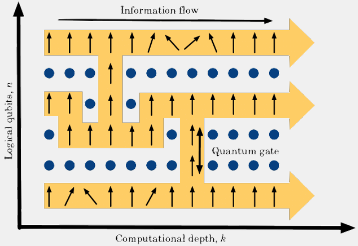

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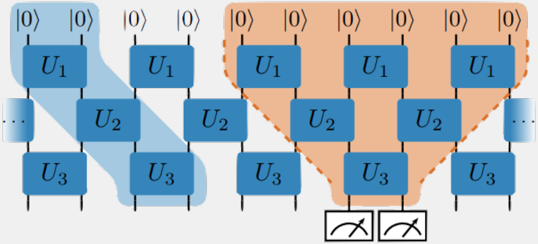
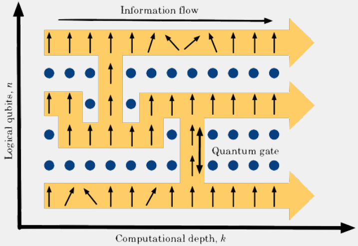
- However, computation is dynamic, with nontrivial **causal flow** between operations.
- This should be taken into account in the analysis.
- Similar motivation applies to basic physics experiments with a given causal background.





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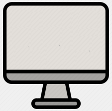
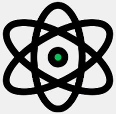
- feed forward in MBQC, and more generally, adaptive computation.

Game semantics of causality

Approach: **two-person game** between Experimenter and Nature

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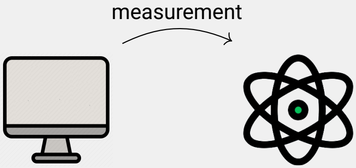

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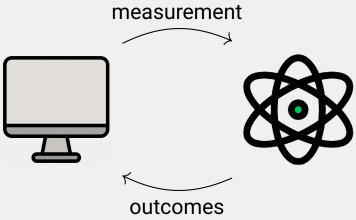

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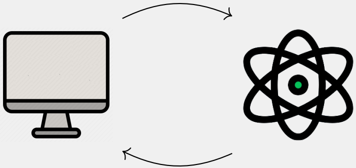

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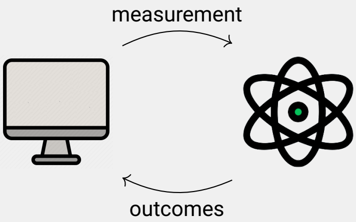

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Note Borow ideas from CS: Kahn–Plotkin concrete domains and their representations.

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  - contextual fraction
  - logical Bell inequalities
  - resource theory
  - topological criteria
  - connections with logic and computation

Dual nature of causality

Causality may be:

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We illustrate these two sources of causality in two basic examples.

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In a standard, "flat" scenario, deterministic outcomes are given by functions

$$s_A : M_A \longrightarrow O_A, \quad s_B : M_B \longrightarrow O_B,$$

With these causal constraints, we have functions

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That is, the responses by Nature to Bob's measurement may depend on the previous measurement made by Alice.

Example I ctd

Given measurements $x_1, x_2 \in M_A$, $y \in M_B$, we can have

$$\{(x_1, 0), (y, 0)\} \text{ and } \{(x_2, 0), (y, 1)\}$$

as valid histories in a single deterministic model.

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Thus **no-signalling is relaxed** in a controlled fashion.

Example II: Anders–Browne

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In terms of parities (product of $+1/-1$ outputs):

$$\begin{array}{ccccc} X_1 & Y_2 & Y_3 & = & -1 \\ Y_1 & X_2 & Y_3 & = & -1 \\ Y_1 & Y_2 & X_3 & = & -1 \\ X_1 & X_2 & X_3 & = & +1 \end{array}$$

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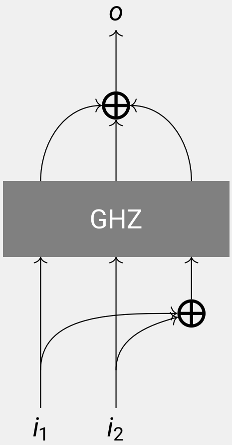
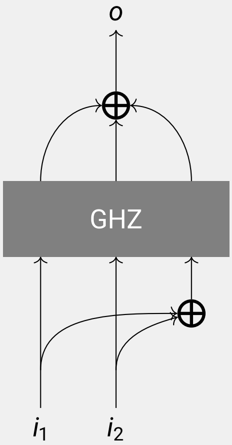


Using GHZ to implement OR

- Taking X as 0, Y as 1, the inputs to an OR-gate determine the measurements for Alice and Bob.



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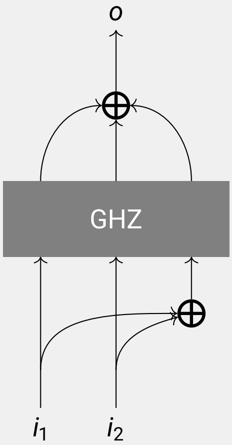

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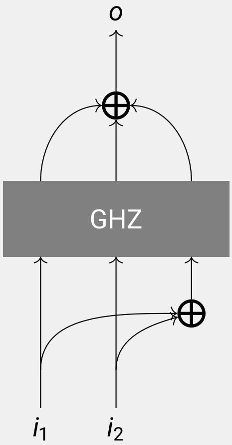

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- The above implements one OR gate. An arbitrary Boolean circuit with embedded OR gates can be represented using (classically computed) **feed-forward of measurement settings**.
- Such **adaptivity** is purely causality employed by the Experimenter; from Nature's point of

Contextuality scenarios

(Flat) contextuality scenario $(X, O, \mathcal{C})$:

- X a finite set of **measurements**.
- $O = \{O_x\}_{x \in X}$ a set of possible **outcomes** for each measurement.
- $\mathcal{C} = \{C_i\}_{i \in I}$ a cover of X, consisting of **contexts** $C_i \subseteq X$ st $\bigcup_{i \in I} C_i = X$.

An **event** has the form (x, o), where $x \in X$ and $o \in O_x$.

It corresponds to the measurement x being performed, with outcome o.

Joint outcome events

- A set s of events is **consistent** if $(x, y), (x, y') \in s$ implies $y = y'$.
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A consistent set of events is a **section**.

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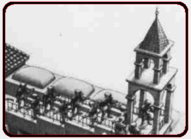
By adding probabilities $\mathcal{D} \circ \mathcal{E}$ contextuality may arise.

The essence of contextuality



Local consistency

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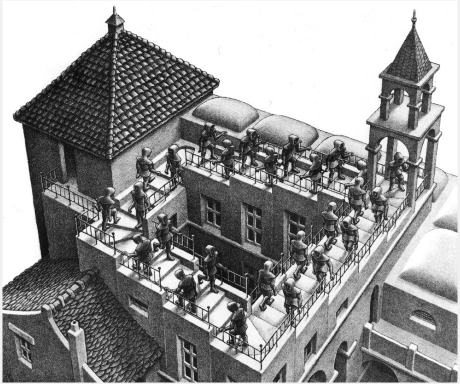

Local consistency *but* **Global inconsistency**

Causal contextuality scenarios

Causal contextuality scenario $(X, O, \mathcal{C}, \vdash)$:

- Additional ingredient: **enabling relation**, which expresses **causal constraints**.

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- meaning it is possible to perform x after the events in s have occurred.

Note that constraints refer to the **measurement outcomes** as well as the measurements which have been performed. This allows adaptive behaviours to be described.

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Sets of events that can happen in a **causally consistent** fashion.

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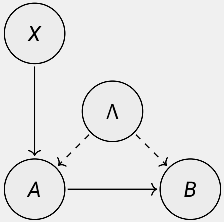
- $\mathcal{H}$, the set of histories over the scenario, is defined inductively:

$$\begin{array}{lcl} H_0 & := & \{\varnothing\} \\ H_{k+1} & := & H_k \ \cup \ \{s \cup \{(x,o)\} \mid s \in H_k, s \triangleright x, o \in O_x\}. \end{array}$$

- With X finite, we have $H_k = H_{k+1}$ for some k, and we take $\mathcal{H} = H_k$ for the least such k.

Example: instrumental scenario



Outcomes: $\{1, 2\}$
Measurement settings

- for Alice: $\{x_1, x_2\}$
- for Bob: $\{y_1, y_2\}$

Enablings:

$$\varnothing \vdash x_i, \qquad (x_i, j) \vdash y_j$$

Thus Alice's measurement outcome determines Bob's measurement setting, without any information as to what Alice's measurement setting was.
The variant where there **is** such information flow can also be represented.

Game

A causal contextuality scenario specifies a **game** between Experimenter and Nature:

- Events (x, o) correspond to the Experimenter choosing a measurement x, and Nature responding with outcome o.
- The histories correspond to the **plays** or runs of the game.

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Strategies

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- σ is **downwards closed**:
  if $s, t \in \mathcal{H}(M)$ and $s \subseteq t \in \sigma$, then $s \in \sigma$.

- σ is **deterministic** and **total**:
  $\varnothing \in \sigma$, and if $s \in \sigma$ and $s \triangleright x$, then there is a unique $o \in O_x$ such that $s \cup \{(x, o)\} \in \sigma$.

In any position s reachable under σ, it specifies a unique response to any measurement that can be chosen by the Experimenter.

The sheaf of strategies

Given a causal contextuality scenario $M = (X, O, \mathcal{C}, \vdash)$, we can define a presheaf

$$\Gamma : \mathcal{P}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

- For $U \subseteq X$, $\Gamma(U)$ is the set of strategies for $M|_U$ (restriction to measurements in U).
- When $U \subseteq V$, the restriction map $\Gamma(U \subseteq V) : \Gamma(V) \longrightarrow \Gamma(U)$ is given by $\sigma \mapsto \sigma|_U := \sigma \cap \mathcal{H}(M_U)$.

Proposition

Γ *is a presheaf, and satisfies the sheaf condition for "causally secured" covers.*

Running the sheaf approach script

Follow Abramsky–Brandenburger, replacing the “flat” event sheaf of local sections by the sheaf of strategies.

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An **empirical model** is a family $\{e_i\}_{i\in I}$, where $e_i \in \mathcal{D}_R\Gamma(C_i)$, subject to the usual compatibility conditions: for all i, j, $e_i|_{C_i\cap C_j} = e_j|_{C_i\cap C_j}$.
Thus e_i assigns a probability to each extensional strategy over M_{C_i}.

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We can show that this recovers

- Standard "flat" contextuality when the enabling is trivial (all measurements initially enabled)
- The Gogioso–Pinzani theory of contextuality for causal Bell scenarios

Experimenter strategies and adaptive computation

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Future choices of the Experimenter can then depend on Nature's responses, allowing for adaptive protocols. We can use Experimenter strategies to capture adaptive MBQC.

The Big Picture

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This provides a basis for exploring a wide range of phenomena.

Where we are

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Plenty left to do!

Thank you for your attention!

Questions...

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Anders–Browne revisited

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First, we have the description of the standard GHZ construction. This is given by a flat measurement scenario with $X = \{A_i, B_j, C_k \mid i, j, k \in \{0,1\}\}$, and $O_x = \{0,1\}$ for all $x \in X$.

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The maximal compatible sets of measurements are all sets of the form $\{A_i, B_j, C_k\}$ with $i, j, k \in \{0, 1\}$, i.e. a choice of one measurement per each site or agent. We regard each measurement as initially enabled. The N-strategies for this scenario form the usual sections assigning an outcome to each choice of measurement for each site, and the GHZ model assigns distributions on these strategies as in the table shown previously.

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To get the Anders–Browne construction, we consider the E-strategy which initially allows any A or B measurement to be performed, and after a history $\{(A_i, o_1), (B_j, o_2)\}$ chooses the C-measurement $C_{i \oplus j}$. Playing this against the GHZ model results in a strategy that computes the OR function with probability 1.

Anders-Browne ctd

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The E-strategy implements the first OR gate as above, with any B' measurement also enabled, being a free input. After that, the A'-measurement can be determined: after a history containing $\{(A_i, o_1), (B_j, o_2), (C_{i \oplus j}, o_3)\}$, the E-strategy chooses the A'-measurement $A'_{o_1 \oplus o_2 \oplus o_3}$. The second OR gate is then implemented like the first.

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Note that the choice of A'-measurement depends not only on previous measurement choices, but on outcomes provided by Nature.