Contextuality in logical form

Duality for transitive partial CABAs

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In a nutshell

$\boxed{\text{Generalise Tarski duality to partial Boolean algebras}}$

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- Duality between **CABA** and **Set** (Tarski, 1935)
  - Simplest of dualities relating algebra and topology
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  - In logic, between syntax and semantics
- partial Boolean algebras (Kochen & Specker, 1965)
  - Algebraic-logical setting for **contextuality**
  - A key signature of **nonclassicality** in quantum theory
  - Includes non-locality (Bell's theorem) as a special case
  - Key role in many instances of **quantum computational advantage**: magic state distillation, MBQC, shallow circuits, VQE, . . .

The mirror of mathematics

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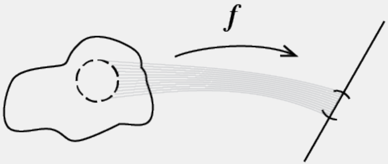

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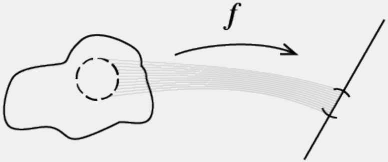

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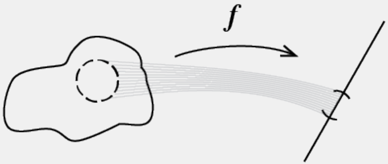

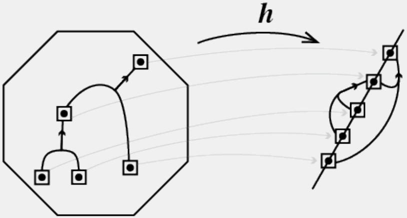

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Here, I mean **commutativity** in a loose, informal sense.
For lattices, this would be **distributivity** (think: idempotents of a ring).

The logic of quantum theory

From classical to quantum

John von Neumann (1932), ‘*Mathematische Grundlagen der Quantenmechanik*’.

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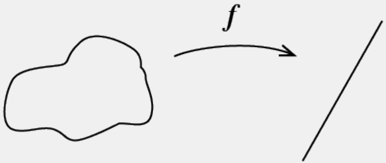

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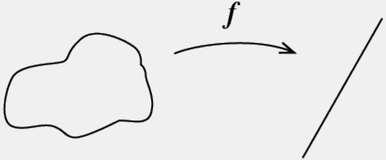

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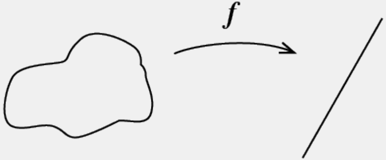

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- Quantum properties or propositions are **projectors** (dichotomic measurements):

$$p : \mathcal{H} \longrightarrow \mathcal{H} \qquad \text{s.t.} \qquad p = p^\dagger = p^2$$

which correspond to closed subspaces of $\mathcal{H}$.

From states to properties

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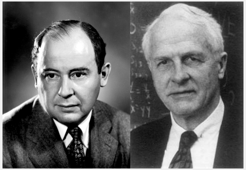

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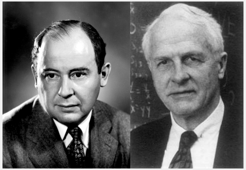

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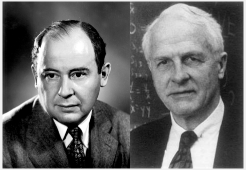

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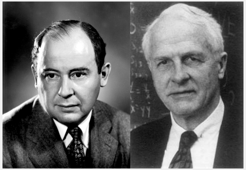

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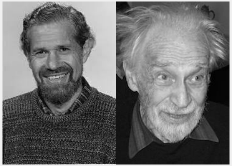
- Distributivity fails: $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$.
- Taking the *phenomenological* requirement seriously:
  in QM, only **commuting** measurements can be performed together.

  So, what is the operational meaning of $p \wedge q$, when p and q **do not commute**?



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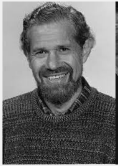
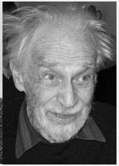

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- Only admit physically meaningful operations,
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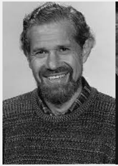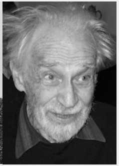

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Kochen (2015), '*A reconstruction of quantum mechanics*'.

- Kochen develops a large part of foundations of quantum theory in this framework.

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- When A, B, C with $C = AB$ are jointly measured on **any** quantum state, the observed outcomes a, b, c satisfy $c = ab$.

- More generally, for $A_1, \ldots, A_n$ pairwise commuting and any Borel $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, then $f(A_1, \ldots, A_n)$ commutes with all A_i and eigenvalues satisty the same functional relation.

The essence of contextuality

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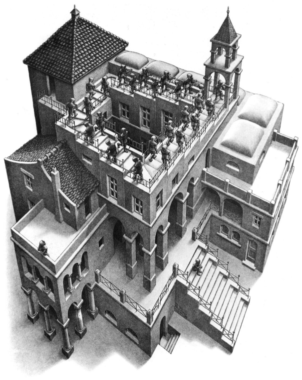

M. C. Escher, *Ascending and Descending*

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Local consistency

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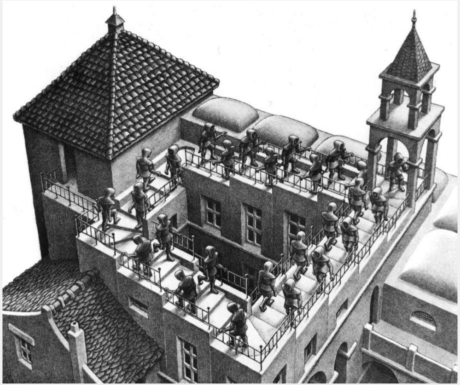

Local consistency *but* **Global inconsistency**

Partial Boolean algebras

Boolean algebras

Boolean algebra $\langle A, 0, 1, \neg, \vee, \wedge \rangle$:

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satisfying the usual axioms: $\langle A, \vee, 0 \rangle$ and $\langle A, \wedge, 1 \rangle$ are commutative monoids,
$\vee$ and $\wedge$ distribute over each other,
$a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

E.g.: $\langle \mathcal{P}(X), \varnothing, X, \cup, \cap \rangle$, in particular $\mathbf{2} = \{0, 1\} \cong \mathcal{P}(\{\star\})$.

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- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
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such that every set S of pairwise-commeasurable elements is contained in a set T of pairwise-commeasurable elements which is a Boolean algebra under the restriction of the operations.

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Conjunction, i.e. meet of projectors, becomes partial, defined only on **commuting** projectors.

A more concrete formulation of the defining axioms is:

- operations preserve commeasurability: for each n-ary operation f,

$$\frac{a_1 \odot c, \ \ldots, \ a_n \odot c}{f(a_1, \ldots, a_n) \odot c}$$

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- for any triple a, b, c of pairwise-commeasurable elements, the axioms of Boolean algebra are satisfied, e.g.

$$\frac{a \odot b}{a \wedge b = b \wedge a} \qquad \frac{a \odot b, \ a \odot c, \ b \odot c}{a \wedge (b \vee c) \ = \ (a \wedge b) \vee (a \wedge c)}$$

The category pBA

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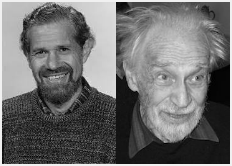
Abramsky & B (2021), ‘*The logic of contextality*’.

- We give a direct construction of colimits.
- More generally, we show how to freely generate from a given partial Boolean algebra A a new one satisfying prescribed additional commeasurability relations $\circ$, denoted $A[\circledcirc]$.

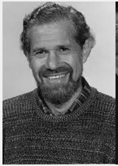
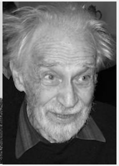
Contextuality, or the Kochen–Specker theorem



Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathrm{P}(\mathcal{H})$ its pBA of projectors.

Contextuality, or the Kochen–Specker theorem

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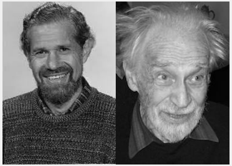

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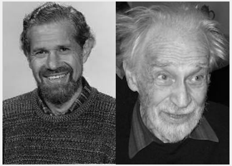

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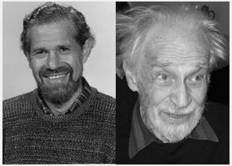

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- Spectrum of a pBA cannot have *points*. . .

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We could say that such a diagram is “implicitly contradictory”, and in trying to combine all the information in a colimit, we obtain the manifestly contradictory **1**.

At the borders of paradox

- There is a Boolean term $\varphi(\vec{x})$ with $\varphi(\vec{x}) \equiv_{\mathrm{Bool}} 0$ and an assignment $\vec{x} \mapsto \vec{a}$ such that $\varphi(\vec{a})$ is well-defined and equals 1.

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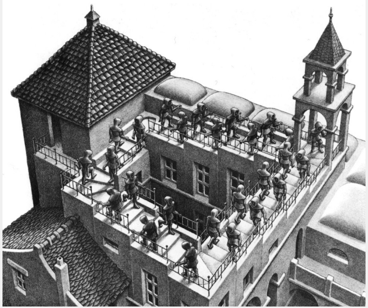

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Let A be a partial Boolean algebra. The following are equivalent:

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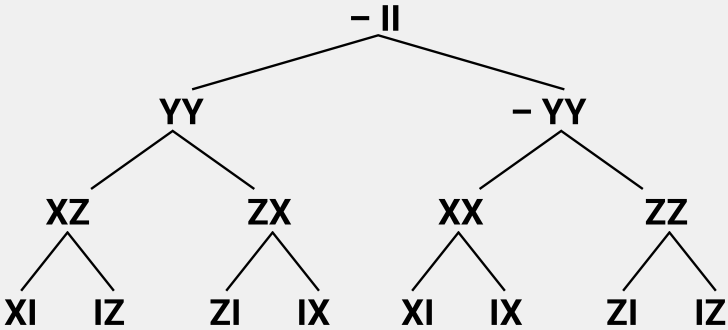

$$\langle \{0, 1\}, \oplus \rangle \quad \longleftrightarrow \quad \langle \{1, -1\}, \cdot \rangle$$

No-go theorems for noncommutative dualities

- Reyes (2012)
  - Any extension of Zariski spectrum to a functor $\mathbf{Rng}^{\mathrm{op}} \longrightarrow \mathbf{Top}$ trivialises on $\mathbb{M}_n(\mathbb{C})$ ($n \geq 3$).
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'What is proved by impossibility proofs is lack of imagination.' – John S. Bell

Summary of results

Duality for partial CABAs: key idea

- Replace **sets** by certain **graphs**.
- Vertices are *possible worlds of maximal information*.
- Adjacency represents **exclusivity**.
- It generalises $\neq$, thus sets embed as **complete graphs**.

- These exclusivity graphs are the 'non-commutative' spaces in this duality.

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- The partial algebra is reconstructed as equivalence classes of **cliques**, or double-neighbourhood closures of cliques.

- Morphisms of exclusivity graphs are certain **relations**, generalising **functional** ones from Tarski duality.

Tarski duality

$$
\begin{tikzcd}
\mathbf{FinBA} \arrow[rr, shift left=2, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left=2, "U"] & & \mathbf{FinSet}^{\mathrm{op}} \arrow[ll, shift left=2, "\mathcal{P}"] \arrow[dl, shift left=2, "\mathcal{P}"] \\
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& \mathbf{FinRGph} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "\mathcal{V}"] &
\end{tikzcd}
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\end{tikzcd}$$

Recap: Tarski duality

Partial order

Let A be a Boolean algebra.

Definition

For $a, b \in A$, we write $a \leq b$ when one (hence all) of the following equivalent conditions hold:

- $a \wedge b = a$
- $a \vee b = b$
- $a \wedge \neg b = 0$
- $\neg a \vee b = 1$

$\leq$ is a partial order.

It determines A as a Boolean algebra: e.g. $\vee$ (resp. $\wedge$) is supremum (resp. infimum) wrt $\leq$.

Definition (Complete Boolean algebra)

A Boolean algebra A is said to be **complete** if any subset of elements $S \subseteq A$ has a supremum $\bigvee S$ in A (and consequently an infimum $\bigwedge S$, too). It thus has additional operations

$$\bigwedge, \bigvee : \mathcal{P}(A) \longrightarrow A \ .$$

Definition (Atomic Boolean algebra)

An **atom** of a Boolean algebra is a minimal non-zero element, i.e. an element $x \neq 0$ such that $a \leq x$ implies $a = 0$ or $a = x$.

A Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

A **CABA** is a complete, atomic Boolean algebra.

Example

Any finite Boolean algebra is trivially a CABA.

The powerset $\mathcal{P}(X)$ of an arbitrary set X is a CABA.

- completeness: closed under arbitrary unions
- atoms: singletons $\{x\}$ for $x \in X$

This is in fact the 'only' (up to iso) example.

Proposition

In a CABA, every element is the join of the atoms below it:

$$a = \bigvee U_a \quad \textit{where } U_a := \{x \in A \mid x \textit{ is an atom and } x \leq a\}\,.$$

Proof.

Suppose $a \not\leq \bigvee U_a$, i.e. $a \wedge \neg \bigvee U_a \neq 0$. Atomicity implies there's an atom $x \leq a \wedge \neg \bigvee U_a$. On the one hand, $x \leq \neg \bigvee U_a$. On the other, $x \leq a$, i.e. $x \in U_a$, hence $x \leq \bigvee U_a$. Hence $x = 0$. ↯ □

Tarski duality

$$\begin{tikzcd} \mathbf{CABA} \arrow[rr, bend right, "\mathrm{At}"'] \arrow[rr, phantom, "\cong"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, bend right, "\mathcal{P}"'] \end{tikzcd}$$

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$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{CABA}$ is the contravariant powerset functor:

- on objects: a set X is mapped to its powerset $\mathcal{P}X$ (a CABA).
- on morphisms: a function $f : X \longrightarrow Y$ yields a complete Boolean algebra homomorphism

$$\begin{aligned} \mathcal{P}(f) : \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X) \\ (T \subseteq Y) &\longmapsto f^{-1}(T) = \{x \in X \mid f(x) \in T\} \end{aligned}$$

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$\mathsf{At} : \mathbf{CABA}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is defined as follows:

- on objects: a CABA A is mapped to its set of atoms.
- on morphisms: a complete Boolean homomorphism $h : A \longrightarrow B$ yields a function

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mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

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Tarski duality

Lemma

Let $h : A \longrightarrow B$ *in* **CABA**. *For all* $y \in \mathrm{At}(A)$, *there is a unique* $x \in \mathrm{At}(A)$ *with* $y \leq h(x)$.

Proof.

Facts about atoms in any BA:

- If $x \neq x'$ are atoms, then $x \wedge_A x' = 0$.
- If x is an atom and $x \leq \bigvee S$, there is $a \in S$ with $x \leq a$.

Existence

A complete atomic implies $1_A = \bigvee \mathrm{At}(A)$. Hence,

$$1_B = h(1_A) = h(\bigvee \mathrm{At}(A)) = \bigvee \{h(x) \mid x \in \mathrm{At}(A)\}$$

Since $y \leq 1_B$, we conclude $y \leq h(x)$ for some $x \in \mathrm{At}(A)$.

Uniqueness

If $y \leq h(x)$ and $y \leq h(x')$, then $y \leq h(x) \wedge_B h(x') = h(x \wedge x')$, hence $x = x'$. $\square$

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- Given a CABA A, the isomorphism $A \cong \mathcal{P}(\mathrm{At}(A))$ maps $a \in A$ to the set of elements

$$U_a = \{x \in \mathrm{At}(A) \mid x \leq a\} .$$

  A property is identified with the set of possible worlds in which it holds.

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- Given a set X, the bijection $X \cong \mathrm{At}(\mathcal{P}(X))$ maps $x \in X$ to the singleton $\{x\}$, which is an atom of $\mathcal{P}(X)$.

  A possible world is identified with its characteristic property (which fully determines it).

Transitive partial CABAs

Logical exclusivity principle

Let A be a partial Boolean algebra.

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Definition

A is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are logically exclusive are also commeasurable, i.e. if $\perp \subseteq \odot$.

Logical exclusivity principle

Note that $\leq$ is always reflexive and antisymmetric.

Definition

A partial Boolean algebra is said to be **transitive** if $a \leq b$ and $b \leq c$ implies $a \leq c$, i.e. $\leq$ is (globally) a partial order on A.

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A partial Boolean algebra satisfies LEP if and only if it is transitive.

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We restrict atention to partial Boolean algebras satisfying LEP in this talk.

Theorem

The category **epBA** *of partial Boolean algebras satisfying LEP is a reflective subcategory of* **pBA***, i.e. the inclusion functor* $I : \mathbf{epBA} \longrightarrow \mathbf{pBA}$ *has a left adjoint* $X : \mathbf{pBA} \longrightarrow \mathbf{epBA}$.

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

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Partial CABAs

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

satisfying the following property: any set $S \in \odot$ is contained in a set $T \in \odot$ which forms a complete Boolean algebra under the restriction of the operations.

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Partial CABAs from their graphs of atoms

Graph

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A **graph** $(X, \#)$ is a set equipped with a symmetric irreflexive relation.

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Given a vertex $x \in X$ and sets of vertices $S, T \subset X$, we write:

- $x \# S$ when for all $y \in S$, $x \# y$;
- $S \# T$ when for all $x \in S$ and $y \in T$, $x \# y$;
- $x^{\#} := \{y \in X \mid y\#x\}$ for the neighbourhood of the vertex x;
- $S^{\#} := \bigcap_{x \in S} x^{\#} = \{y \in X \mid y \# S\}$ for the common neighbourhood of the set S.

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A **clique** is a set of pairwise-adjacent vertices, i.e. a set $K \subset X$ with $x \# K \setminus \{x\}$ for all $x \in K$.

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

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Recall that in a CABA, any element is uniquely written as a join of atoms, viz. $a = \bigvee U_a$ with

$$U_a := \{x \in \mathsf{At}(A) \mid x \leq a\}$$

In a pBA, U_a may not be pairwise commeasurable, hence their join need not even be defined.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

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Proof.

Let $a \in A$ and K be a clique of $\mathrm{At}(A)$ maximal in U_a.

Being a clique in $\mathrm{At}(A)$, $K \in \bigodot$ and thus $\bigvee K$ is defined.

Since $K \subset U_a$, all $k \in K$ satisfy $k \leq a$ and in particular $k \odot a$. Hence, $K \cup \{a\} \in \bigodot$, implying that it is contained in a complete Boolean subalgebra. Consequently, $\bigvee K \leq a$.

Now, suppose $a \not\leq \bigvee K$, i.e. $a \wedge \neg \bigvee K \neq 0$. Then atomicity implies there is an atom $x \leq a \wedge \neg \bigvee K$. By transitivity, $x \leq a$ and $x \leq \neg k$ (hence $x \perp k$) for all $k \in K$. This makes $K \cup \{x\}$ a clique of atoms contained in U_a, contradicting maximality of K. □

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Proposition

Let K and L be cliques in $\mathrm{At}(A)$. Then $\bigvee K \leq \bigvee L$ *iff* $L^{\#} \subseteq K^{\#}$ *iff* $K \subseteq L^{\#\#}$.

Corollary

$\bigvee K = \bigvee L$ *iff* $K^{\#} = L^{\#}$.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \ :\Leftrightarrow\ K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

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- $[K] \vee [L] = [K' \cup L']$.
- $[K] \wedge [L] = [K' \cap L']$.

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Which conditions on a graph $(X, \#)$ allow for such reconstruction?

Exhaustive exclusivity graphs

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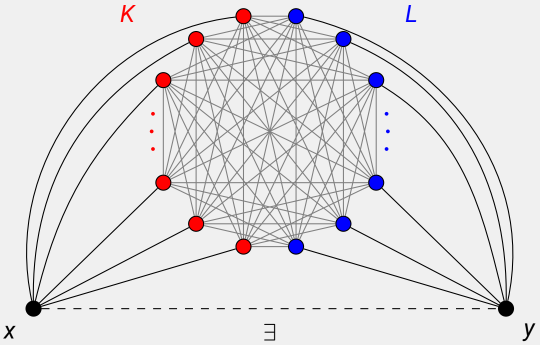
Definition

An **exhaustive exclusivity graph** is a graph $(X, \#)$ such that for K, L cliques and $x, y \in X$:

1. If $K \sqcup L$ is a maximal clique, then $K^\# \,\#\, L^\#$, i.e. $x \# K$ and $y \# L$ implies $x \# y$.
2. $x^\# \subseteq y^\#$ implies $x = y$.



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A helpful intuition is to see these as generalising sets with a $\neq$ relation (the complete graph).

- A graph is symmetric and irreflexive.
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- A graph is symmetric and irreflexive.
- To be an inequivalence relation, we need cotransitivity: $x \# z$ implies $x \# y$ or $x \# z$.
- Condition 1 is a weaker version of cotransitivity.
- Condition 2 eliminates redundant elements: cotransitive + 2 imply $\neq$.

Graph of atoms is an exhaustive exclusivity graph

Proposition

Let A be a partial Boolean algebra. Then $\mathrm{At}(A)$ is an exhaustive exclusivity graph.

Proof.

Let $K, L \subset X$ such that $K \sqcup L$ is a maximal clique, and let x, y be atoms of A.

Write $c := \bigvee K = \neg \bigvee L$.

$x \# K$ means $x \leq \neg \bigvee K = \neg c$ and $x \# L$ means $y \leq \neg \bigvee L = c$.

By transitivity, we conclude that $x \odot y$, hence $x \perp y$. □

The ‘clique powerset’ of an exclusivity graph

Proposition

Let K, L be cliques in an exhaustive exclusivity graph. The following are equivalent:

- $[K] \odot [L]$, *i.e. there exist* K', L' *with* $K' \equiv K$ *and* $L' \equiv L$ *such that* $K' \cup L'$ *is a clique.*
- *The four sets*
$$K^{\#\#} \cap L^{\#\#}, \quad K^{\#\#} \cap L^{\#}, \quad K^{\#} \cap L^{\#\#}, \quad K^{\#} \cap L^{\#},$$
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Choose maximal cliques

$$M_{11} \subset K^{\#\#} \cap L^{\#\#}, \quad M_{10} \subset K^{\#\#} \cap L^{\#}, \quad M_{01} \subset K^{\#} \cap L^{\#\#}, \quad M_{00} \subset K^{\#} \cap L^{\#},$$

and set

$$[K] \wedge [L] := [M_{11}] \qquad \text{and} \qquad [K] \vee [L] := [M_{11} \cup M_{10} \cup M_{11}] \,.$$

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Proposition

Let K, L, M be cliques in an exclusivity graph with $[K] \odot [L]$, $[K] \odot [M]$, $[L] \odot [M]$. The eight sets

$$K^{\square_1} \cap L^{\square_2} \cap M^{\square_3}\,, \qquad \square_i \in \{\#, \#\#\}$$

are pairwise non-intersecting and have empty common neighbourhood.

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Proposition

Let $\{K_i\}_{i\in I}$ be a set of cliques in an exclusivity graph whose equivalence classes are pairwise commeasurable. The sets

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Morphisms

Morphisms of exhaustive exclusivity graphs

$\boxed{\text{What about morphisms?}}$

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \mathrel{R} y$, $x' \mathrel{R} y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
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3. trivialises.

Morphisms of exclusivity graphs and pCABA homomorphisms

Proposition

Let A and B be transitive partial CABAs. Given $h : A \longrightarrow B$ a partial complete Boolean algebra homomorphism, the relation $R_h : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$ given by

$$xR_hy \qquad \textit{iff} \qquad x \leq h(y)$$

is a morphism of exclusivity graphs. Moreover, the assignment $h \mapsto R_h$ is functorial.

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Proposition

For any A and B be transitive partial CABAs, $\mathbf{epCABA}(A, B) \cong \mathbf{XGph}(\mathrm{At}(B), \mathrm{At}(A))$.

Revisiting contextuality

Global points

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Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

i.e. a subset of atoms of A satisfying:

1. it is an independent (or stable) set
2. it is a maximal clique transversal, i.e. it has a vertex in each maximal clique

Outlook

Reconstruction via double-neighbourhood-closed sets

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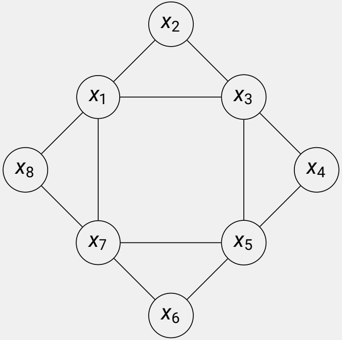

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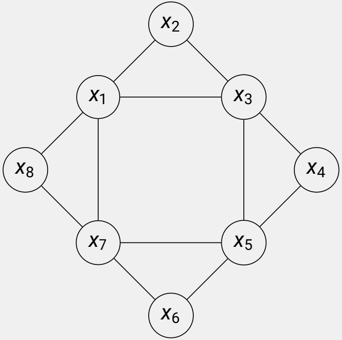

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Can we characterise which ${}^{\#\#}$-closed sets arise from cliques?

Free-forgetful adjunction for CABAs

$$
\begin{tikzcd}
\mathbf{CABA} \arrow[rr, shift left, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{Set}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{P}"] \arrow[dl, shift left, "\mathcal{P}"] \\
& \mathbf{Set} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{P}"] \arrow[ur, phantom, "\dashv"] &
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& \mathbf{Set} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\top"] \arrow[ur, shift left, "\mathcal{P}"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}
$$

- Under the duality, it corresponds to the contravariant powerset self-adjunction.
- It gives the construction of the free CABA as a double powerset.

Free-forgetful adjunction for partial CABAs

$$
\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \\
& \mathbf{RGph} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{V}"] \arrow[ur, phantom, "\vdash"] &
\end{tikzcd}
$$

Free-forgetful adjunction for partial CABAs

$$
\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left=2, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left=2, "U"] \arrow[dr, phantom, "\dashv"] & & \mathbf{XGph}^{\mathsf{op}} \arrow[ll, shift left=2, "\mathcal{K}"] \arrow[dl, shift left=2, "\mathcal{K}"] \arrow[dl, phantom, "\dashv"] \\
& \mathbf{RGph} \arrow[ul, shift left=2, "F"] \arrow[ur, shift left=2, "\mathcal{V}"] &
\end{tikzcd}
$$

- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$

Free-forgetful adjunction for partial CABAs

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\mathbf{epCABA} \arrow[rr, shift left=2, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, shift left=2, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \\
& \mathbf{RGph} \arrow[ul, shift left, "F"] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "\mathcal{V}"] \arrow[ur, phantom, "\vdash"] &
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- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$
- Under duality it corresponds to adjunction between **compatibility** and **exclusivity** graphs.

Free-forgetful adjunction for partial CABAs

$$\begin{tikzcd}
\mathbf{epCABA} \arrow[rr, shift left, "\mathsf{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] \arrow[dr, phantom, "\dashv"] & & \mathbf{XGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \arrow[dl, phantom, "\vdash"] \\
& \mathbf{RGph} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "\mathcal{V}"] &
\end{tikzcd}$$

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$\boxed{\text{Can we give a concrete construction of the free CABA?}}$

Free-forgetful adjunction for partial CABAs

$$
\begin{tikzcd}
\textbf{epCABA} \arrow[rr, shift left, "\mathrm{At}"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift left, "U"] \arrow[dr, phantom, "\top"] & & \textbf{XGph}^{\mathrm{op}} \arrow[ll, shift left, "\mathcal{K}"] \arrow[dl, shift left, "\mathcal{K}"] \arrow[dl, phantom, "\perp"] \\
& \textbf{RGph} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "\mathcal{V}"] &
\end{tikzcd}
$$

- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$
- Under duality it corresponds to adjunction between **compatibility** and **exclusivity** graphs.

Can we give a concrete construction of the free CABA?

- First attempt: Given $\langle P, \odot \rangle$ build a graph with vertices $\langle C, \gamma : C \longrightarrow \{0,1\} \rangle$ where C maximal compatible set, and edges $\langle C, \gamma \rangle \# \langle D, \delta \rangle$ iff $\exists x \in C \cap D.\ \gamma(x) \neq \delta(x)$.

The spatial landscape of partial Boolean algebra

$$
\begin{tikzcd}
 & \mathbf{epDLat}_{\mathbf{bs}} \arrow[rr, tail] & & \mathbf{pDLat}_{\mathbf{bs}} \\
\mathbf{epDLat} \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pDLat} \arrow[ur, tail] & \\
 & \mathbf{epBA}_{\mathbf{bs}} \arrow[uu, hook] \arrow[rr, tail] & & \mathbf{pBA}_{\mathbf{bs}} \arrow[uu, hook] \\
\mathbf{epBA} \arrow[uu, hook] \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pBA} \arrow[uu, hook] \arrow[ur, tail] & \\
 & \mathbf{epCABA}_{\mathbf{bs}} \arrow[uu, hook] \arrow[rr, tail] & & \mathbf{pCABA}_{\mathbf{bs}} \arrow[uu, hook] \\
\mathbf{epCABA} \arrow[uu, hook] \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pCABA} \arrow[uu, hook] \arrow[ur, tail] &
\end{tikzcd}
$$

The spatial landscape of partial Boolean algebra

$$\begin{tikzcd}
 & \mathbf{epDLat_{bs}} \arrow[rr, tail] & & \mathbf{pDLat_{bs}} \\
\mathbf{epDLat} \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pDLat} \arrow[ur, tail] & \\
 & \mathbf{epBA_{bs}} \arrow[uu, hook] \arrow[rr, tail] & & \mathbf{pBA_{bs}} \arrow[uu, hook] \\
\mathbf{epBA} \arrow[uu, hook] \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pBA} \arrow[uu, hook] \arrow[ur, tail] & \\
 & \mathbf{epCABA_{bs}} \arrow[uu, hook] \arrow[rr, tail] & & \mathbf{pCABA_{bs}} \arrow[uu, hook] \\
\mathbf{epCABA} \arrow[uu, hook] \arrow[ur, tail] \arrow[rr, tail, red] & & \mathbf{pCABA} \arrow[uu, hook] \arrow[ur, tail] &
\end{tikzcd}$$

- Drop transitivity / LEP

The spatial landscape of partial Boolean algebra

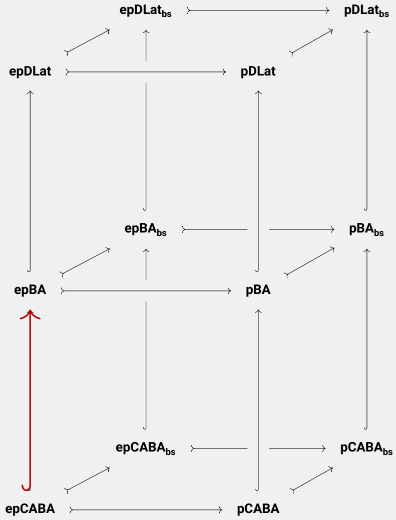
$$
\begin{tikzcd}
& \mathbf{epDLat}_{\mathbf{bs}} \arrow[rr, tail] & & \mathbf{pDLat}_{\mathbf{bs}} \\
\mathbf{epDLat} \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pDLat} \arrow[ur, tail] & \\
& \mathbf{epBA}_{\mathbf{bs}} \arrow[uu, hook] \arrow[rr, tail] & & \mathbf{pBA}_{\mathbf{bs}} \arrow[uu, hook] \\
\mathbf{epBA} \arrow[uu, hook] \arrow[ur, tail] \arrow[rr, tail] & & \mathbf{pBA} \arrow[uu, hook] \arrow[ur, tail] & \\
& \mathbf{epCABA}_{\mathbf{bs}} \arrow[uu, hook] \arrow[rr, tail] & & \mathbf{pCABA}_{\mathbf{bs}} \arrow[uu, hook] \\
\mathbf{epCABA} \arrow[uu, hook] \arrow[ur, tail, red] \arrow[rr, tail] & & \mathbf{pCABA} \arrow[uu, hook] \arrow[ur, tail] &
\end{tikzcd}
$$

- Drop transitivity / LEP
- Relax binary to simplicial compatibility

$$
\begin{tikzcd}
\mathbf{RGph} \arrow[rr, bend left, "\mathsf{Kl}"] \arrow[rr, bend right, "\iota"'] & & \mathbf{Simp} \arrow[ll, "\mathsf{sk}_1"'{name=S}]
\end{tikzcd}
\qquad \top \ \top
$$

$\rightsquigarrow$Czelakowski's *pBAs in a broader sense*

The spatial landscape of partial Boolean algebra



- Drop transitivity / LEP
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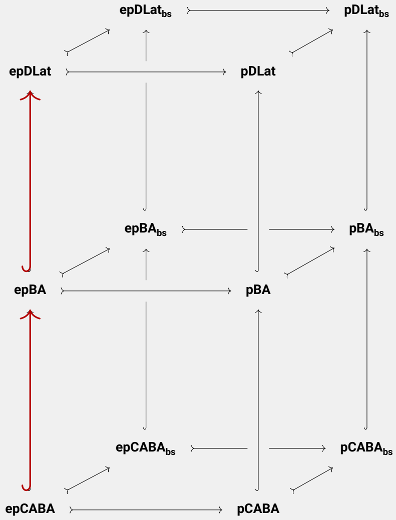
$$\begin{tikzcd} \mathbf{RGph} \arrow[rr, bend left, "\mathrm{Kl}"] \arrow[rr, bend right, "\iota"'] & \arrow[phantom, u, "\perp"] \arrow[phantom, d, "\perp"] & \mathbf{Simp} \arrow[ll, "\mathrm{sk}_1"'] \end{tikzcd}$$

$\rightsquigarrow$Czelakowski's *pBAs in a broader sense*

- Dropping completeness and atomicity (e.g. $\mathrm{P}(A)$ for vN algebra A with factor not of type I)

The spatial landscape of partial Boolean algebra



- Drop transitivity / LEP

- Relax binary to simplicial compatibility

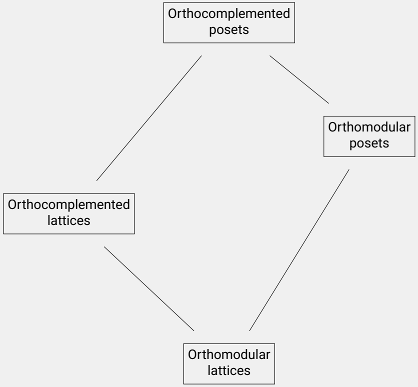
$$\begin{tikzcd} \mathbf{RGph} \arrow[r, bend left, "\mathrm{Kl}"] \arrow[r, bend right, "\iota"'] & \mathbf{Simp} \arrow[l, "\mathrm{sk}_1"] \end{tikzcd}$$

$\rightsquigarrow$Czelakowski's *pBAs in a broader sense*

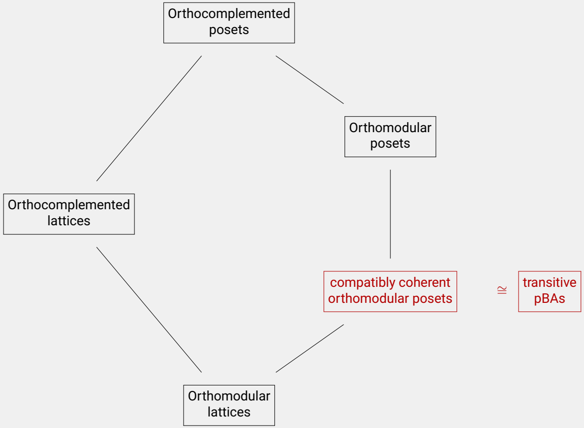
- Dropping completeness and atomicity
(e.g. $\mathrm{P}(A)$ for vN algebra A with factor not of type I)

$\rightsquigarrow$ analogues of Stone, Priestley, ...
Stone's motto: *'always topologise'* – but how?

The wider spatial landscape of ‘quantum’ logics

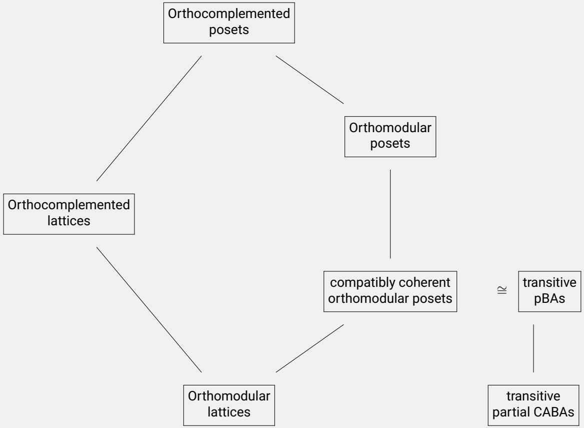


The wider spatial landscape of ‘quantum’ logics



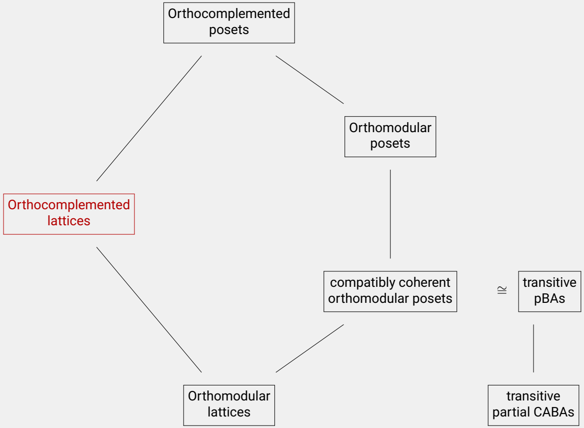
(Gudder, 1972)

The wider spatial landscape of 'quantum' logics



(Gudder, 1972)

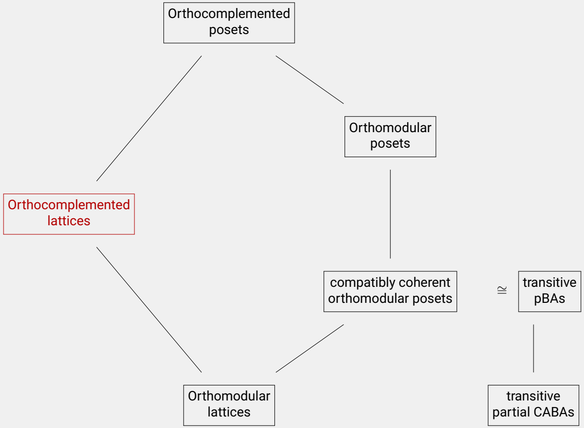
The wider spatial landscape of ‘quantum’ logics



(Gudder, 1972)

OLs $\leftrightsquigarrow$ Minimal quantum logic
(Dishkant, Goldblatt, Dalla Chiara, 1970s)

The wider spatial landscape of ‘quantum’ logics



(Gudder, 1972)

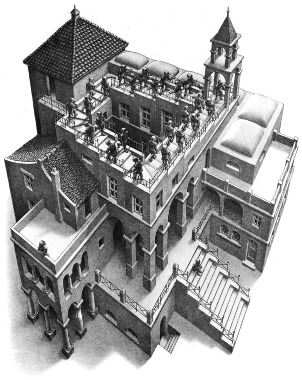
OLs $\leftrightsquigarrow$ Minimal quantum logic
(Dishkant, Goldblatt, Dalla Chiara, 1970s)

Stone representation for OLs
(Goldblatt, 1975)

- related to our construction
- all graphs, all nhood-regular sets
- nothing on morphisms

Towards noncommutative dualities?

- Can one find a more encompassing duality theory for ‘noncommutative’ or ‘quantum’ structures by viewing them through multiple partial classical snapshots?



Thank you for your attention!

Questions...

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