An introduction to contextuality and quantum advantage

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INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

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- A hypothesis: this is related to **non-classical** features of quantum mechancics.
- In this talk, we focus on **non-local** and **contextual** behaviours as quantum resources.
- Contextuality is a feature of **empirical data** that is a key signature of non-classicality.

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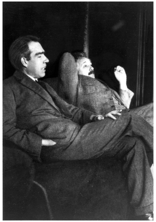

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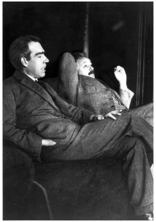

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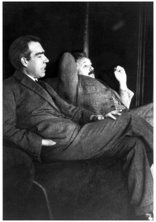

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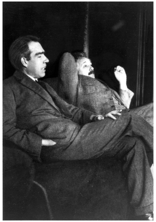

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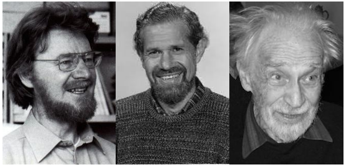
- Bell–Kochen–Specker (60s):
  Non-locality and contextuality as fundamental **empirical** phenomena rather than shortcomings of the formalism.



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- How can we make the most of quantum systems as informatic resources?
- How can we reason systematically and compositionally about them?
- What extra power do they offer vis-à-vis classical systems?

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- $\rightsquigarrow$ Renewed interest in quantum foundations.

Non-local games

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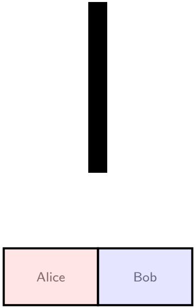
Alice and Bob cooperate in solving a task set by Verifier.

They may share prior information,



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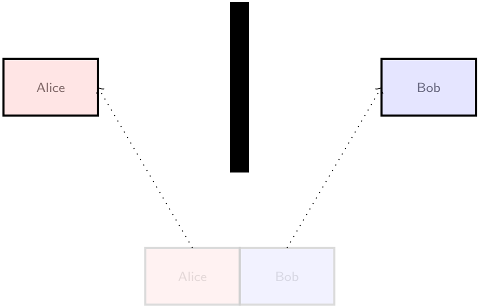

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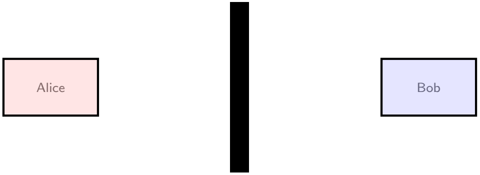

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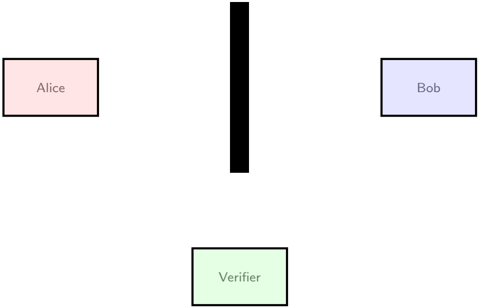

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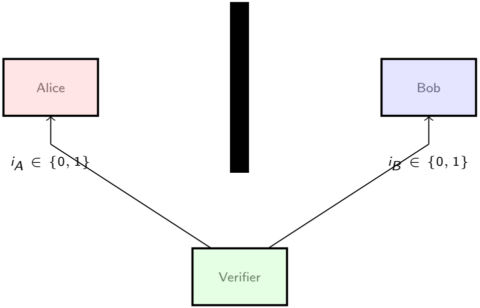

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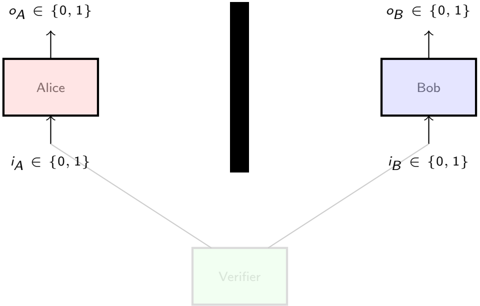

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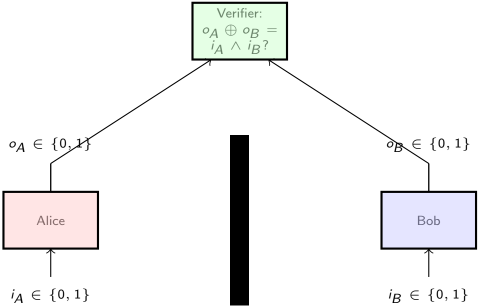

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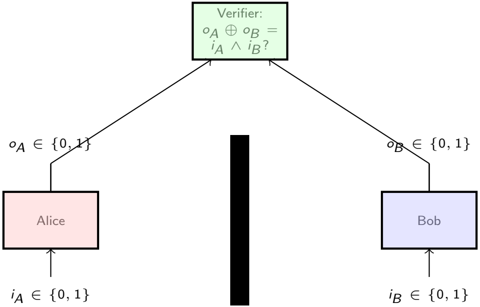

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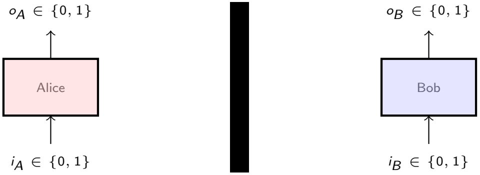

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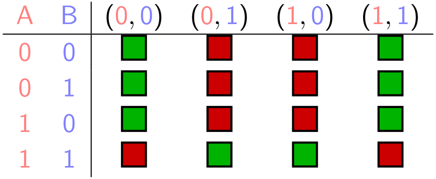

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- Can they do any better?

Using quantum resources

- Alice and Bob share a quantum state $|\psi\rangle$ on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ (e.g. two qubits).

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- Bob chooses to perform B_0 or B_1 on his side of the system depending on his input i_B, and chooses his output o_B according to the observed outcome.
- The probabilities are given by the Born rule

$$p(o_A, o_B | i_A, i_B) = \langle \psi | A_{i_A}^{o_B} \otimes B_{i_B}^{o_B} | \psi \rangle \ .$$

A quantum-realisable strategy

Sharing a pair of qubits and performing quantum measurements, Alice and Bob can realise:

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
0	0	$1/2$	0	0	$1/2$
0	1	$3/8$	$1/8$	$1/8$	$3/8$
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P(o_A \oplus o_B = i_A \wedge i_B) \;\; = \;\; & \frac{1}{4}\, P(o_A = o_B \mid i_A = 0, i_B = 0) + \frac{1}{4}\, P(o_A = o_B \mid i_A = 0, i_B = 1) \\
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In fact, one can achieve a winning probability of $\frac{2+\sqrt{2}}{4} \approx 0.85$!

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Logical Bell inequalities

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- Hence,

$$\sum_{i=1}^{N} p_i \leq N - 1 \quad .$$

Analysis of the Bell table

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_0	b_0	$1/2$	0	0	$1/2$
a_0	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_1	b_0	$3/8$	$1/8$	$1/8$	$3/8$
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\begin{array}{lcccccccc}
\phi_1 & = & a_0 & \leftrightarrow & b_0 & = & (a_0 \wedge b_0) & \vee & (\neg a_0 \wedge \neg b_0) \\
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$$
\begin{array}{ccccccccc}
\phi_1 & = & a_0 & \leftrightarrow & b_0 & = & (a_0 \wedge b_0) & \vee & (\neg a_0 \wedge \neg b_0) \\
\phi_2 & = & a_0 & \leftrightarrow & b_1 & = & (a_0 \wedge b_1) & \vee & (\neg a_0 \wedge \neg b_1) \\
\phi_3 & = & a_1 & \leftrightarrow & b_0 & = & (a_1 \wedge b_0) & \vee & (\neg a_1 \wedge \neg b_0) \\
\phi_4 & = & a_1 & \oplus & b_1 & = & (\neg a_1 \wedge b_1) & \vee & (a_1 \wedge \neg b_1) .
\end{array}
$$

These formulae are contradictory.

Analysis of the Bell table

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_0	b_0	$1/2$	0	0	$1/2$
a_0	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_1	b_0	$3/8$	$1/8$	$1/8$	$3/8$
a_1	b_1	$1/8$	$3/8$	$3/8$	$1/8$

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These formulae are contradictory. But $p_1 + p_2 + p_3 + p_4 = 3.25$. The inequality is violated by $1/4$.

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- So, what was our unwarranted assumption?
- That all variables could *in principle* be observed simultaneously,
- i.e. that one should be able to assign probabilities to empirically unobserved events such as $a_0 \wedge a_1$.

The essence of contextuality

- Not all properties may be observed at once.
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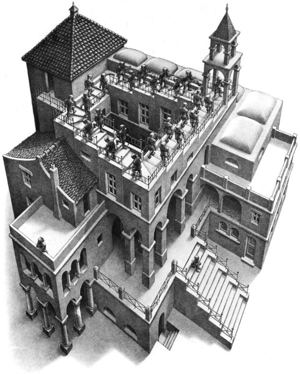

M. C. Escher, *Ascending and Descending*

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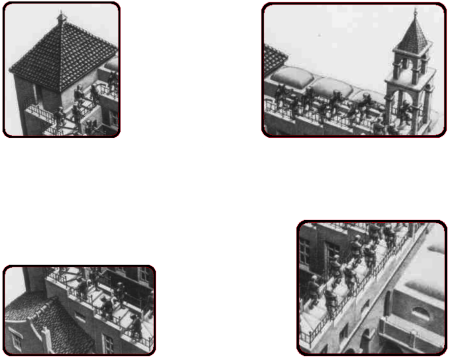

Local consistency

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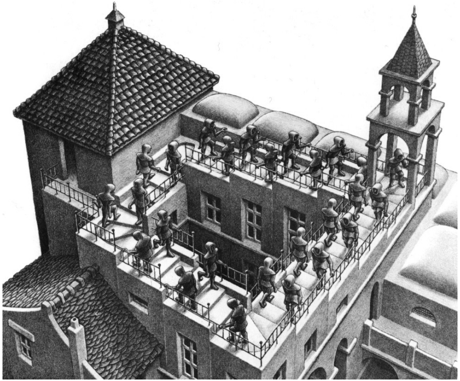

Local consistency *but* **Global inconsistency**

General framework for contextuality

'*The sheaf-theoretic structure of non-locality and contextuality*'

Abramsky & Brandenburger, New Journal of Physics, 2011.

'*Contextuality, cohomology, and paradox*'

Abramsky, B, Kishida, Lal, & Mansfield, CSL 2015.

(cf. Cabello–Severini–Winter, Acín–Fritz–Leverrier–Sainz)

Formalising empirical data

A **measurement scenario** $\mathbf{X} = \langle X, \Sigma, O \rangle$:

- X – a finite set of measurements
- Σ – a simplicial complex on X
  faces are called the **measurement contexts**
- $O = (O_x)_{x \in X}$ – for each $x \in X$ a non-empty set of possible outcomes O_x

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_0	b_0	$\square$	$\square$	$\square$	$\square$
a_0	b_1	$\square$	$\square$	$\square$	$\square$
a_1	b_0	$\square$	$\square$	$\square$	$\square$
a_1	b_1	$\square$	$\square$	$\square$	$\square$

$X = \{a_0, a_1, b_0, b_1\}, \; O_x = \{0, 1\}$

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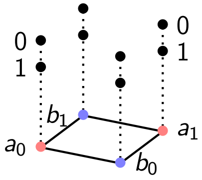

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An **empirical model** $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on $\mathbf{X}$:

- each $e_\sigma \in \mathrm{Prob}\left(\prod_{x \in \sigma} O_x\right)$ is a probability distribution over joint outcomes for σ.
- *generalised no-signalling* holds: for any $\sigma, \tau \in \Sigma$, if $\tau \subseteq \sigma$,

$$e_\sigma|_\tau = e_\tau$$

(i.e. marginals are well-defined)

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a_0	b_1	1/2	0	0	1/2
a_1	b_0	1/2	0	0	1/2
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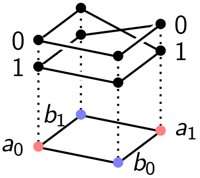

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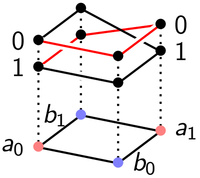

Contextuality

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The import of Bell's and Kochen–Spekker's theorems is that there are behaviours arising from quantum mechanics that are contextual.

Hierarchy of contextuality

Possibilistic collapse

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$$
\begin{array}{cc|cccc}
A & B & (0,0) & (0,1) & (1,0) & (1,1) \\
\hline
a_0 & b_0 & 1/2 & 0 & 0 & 1/2 \\
a_0 & b_1 & 3/8 & 1/8 & 1/8 & 3/8 \\
a_1 & b_0 & 3/8 & 1/8 & 1/8 & 3/8 \\
a_1 & b_1 & 1/8 & 3/8 & 3/8 & 1/8
\end{array}
\longmapsto
\begin{array}{cc|cccc}
A & B & (0,0) & (0,1) & (1,0) & (1,1) \\
\hline
a_0 & b_0 & 1 & 0 & 0 & 1 \\
a_0 & b_1 & 1 & 1 & 1 & 1 \\
a_1 & b_0 & 1 & 1 & 1 & 1 \\
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\end{array}
$$

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Hardy model

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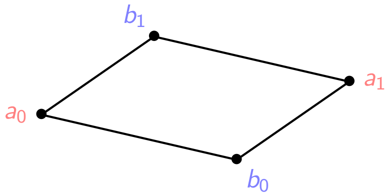

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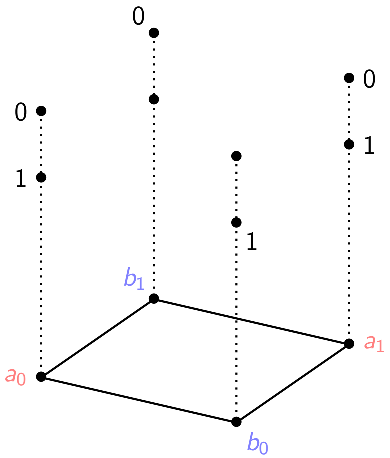

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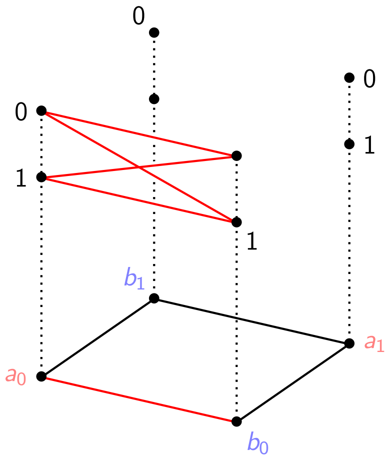

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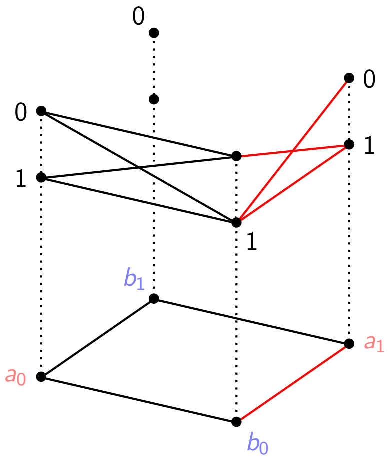
$a_1 \vee b_0$



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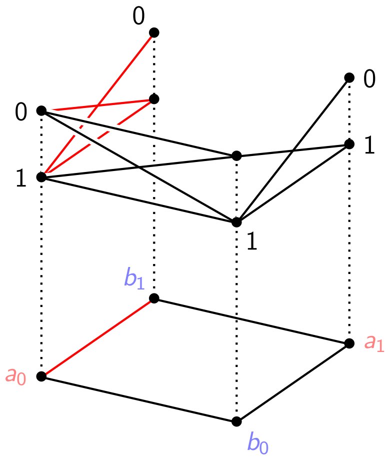

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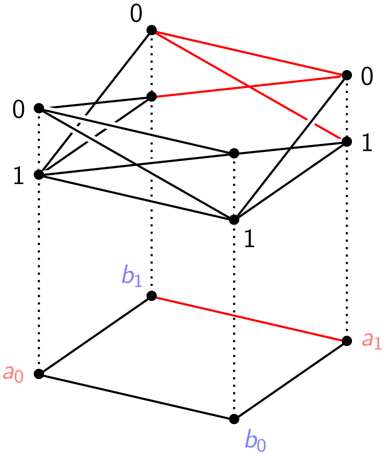

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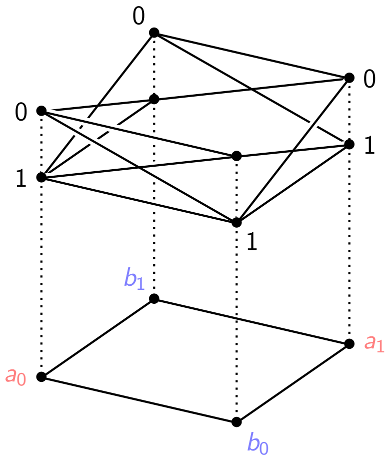

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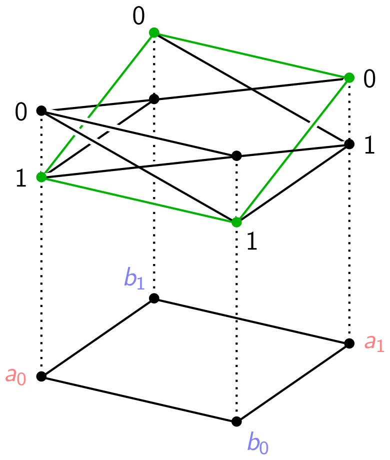

There are some global sections,

Classical assignment: $[a_0 \mapsto 1, a_1 \mapsto 0, b_0 \mapsto 1, b_1 \mapsto 0]$

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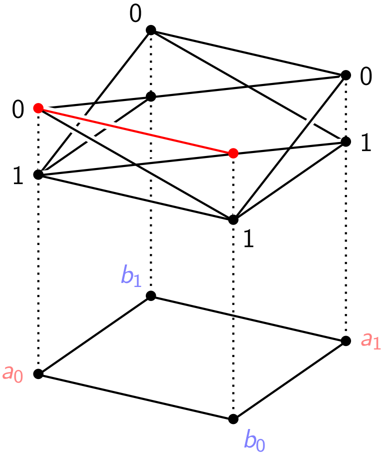
There are some global sections, but . . .



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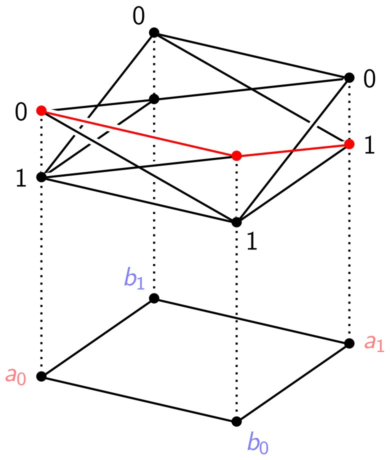

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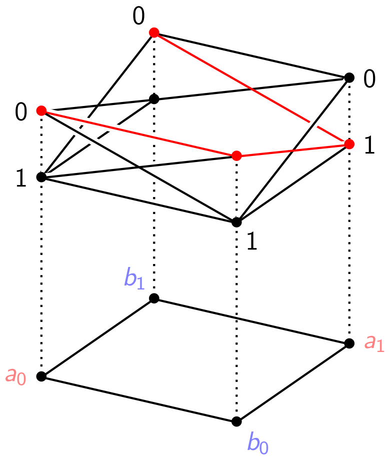

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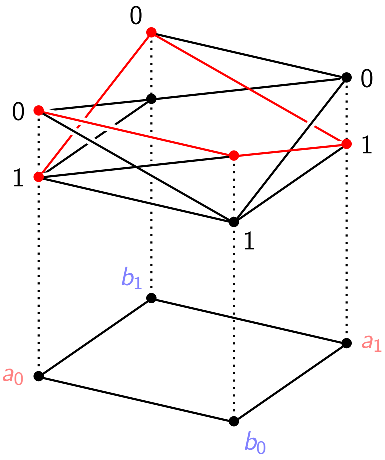

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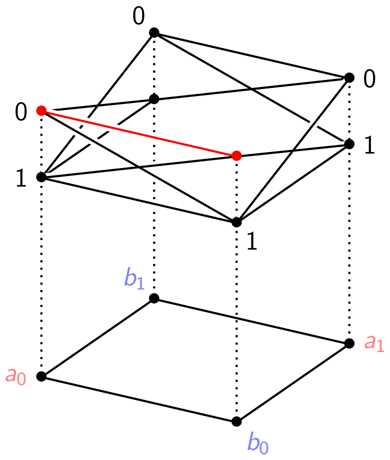

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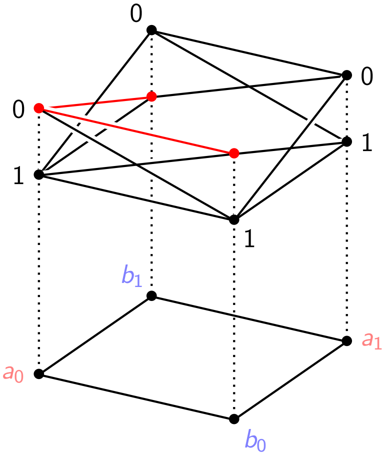

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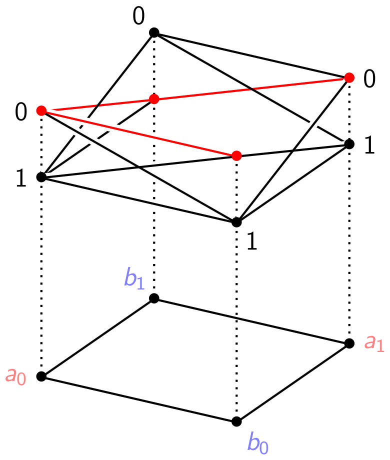

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a_1	b_0	0	1	1	1
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$a_1 \vee b_0$ $\quad$ $a_0 \vee b_1$ $\quad$ $\neg(a_1 \wedge b_1)$

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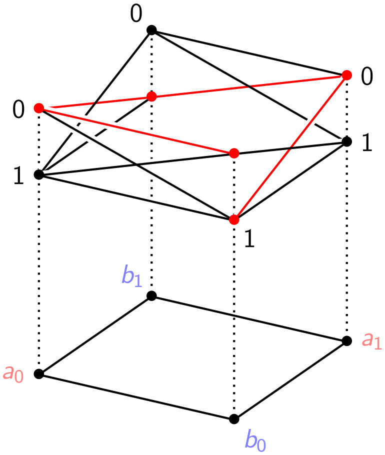
There are some global sections, but . . .



Hierarchy of contextuality

Hardy model

A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_0	b_0	1	1	1	1
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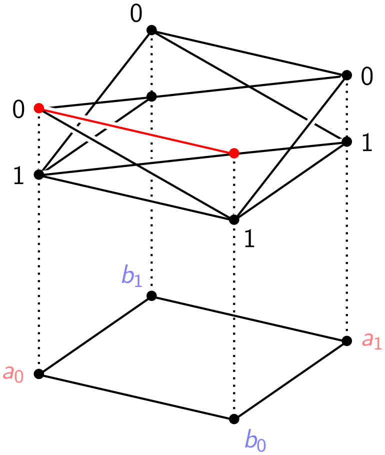

There are some global sections, but . . .

Logical contextuality: Not all sections extend to global ones.

Hierarchy of contextuality

Popescu–Rohrlich box

A	B	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
a_0	b_0	1	0	0	1
a_0	b_1	1	0	0	1
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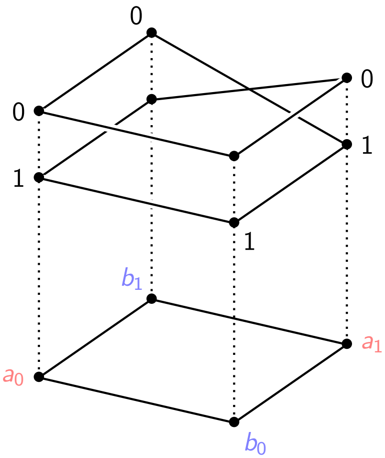

Strong contextuality:
no event can be extended to a global assignment.

$a_0 \leftrightarrow b_0$ $\quad a_0 \leftrightarrow b_1$ $\quad a_1 \leftrightarrow b_0$ $\quad a_1 \oplus b_1$

Peres–Mermin magic square

Magic square:

- Fill with 0s and 1s
- rows and first two columns: even parity
- last column: odd parity

Peres–Mermin magic square

A	B	C
D	E	F
G	H	I

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System of linear equations over $\mathbb{Z}_2$:

$$\begin{aligned} A \oplus B \oplus C &= 0 \\ D \oplus E \oplus F &= 0 \\ G \oplus H \oplus I &= 0 \end{aligned} \qquad \begin{aligned} A \oplus D \oplus G &= 0 \\ B \oplus E \oplus H &= 0 \\ C \oplus F \oplus I &= 1 \end{aligned}$$

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Clearly, this is not satisfiable in $\mathbb{Z}_2$. But it has a "quantum solution"!

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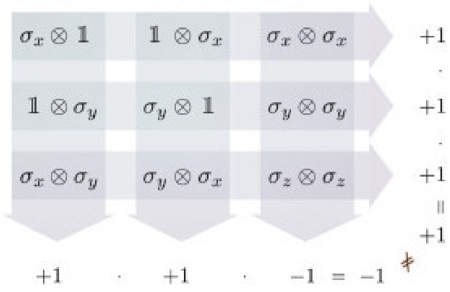

Quantifying contextuality and quantum advantages

Contextuality and advantages

- Contextuality has been associated with quantum advantage in information-processing and computational tasks.

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- Measure of contextuality $\rightsquigarrow$ **quantify such advantages**.

'*Contextuality fraction as a measure of contextuality*'
Abramsky, B, & Mansfield, Physical Review Letters, 2017.

The contextual fraction

Non-contextuality: global distribution $d \in \mathrm{Prob}(O^X)$ such that:

$$\forall_{C \in \mathcal{M}}.\ d|_C \ = \ e_C \ .$$

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$$e = \lambda e^{NC} + (1 - \lambda)e'$$

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$$\mathrm{NCF}(e) = \lambda \qquad \mathrm{CF}(e) = 1 - \lambda$$

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- $\mathsf{CF}(e)$ is calculated via linear programming, the dual LP yields this inequality.

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- We have

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Contextuality and advantage in quantum computation

- Measurement-based quantum computation (MBQC)

  '*Contextuality in measurement-based quantum computation*'
  Raussendorf, Physical Review A, 2013.

- Magic state distillation

  '*Contextuality supplies the 'magic' for quantum computation*'
  Howard, Wallman, Veitch, Emerson, Nature, 2014.

- Shallow circuits

  '*Quantum advantage with shallow circuits*'
  Bravyi, Gossett, Koenig, Science, 2018.

  - Contextuality analysis: Aasnæss, Forthcoming, 2020.

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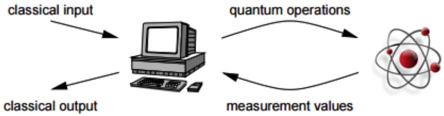

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- Probabilistic version: non-linear function computed with sufficently large probability of success implies contextuality.

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- Then,

$$\boxed{1 - \bar{p}_S \quad \geq \quad \mathsf{NCF}(e)\, \nu(f)}$$

Some further topics

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- Monogamy relations limiting contextuality

  '*On monogamy of non-locality and macroscopic averages*', B, QPL, 2014.

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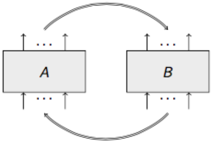

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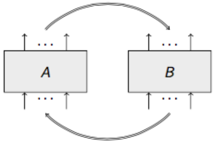

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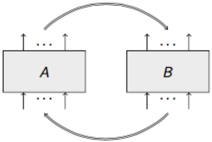

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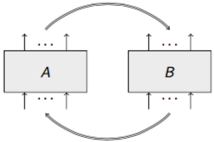

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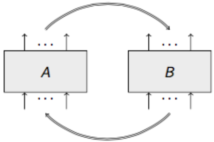

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Questions...

?