Witnessing Wigner Negativity

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ongoing work with Ulysse Chabaud and Frédéric Grosshans

December 2, 2020

Why study Wigner negativity?

Motivations

- Mari and Eisert's generalisation of Gottesman-Knill theorem[1].

[1] Andrea Mari and Jens Eisert. "Positive Wigner functions render classical simulation of quantum computation efficient". In: *Physical review letters* 109.23 (2012), p. 230503.

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- Hard to characterise Wigner negativity in CV for mixed states.
- *Can we get an experimentally accessible witness for Wigner negativity?*

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Fidelity with Fock states as a witness

Fidelity with Fock states

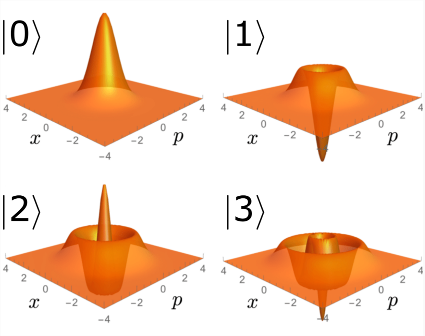

Figure 1: Wigner function for the first four Fock states. [3]

[3] Picture from Andreas Ketterer's thesis.

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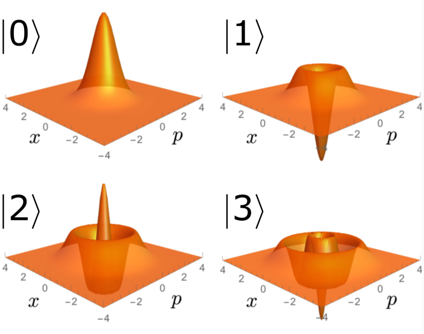

Figure 1: Wigner function for the first four Fock states. [3]

$$\langle n|\rho|n\rangle > F_n^* \Rightarrow \exists \alpha \in \mathbb{C} : W_\rho(\alpha) < 0$$

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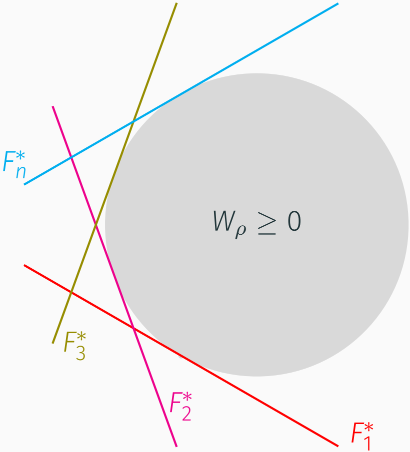
Geometric intuition on the witness



Linear programming approach for computing the witness

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- does not increase Wigner negativity.

$\Rightarrow$ Wlog restrict to mixtures of Fock states to detect Wigner negativity with this witness.

$$
(\mathrm{P}^n)\begin{cases}
\text{Find} & (F_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \\
\text{maximising} & F_n \\
\text{subject to} & \sum_k F_k = 1 \\
\text{and} & \forall k \in \mathbb{N},\ F_k \geq 0 \\
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where $W_k(\alpha) = \frac{2}{\pi}(-1)^k \mathrm{e}^{-2|\alpha|^2} L_k(4|\alpha|^2)$.

Linear Program to compute F_n^*

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(\mathsf{D}^n) \begin{cases} \text{Find} & y_0 \in \mathbb{R}, \mu \in \mathbb{M}_{\pm}(\mathbb{R}^+) \\ \text{minimising} & y_0 \\ \text{subject to} & \forall k \neq n \in \mathbb{N},\ y_0 \geq \int_{\mathbb{R}+} W_k \ \mathrm{d}\,\mu \\ \text{and} & y_0 \geq 1 + \int_{\mathbb{R}+} W_n \ \mathrm{d}\,\mu \\ \text{and} & \mu \geq 0 \end{cases}
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(Partially) solving the problem

Getting interesting results from an infinite linear program?

- **Relaxation**: eliminating some constraints (outside feasible solutions).

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- **Restriction**: imposing a form for your solution (inside feasible solutions):

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An interesting restriction for (P^n)

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\text{and} & \displaystyle\sum_{k=0}^{n} (-1)^k F_k L_k(x^2) = X^T Q X \underset{\text{Cholesky}}{=} ||VX||^2 \\
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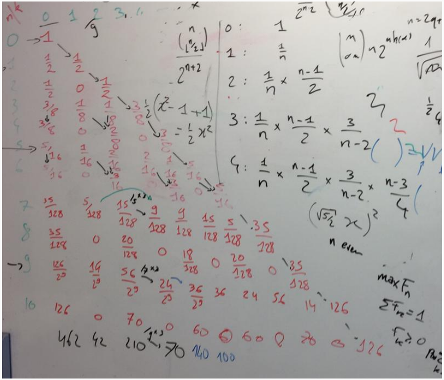

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n even:

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Check that these solutions are feasible for (P^n).

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- Checking (1) is relatively easy.

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- Checking (1) is relatively easy.
- Checking (2) is straightforward.
- Checking (3) is hard. Why? Because you have to find an analytical SOS decomposition (sadly OEIS failed here).

Once you find what should be the SOS decomposition, you have to show that it is valid:

$$\sum_{k=0}^{n}(-1)^k F_k L_k(x^2) = X^T Q X$$

An analytical SOS decomposition for constraint (3)

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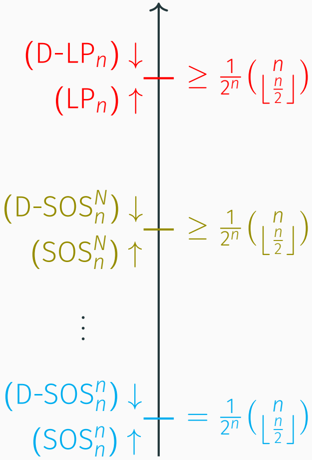
My dear Zeilberger.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Conclusion & ongoing work

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- insightful upperbounds?
- how tight are the lower bounds?

Thank you