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INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

Quantum Error Correction – an Introduction

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Contents

The classical repetition code

1 0 1 1 0

111 000 111 111 000

1 0 1 1 0

111 000 111 111 000

$$0 \rightarrow 000$$

Logical 0

$$1 \rightarrow 111$$

Logical 1

111 000 101 111 000

↓

"Majority Voting" Decoding

↓

111 000 111 111 000

111 000 001 111 000

↓

"Majority Voting" Decoding

↓

111 000 000 111 000

↓

1 0 0 1 0

$$p$$

Bit flip probability

Logical error probability

$$p_e = 3p^2(1-p) + p^3$$

Error threshold

$$p_e < p \text{ if } p < 1/2$$

$$0 \to 00000$$

$$1 \to 11111$$

Distance
d = 5
code

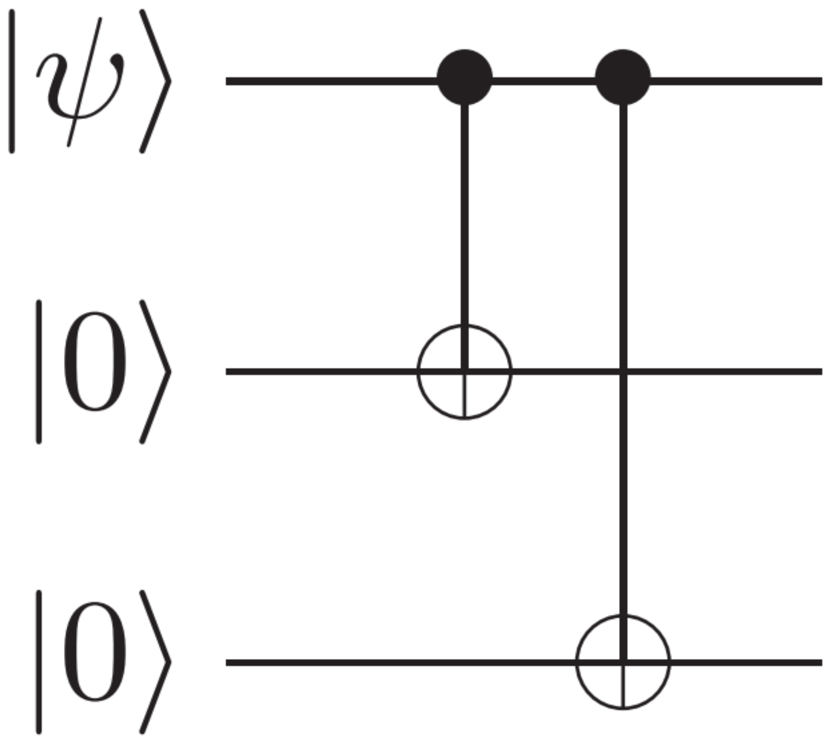
Lower logical error probability
(but same error threshold)

The three-qubit repetition code

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle$$

$$|0\rangle \rightarrow |0_L\rangle \equiv |000\rangle$$
$$|1\rangle \rightarrow |1_L\rangle \equiv |111\rangle$$

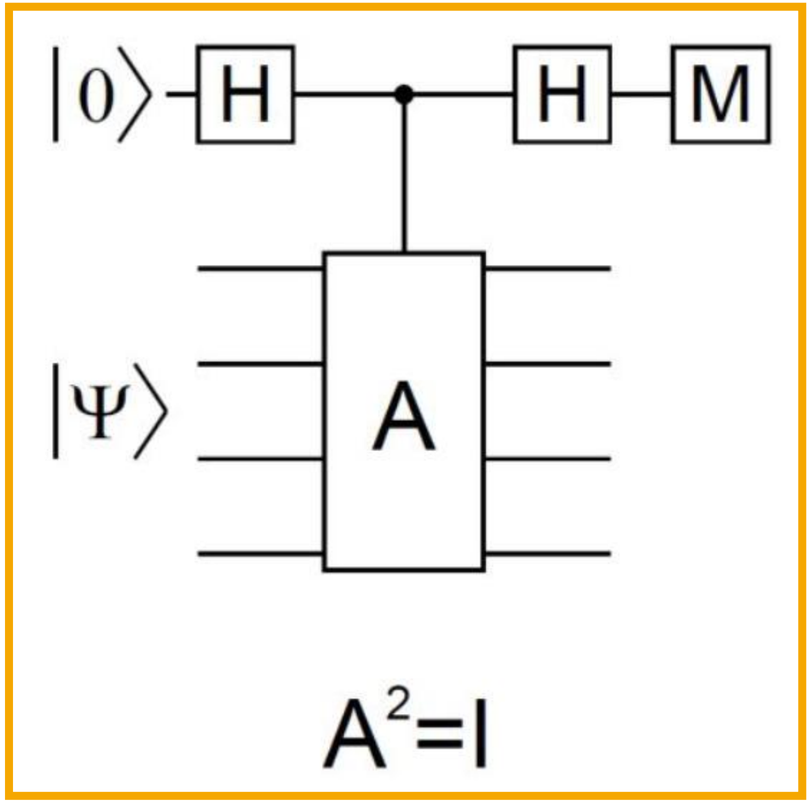
$$|\psi\rangle \rightarrow a|000\rangle + b|111\rangle$$



$$a\,|100\rangle + b\,|011\rangle$$

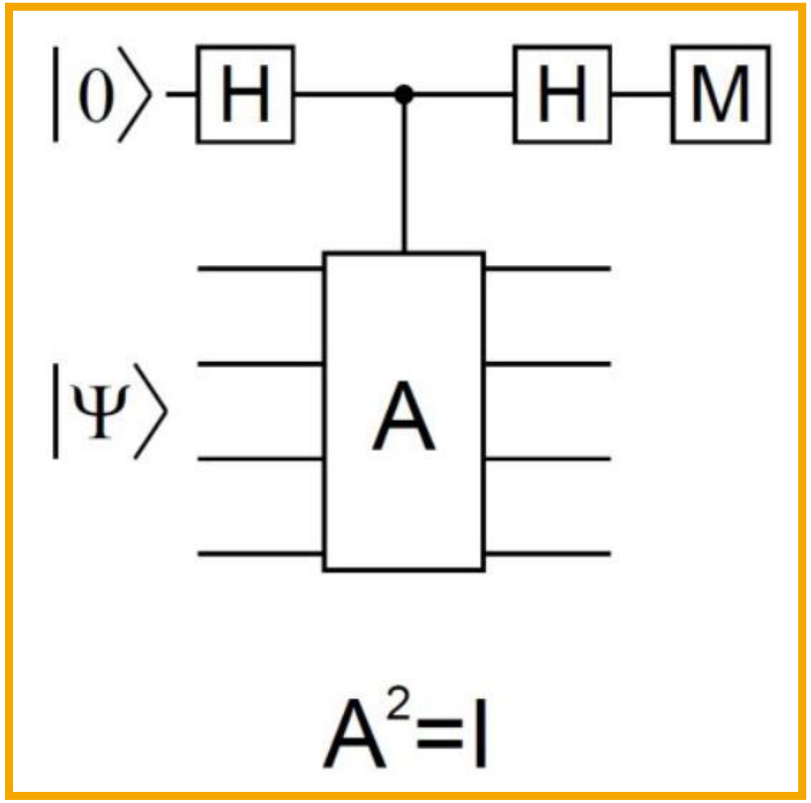
$$Z_1 Z_2 = (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I - (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I$$

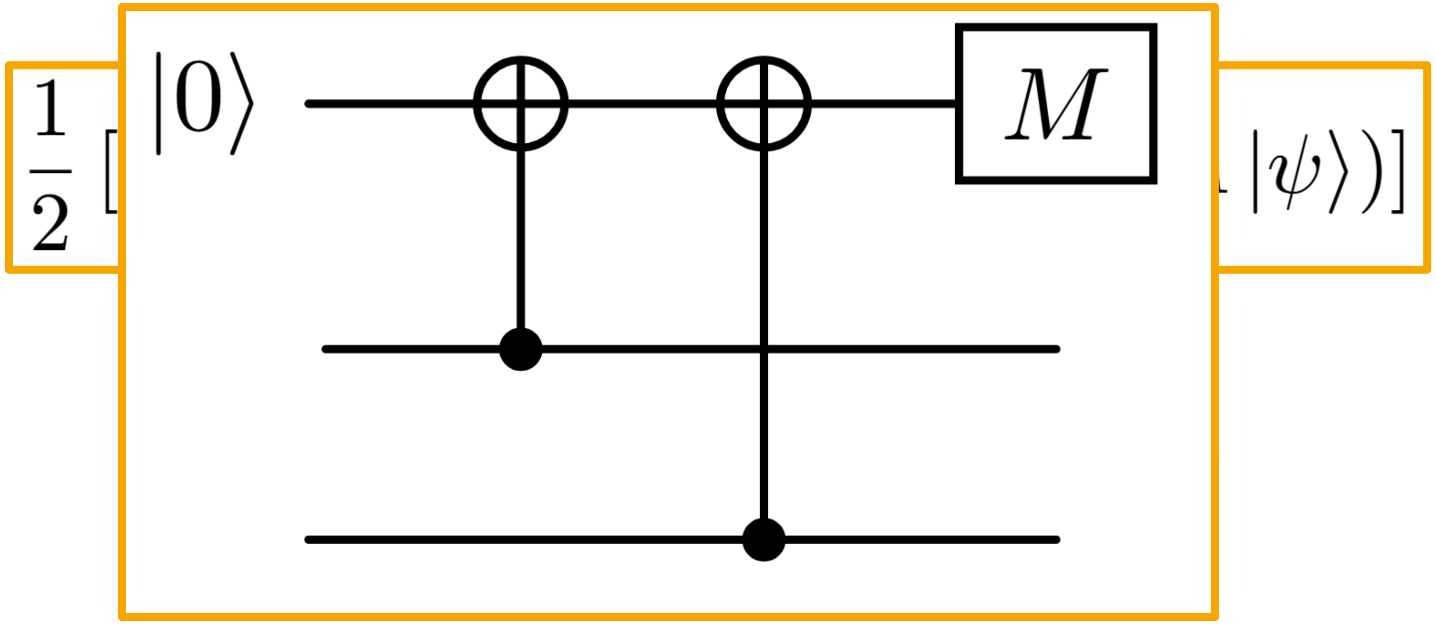
Error	$Z \otimes Z \otimes I$	$Z \otimes I \otimes Z$	$I \otimes Z \otimes Z$
$X \otimes I \otimes I$	-1	-1	+1
$I \otimes X \otimes I$	-1	+1	-1
$I \otimes I \otimes X$	+1	-1	-1



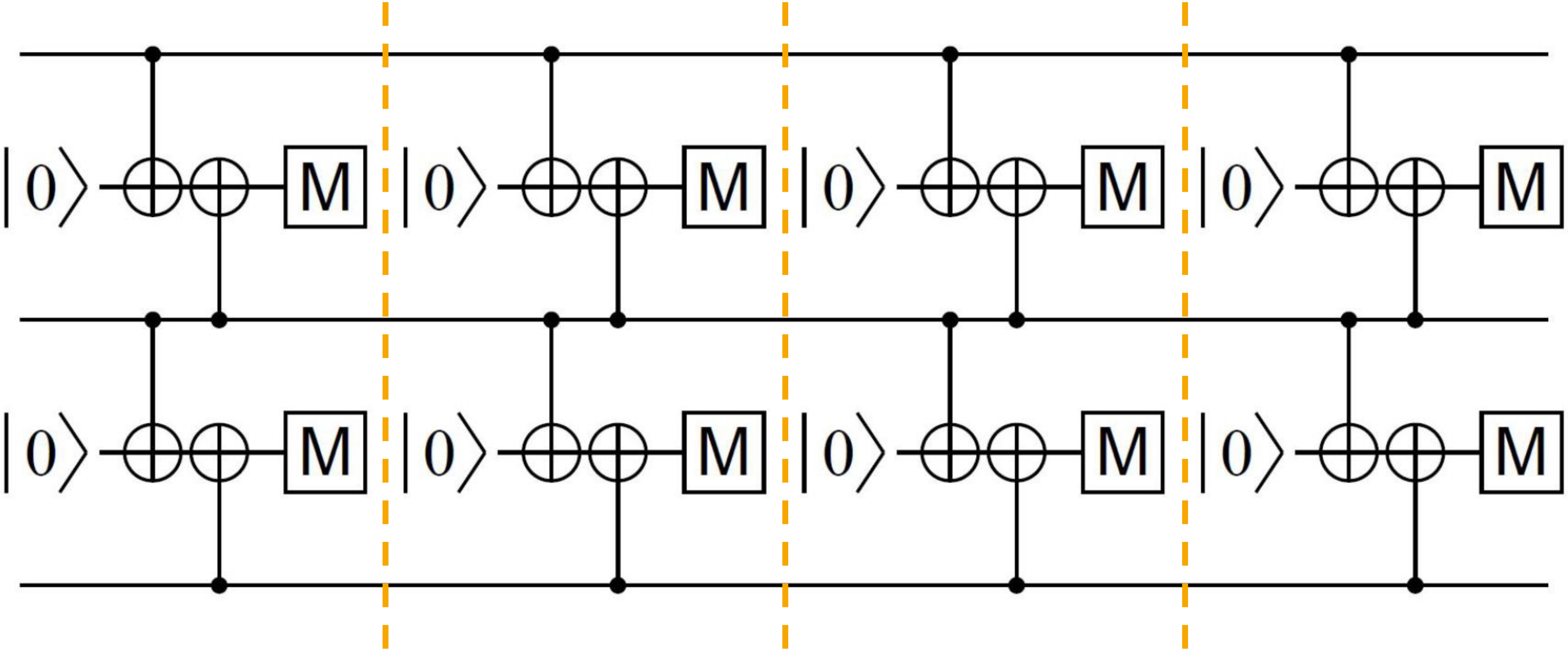
$$\frac{1}{2}\left[|0\rangle\left(|\psi\rangle + A|\psi\rangle\right) + |1\rangle\left(|\psi\rangle - A|\psi\rangle\right)\right]$$

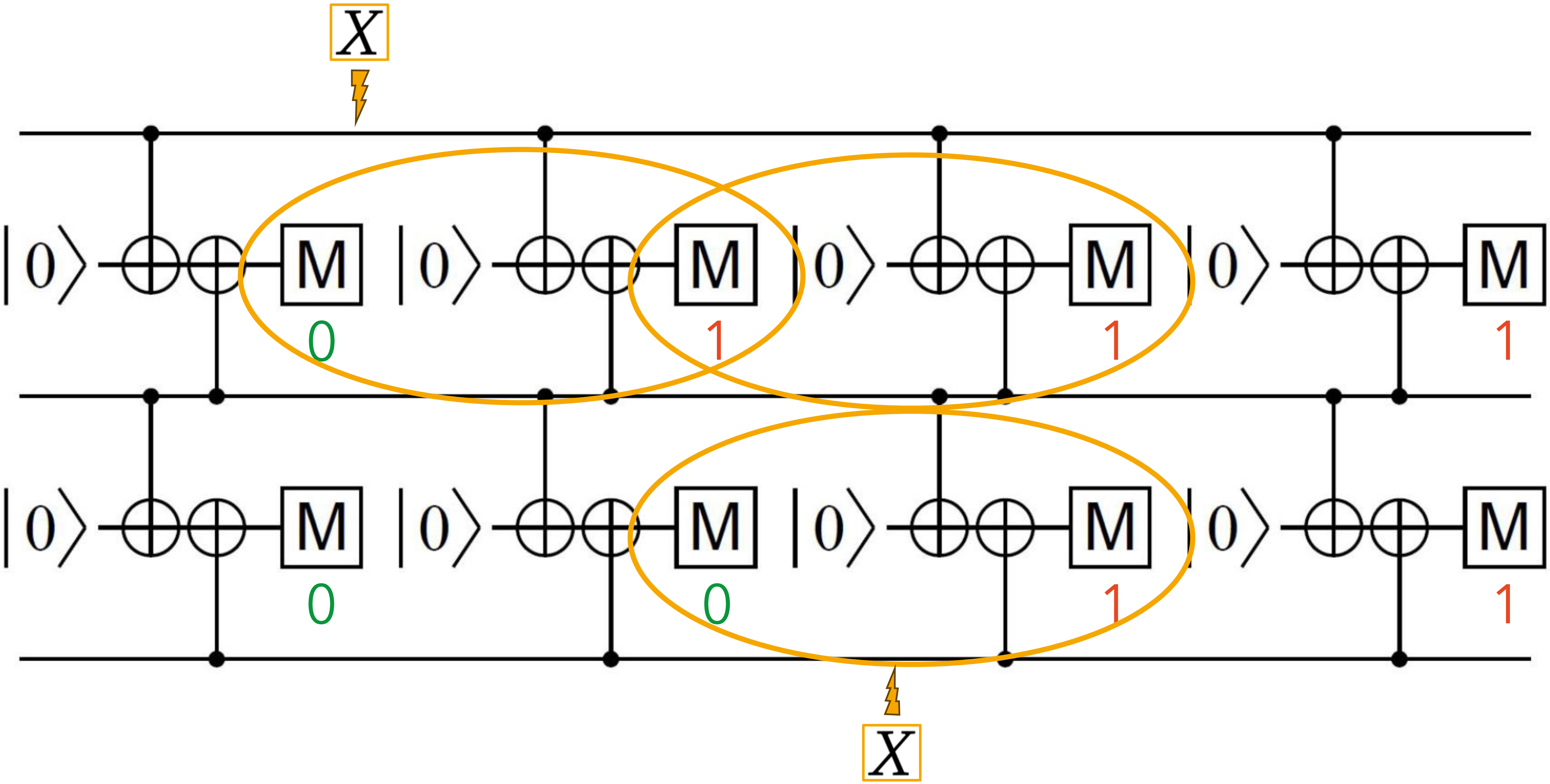
$$|0\rangle P_+ |\psi\rangle + |1\rangle P_- |\psi\rangle$$


|0⟩
H
H
M
|Ψ⟩
A
A²=I


|0⟩
M

Three qubit memory repetition code circuit




X
|0⟩
M
0
1
1
1
0
0
1
1
X

$$a\,|+++\rangle + b\,|---\rangle$$

Detecting Phase flips?

$$X_1X_2 = (|++\rangle\langle++| + |--\rangle\langle--|)\otimes I - (|++\rangle\langle--| + |--\rangle\langle++|)\otimes I$$

$$M = \alpha I + \beta X + \gamma Y + \delta Z$$

➡ **Any error can be decomposed into bit flips and phase flips**

Stabilizer formalism for Quantum Error Correction

$$S|\psi\rangle = |\psi\rangle$$

DEFINITION OF STABILIZER

$$a|000\rangle + b|111\rangle$$

Stabilizer group: $\{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$

A set of generators (Z_1Z_2, Z_2Z_3)

Error	$Z \otimes Z \otimes I$	$Z \otimes I \otimes Z$	$I \otimes Z \otimes Z$
$X \otimes I \otimes I$	-1	-1	+1
$I \otimes X \otimes I$	-1	+1	-1
$I \otimes I \otimes X$	+1	-1	-1

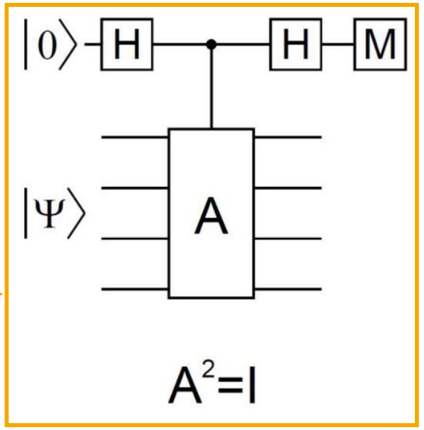
$$a|000\rangle + b|111\rangle$$

Stabilizer group: $\{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$

Error	$Z\otimes Z\otimes I$	$Z\otimes I\otimes Z$	$I\otimes Z\otimes Z$
$X\otimes I\otimes I$	-1	-1	+1
$I\otimes X\otimes I$	-1	+1	-1
$I\otimes I\otimes X$	+1	-1	-1

An n qubit code with m independent stabilizer generators defines a 2^{n-m} dim stabilizer space, encoding $n-m$ logical qubits.

To detect an error, it suffices to measure m independent stabilizer generators.



Given an error E, the measurement of stabilizer S returns:

- $+1$, if $ES = SE$
- -1, if $ES = -SE$

If $ES = SE$, then
$E|\psi\rangle = ES|\psi\rangle = SE|\psi\rangle$;

If $ES = -SE$, then
$E|\psi\rangle = ES|\psi\rangle = -SE|\psi\rangle$;

What about the logical operators?

$$|0_L\rangle \equiv |000\rangle$$
$$|1_L\rangle \equiv |111\rangle$$

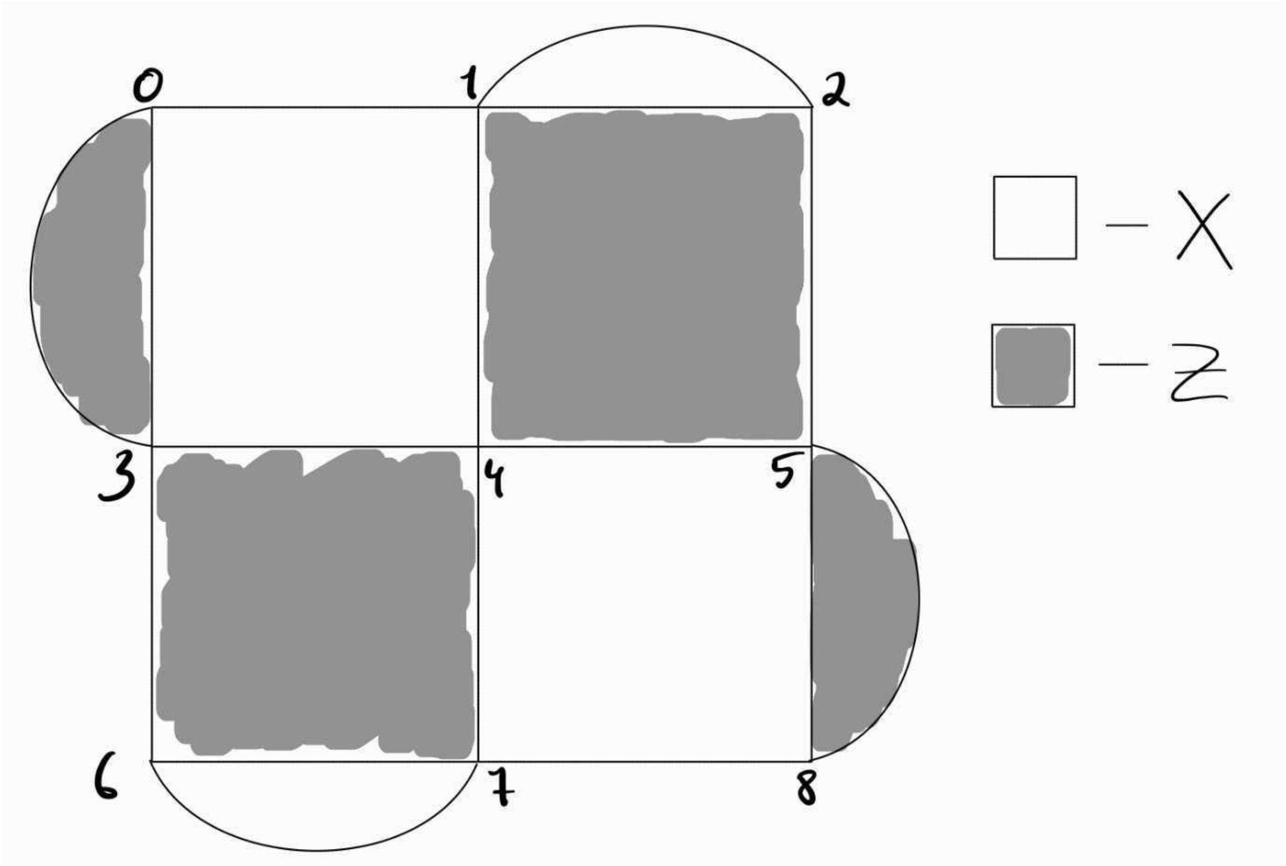
$$\{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$$

$$LS_j|\psi\rangle = L|\psi\rangle$$

$$X_L = X_1X_2X_3$$
$$Z_L \equiv Z_1$$

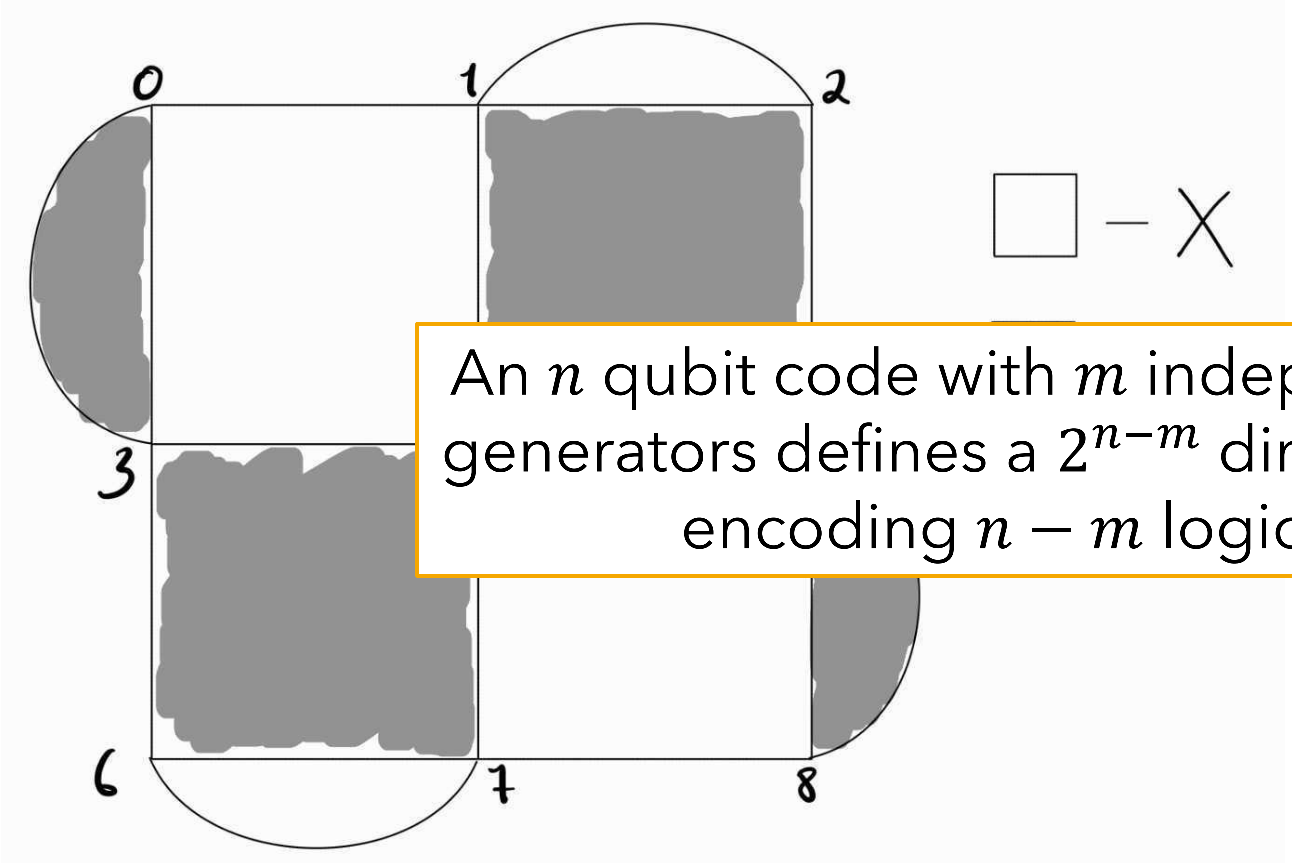
$$I_L = \{III,\ ZZI,\ IZZ,\ ZIZ\}$$
$$X_L = \{XXX,\ -YYX,\ -YXY,\ -XYY\}$$
$$Z_L = \{ZII,\ IZI,\ IIZ,\ ZZZ\}$$
$$Y_L = \{YXX,\ XYX,\ XXY,\ -YYY\}$$

Surface code



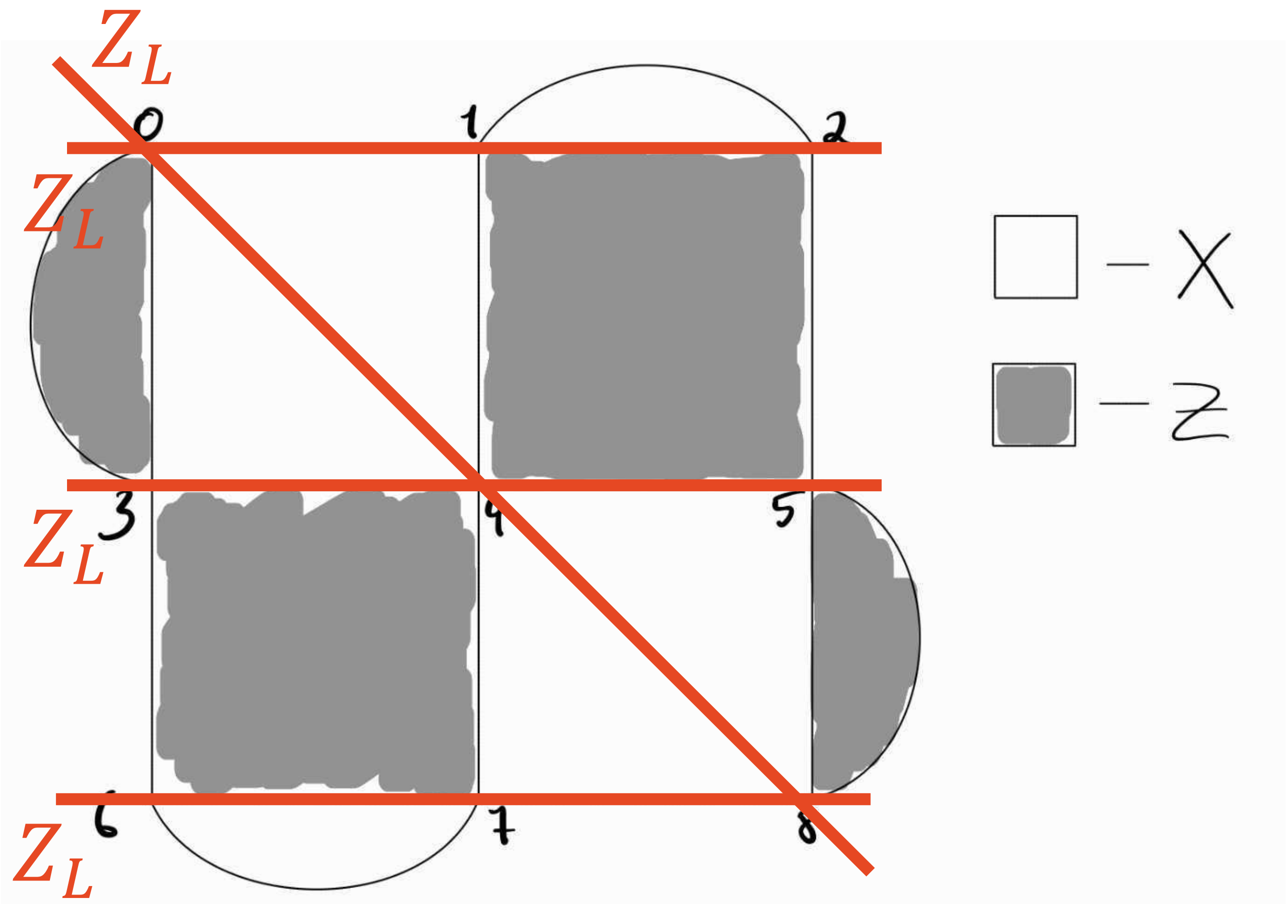
- Distance $d = 3$ surface code
- $n = d^2 = 9$ data qubits in a $d \times d$ lattice
- $n - 1$ stabilizer generators

$$\{X_1X_2, X_0X_1X_3X_4, X_4X_5X_7X_8, X_6X_7,$$
$$Z_0Z_3, Z_1Z_2Z_4Z_5, Z_3Z_4Z_6Z_7, Z_5Z_8\}$$



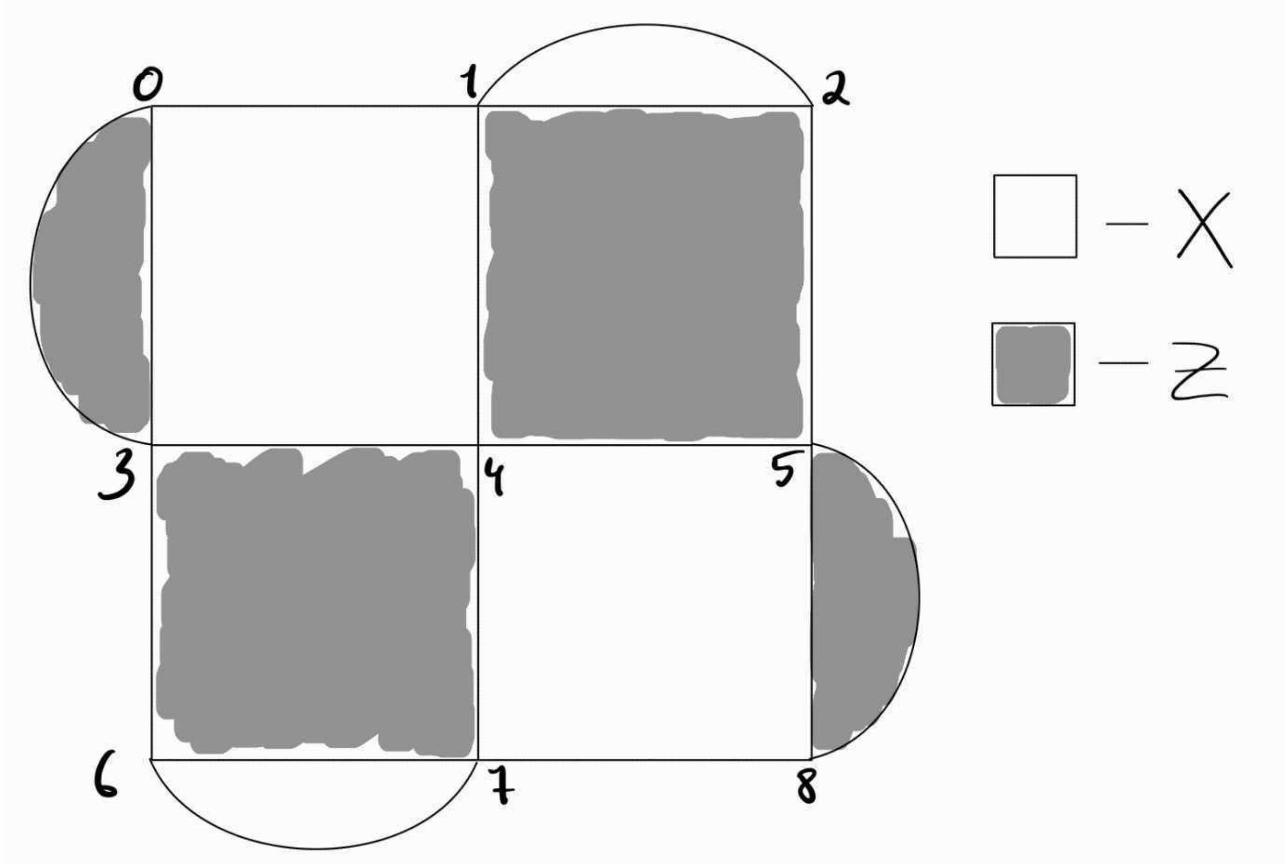
An n qubit code with m independent stabilizer generators defines a 2^{n-m} dim stabilizer space, encoding $n - m$ logical qubits.

- Distance $d = 3$ surface code
- $n = d^2 = 9$ data qubits in a $d \times d$ lattice
- enerators
- stabilizer space – 1 logical qubit



Logical operators?

- Must commute with all stabilizers
- Must not belong to the stabilizer
- Must satisfy anti-commutation properties of X and Z



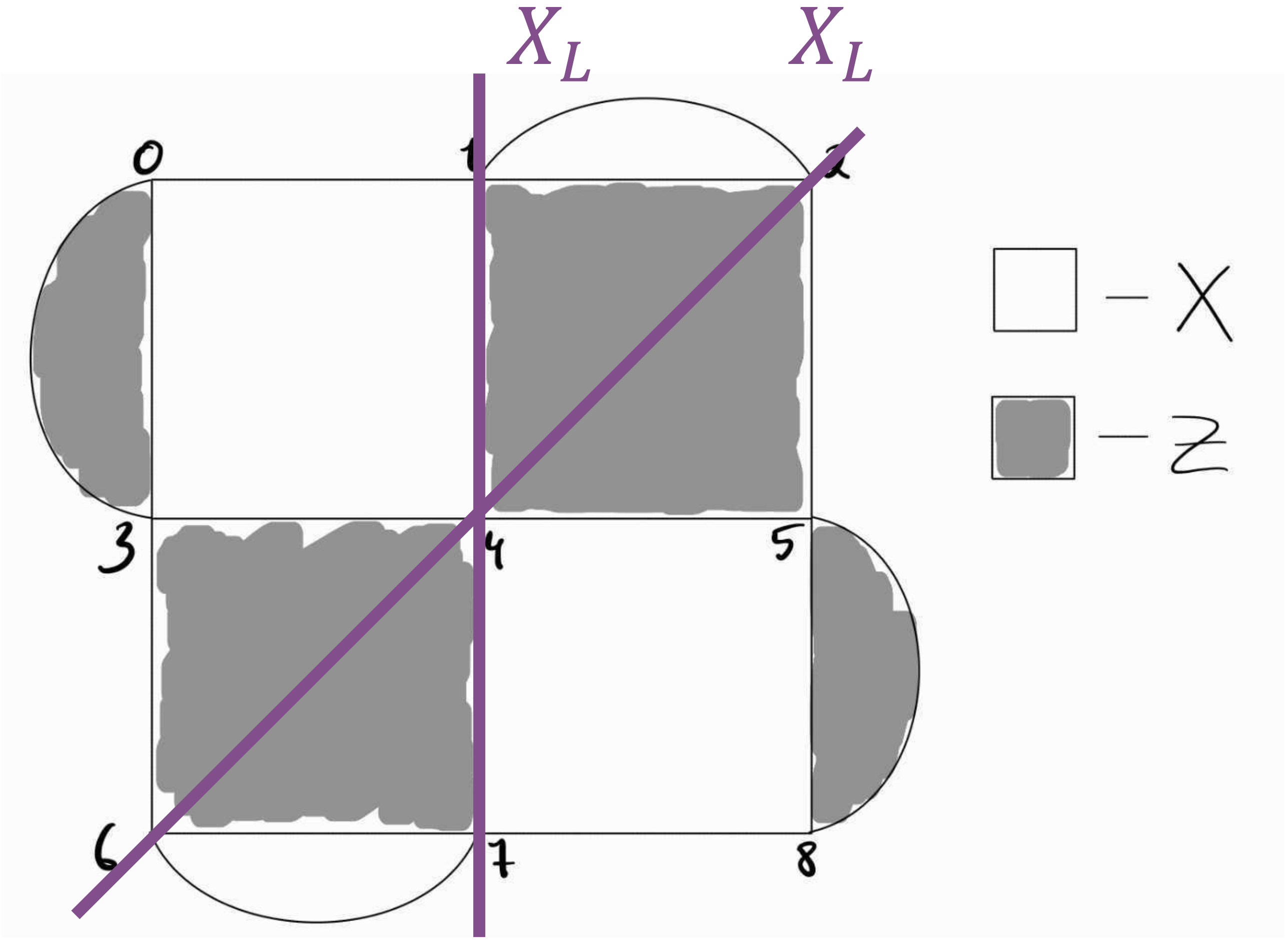
$$|0_L\rangle \equiv$$

$$|000\,000\,000\rangle + |011\,000\,000\rangle + |110\,110\,000\rangle + |101\,110\,000\rangle$$

$$+ |000\,011\,011\rangle + |011\,011\,011\rangle + |110\,101\,011\rangle + |101\,101\,011\rangle$$

$$|000\,000\,110\rangle + |011\,000\,110\rangle + |110\,110\,110\rangle + |101\,110\,110\rangle$$

$$+ |000\,011\,101\rangle + |011\,011\,101\rangle + |110\,101\,101\rangle + |101\,101\,101\rangle$$



Logical operators?

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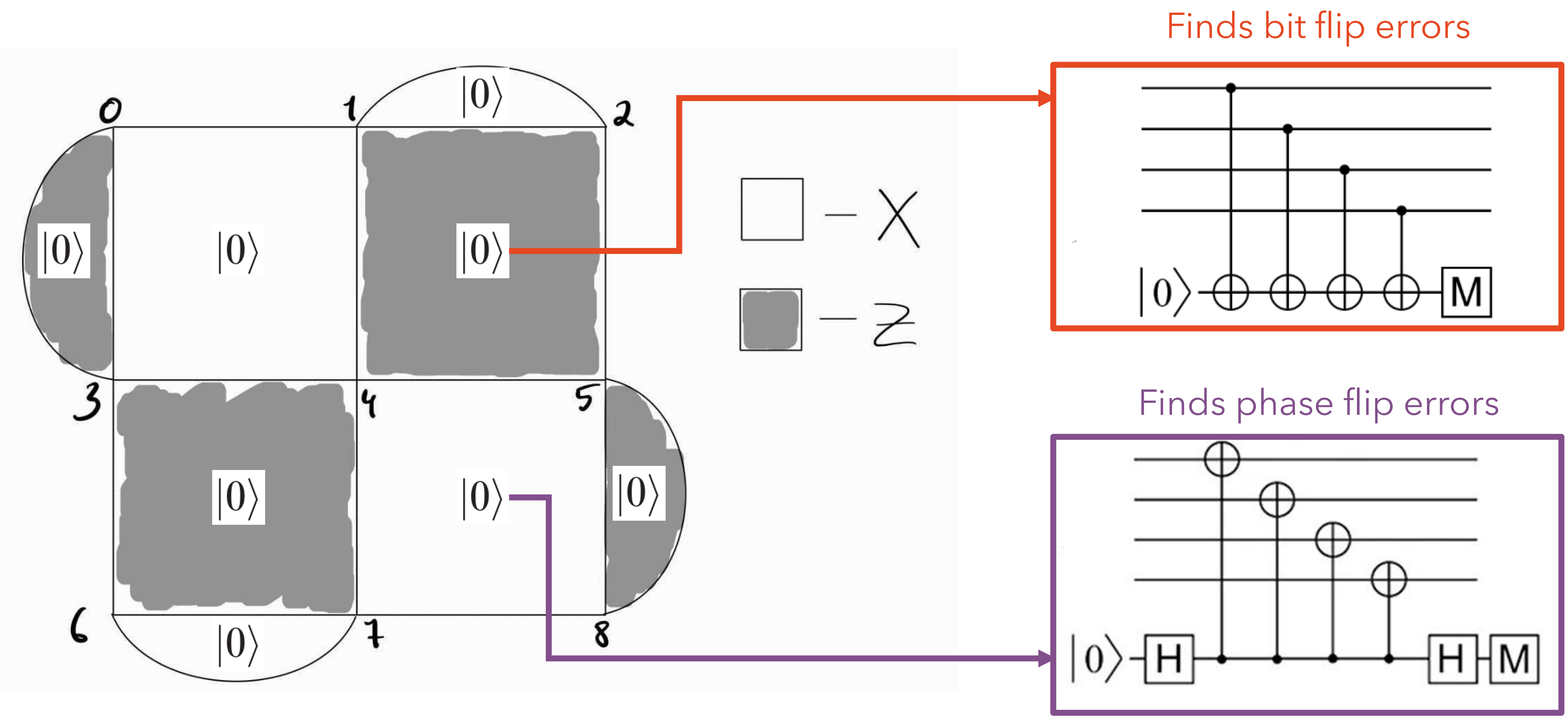
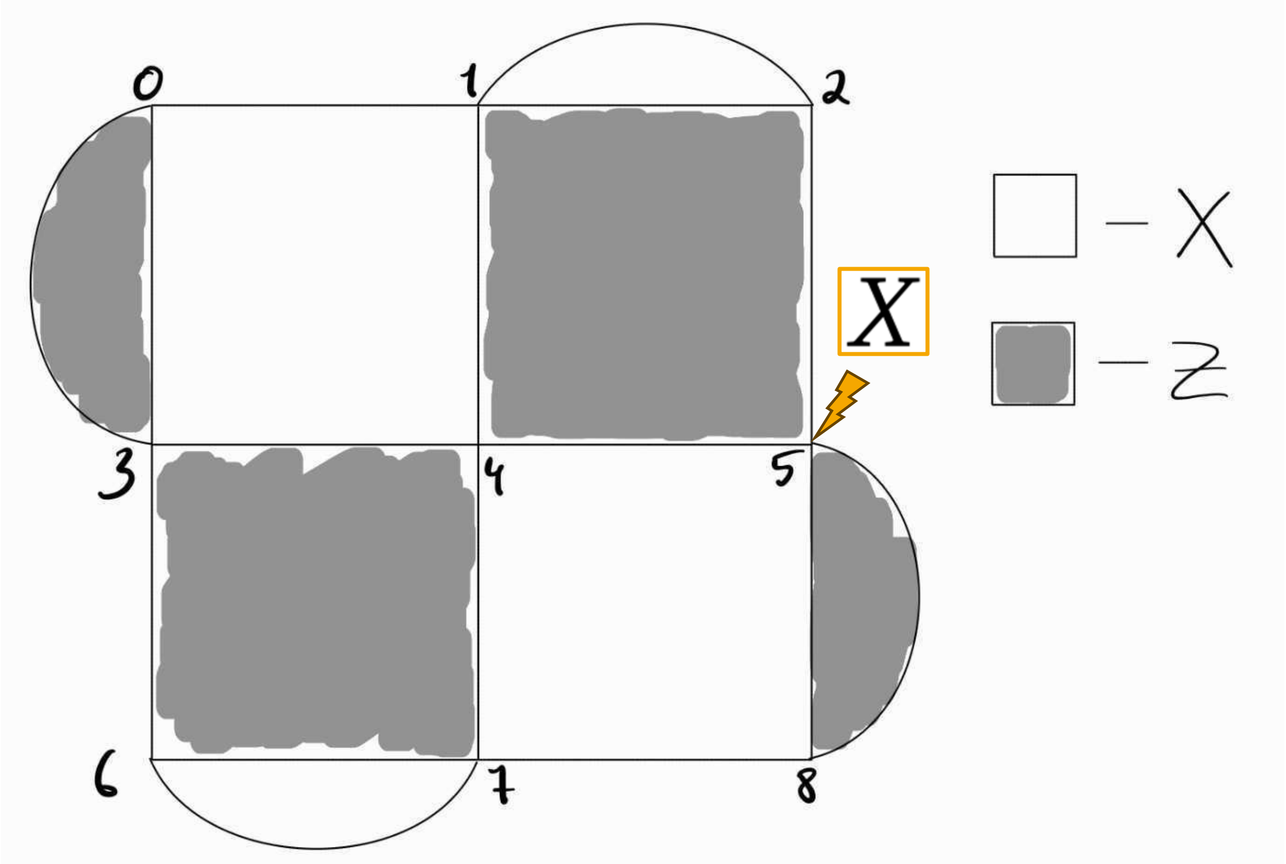

Finds bit flip errors
Finds phase flip errors
0
1
2
3
4
5
6
7
8
|0⟩
– X
– Z
H
M

Code degeneracy

The relationship between errors and syndromes is not one-to-one:

Given an error E, any error of the form $E' = EL$, where L commutes with the stabilizer, produces the same error syndrome.

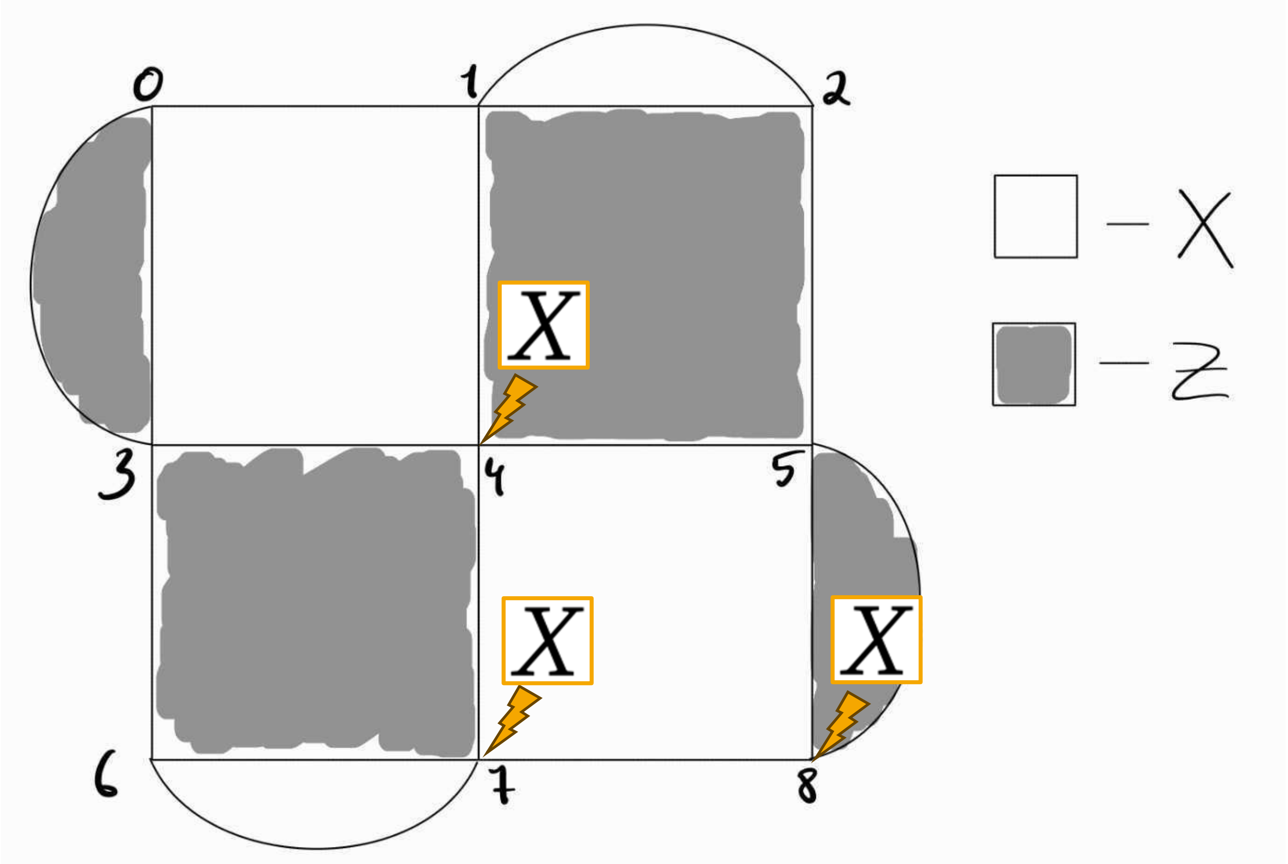


$$X_5$$

$$Z_1 Z_2 Z_4 Z_5$$

$$Z_5 Z_8$$

Given an error E, any error of the form $E' = EL$, where L commutes with the stabilizer, produces the same error syndrome.



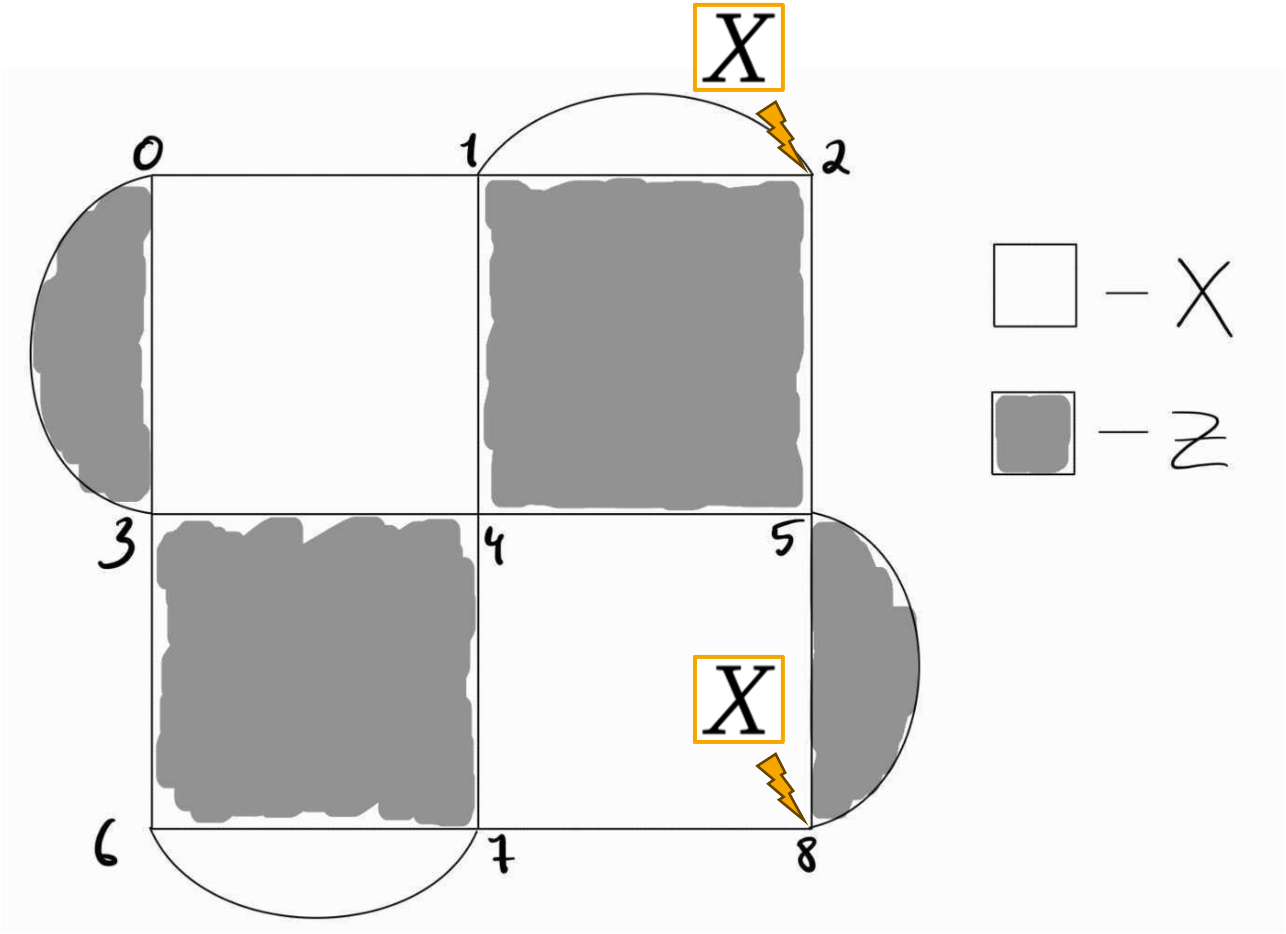
$$L = X_4 X_5 X_7 X_8$$

$$X_4 X_7 X_8$$

$$Z_1 Z_2 Z_4 Z_5$$

$$Z_5 Z_8$$

Given an error E, any error of the form $E' = EL$, where L commutes with the stabilizer, produces the same error syndrome.



$$L = X_2 X_5 X_8$$

$$X_2 X_8$$

$$Z_1 Z_2 Z_4 Z_5$$

$$Z_5 Z_8$$

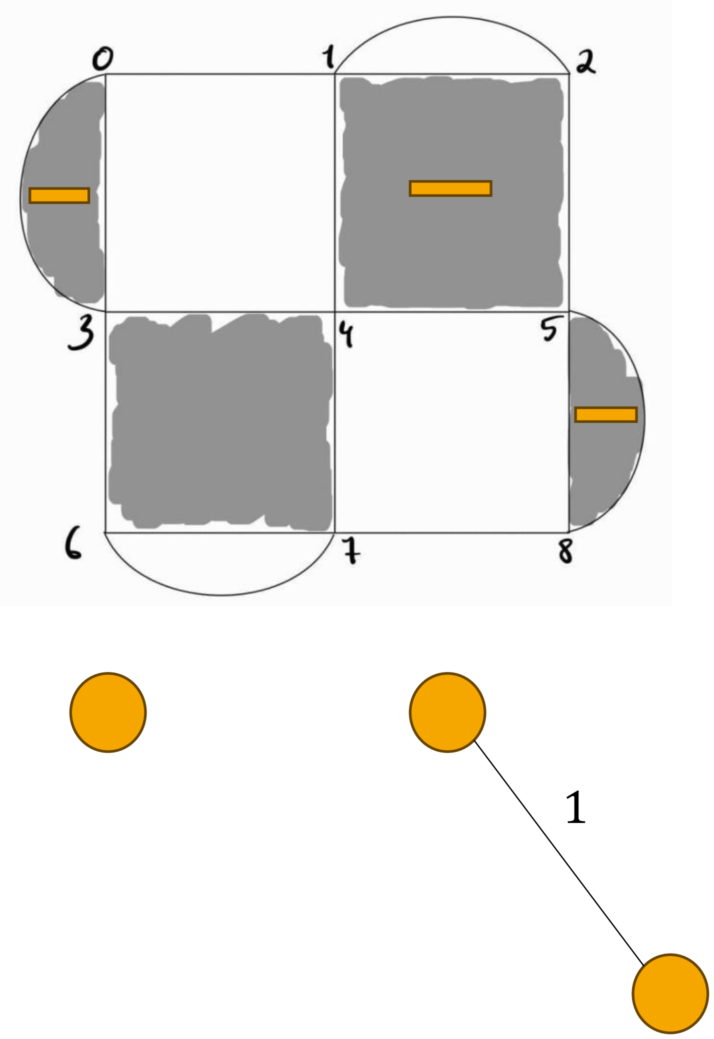
Decoders

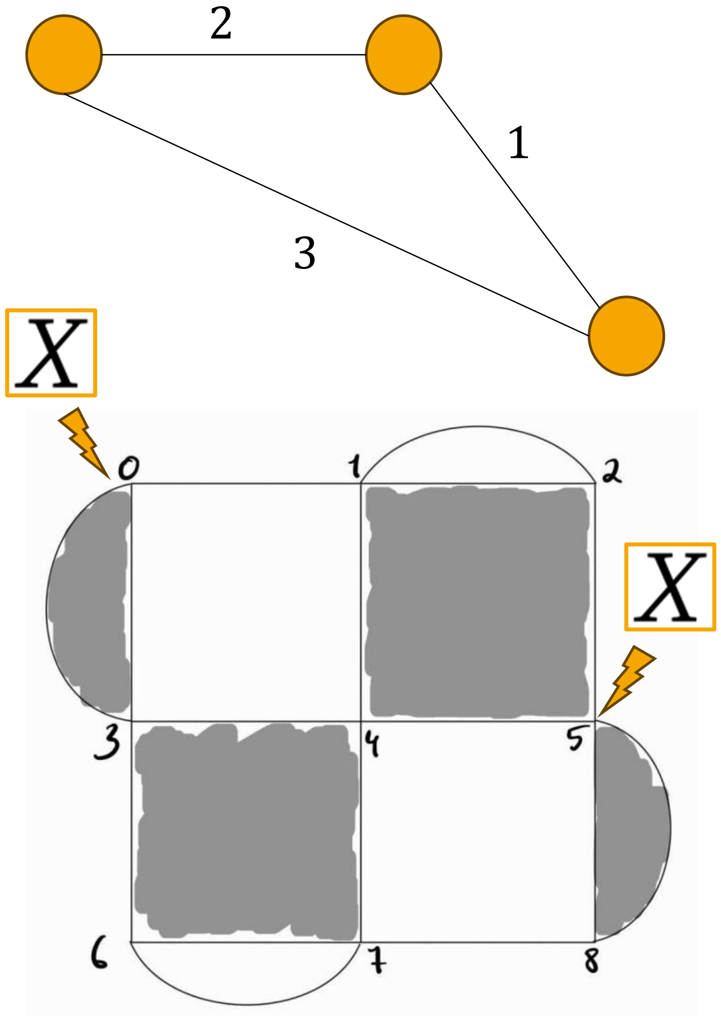
Algorithms that automate the choice of error correction operator given an error syndrome.

Ideally, given an error E, a decoder suggests a correction operator of the form $C = SE$.

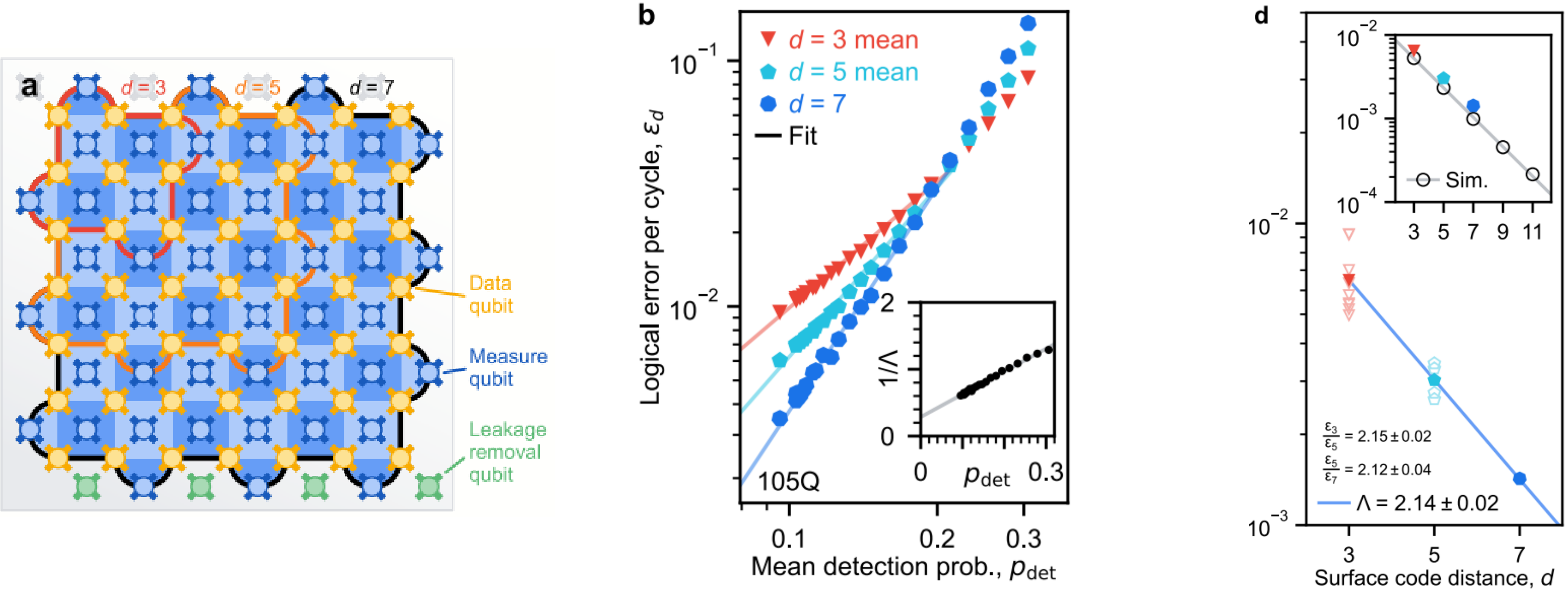
The error threshold of a QEC code depends on the choice of decoder.

Minimum Weight Perfect Matching (MWPM)





Google Quantum AI 2024 paper



Google Quantum AI and Collaborators. Quantum error correction below the surface code threshold. *Nature* (2024). https://doi.org/10.1038/s41586-024-08449-y

References

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- For understanding surface codes:
  - Dan Browne, Lecture notes on Topological Codes and Quantum Computation
  - Austin Fowler et.al. Surface codes: Towards practical large-scale quantum computation. arXiv:1208.0928
  - Lecture notes of the Quantum Error Correction course by Prof. Kastoryano at University of Cologne
- For a tutorial on how to simulate QEC codes with STIM: Hands-on quantum error correction with Google Quantum AI, available for free on Coursera
- Description of Stim software for simulation of QEC codes: Craig Gidney. Stim: a fast stabilizer circuit simulator. arXiv:2103.02202

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Thank you!

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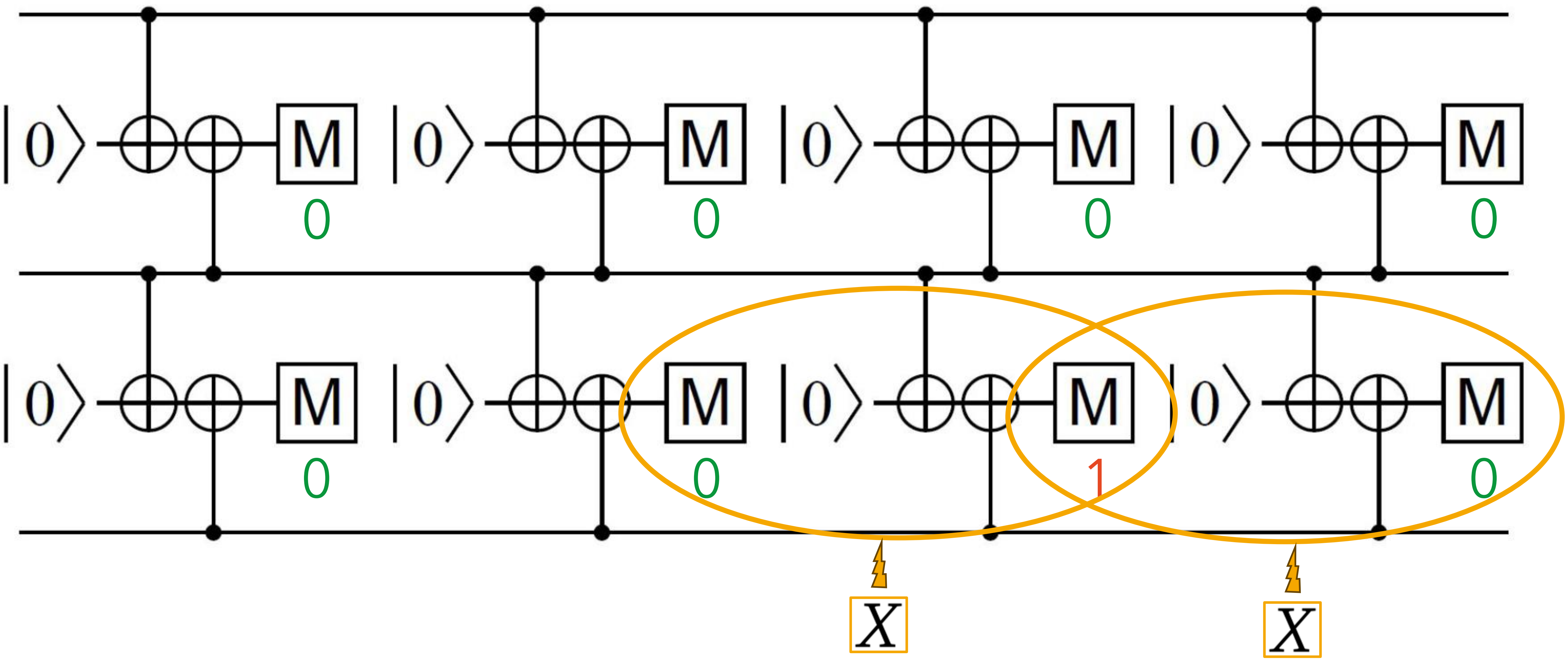
|0⟩
M
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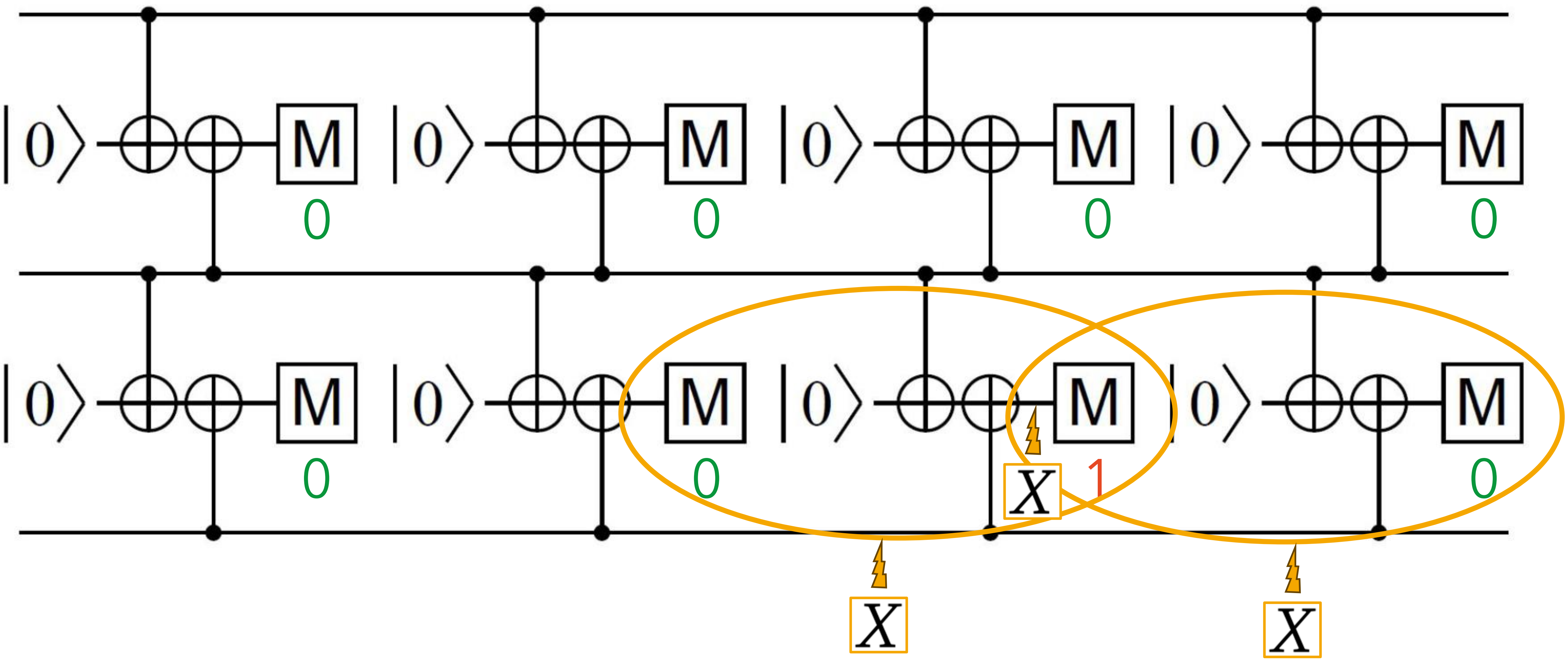
1
X
X



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M
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X
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X
X