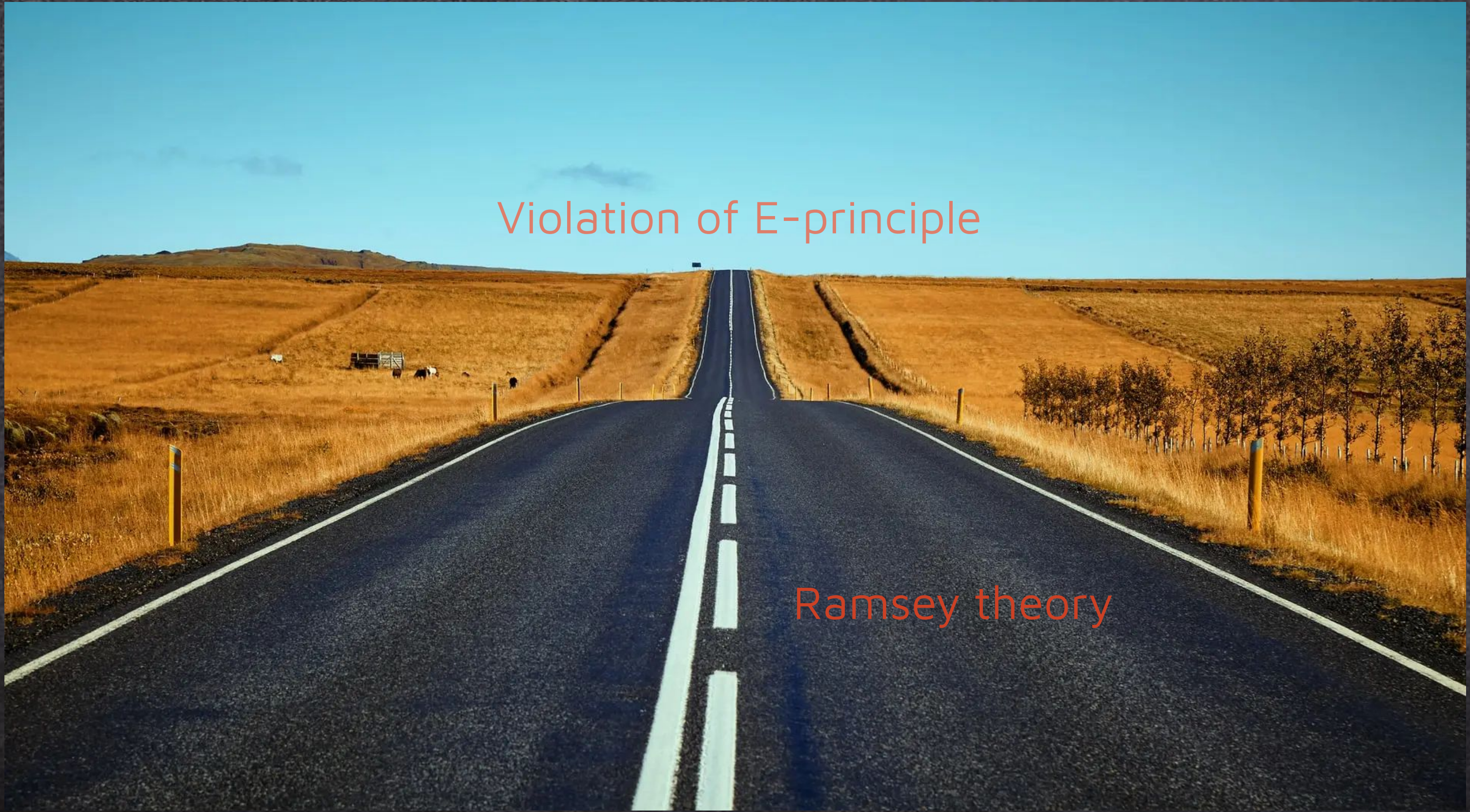
The Exclusivity principle and Ramsey numbers

(within n-cycle KS contextuality scenarios)

a joint work with:

Rui Soares Barbosa and Som Kanjilal

Take Home Message



Exclusivity principle?

Two events* are exclusive if there exists at least one common measurement that has different outcomes:

000|ABC 010|ABD : Exclusive due to measurement B only

000|ABC 101|ADC : Exclusive due to measurements A & C

010|ABC 111|DEF : not exclusive since there is no common measurement

* event ≡ joint outcome of a set of compatible measurements

Exclusivity principle?

Given a set of pairwise exclusive events, the sum of probabilities of all these events must be ≤ 1.

Exclusivity principle?

The non-trivial constraint imposed by E-principle is on probabilities of events coming from different joint measurements and therefore doesn’t follow from Kolmogorov's axioms of probability.

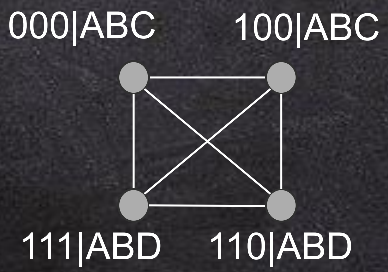
Trivial case: probabilities of all outcomes of a given joint measurement sum to 1.

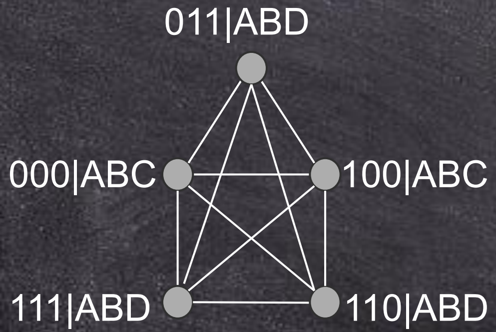
Graphs and Exclusivity principle?

Given a set of pairwise exclusive events, the sum of probabilities of all these events must be ≤ 1.

represent them on a graph

We obtain a complete graph (clique)



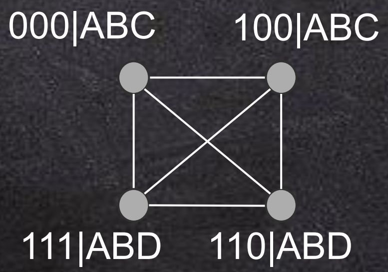


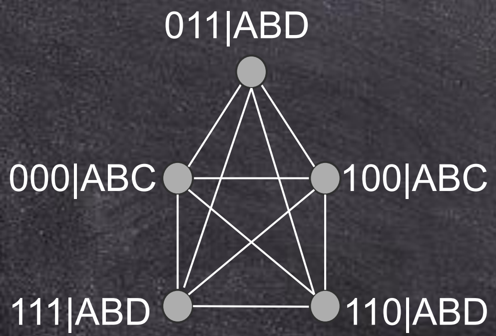
Graphs and Exclusivity principle?

Given a set of pairwise exclusive events, the sum of probabilities of all these events must be ≤ 1.

represent them on a graph

We obtain a complete graph (clique)





E-principle can be only tested on a clique

What is known about E-principle?

1. Quantum theory follows E-principle.
2. For bi-partite Bell scenarios, No-signalling $\Longleftrightarrow$ E-principle.
3. For tri-partite Bell scenarios, No-signalling $\not\Longleftrightarrow$ E-principle.

.

.

.

What is known about E-principle?

1. Quantum theory follows E-principle.
2. For bi-partite Bell scenarios: No-signalling $\Longleftrightarrow$ E-principle.
3. For tri-partite Bell scenarios: No-signalling $\not\Longleftrightarrow$ E-principle.

For two independent copies of CHSH scenario: No-signalling $\not\Longleftrightarrow$ E-principle.

Activation effects

A joint scenario of two independent copies of CHSH scenario

No-signalling $\not\Longleftrightarrow$ E-principle.

Violation gets activated when considering independent copies of the scenario

How to study activation effects?

	00	01	10	11
A_0B_0	1/2	0	0	1/2
A_0B_1	1/2	0	0	1/2
A_1B_0	1/2	0	0	1/2
A_1B_1	0	1/2	1/2	0

PR box in Lisbon

	00	01	10	11
A_0B_0	1/2	0	0	1/2
A_0B_1	1/2	0	0	1/2
A_1B_0	1/2	0	0	1/2
A_1B_1	0	1/2	1/2	0

PR box in Braga

How to study activation effects?

	00	01	10	11
A_0B_0	1/2	0	0	1/2
A_0B_1	1/2	0	0	1/2
A_1B_0	1/2	0	0	1/2
A_1B_1	0	1/2	1/2	0

↓

Exclusivity graph : E_1

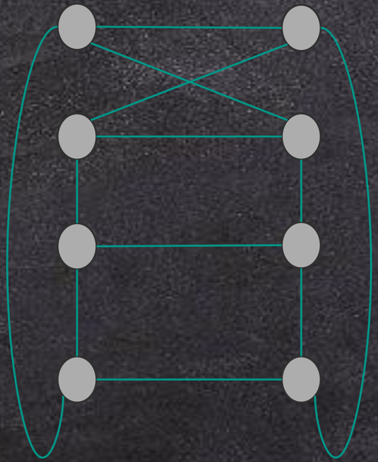
	00	01	10	11
A_0B_0	1/2	0	0	1/2
A_0B_1	1/2	0	0	1/2
A_1B_0	1/2	0	0	1/2
A_1B_1	0	1/2	1/2	0

↓

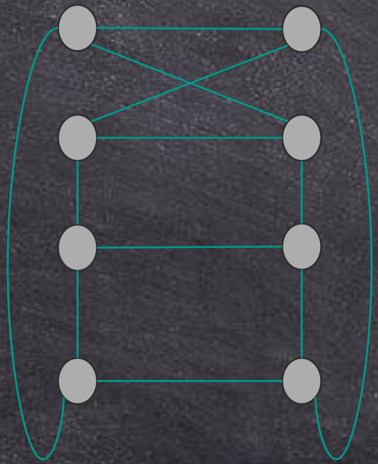
Exclusivity graph : E_2

How to study activation effects?

Exclusivity graph : E_1 (Lisbon)

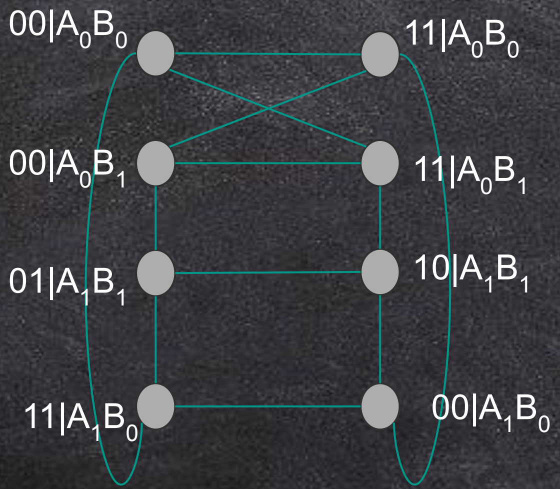


Exclusivity graph : E_2 (Braga)

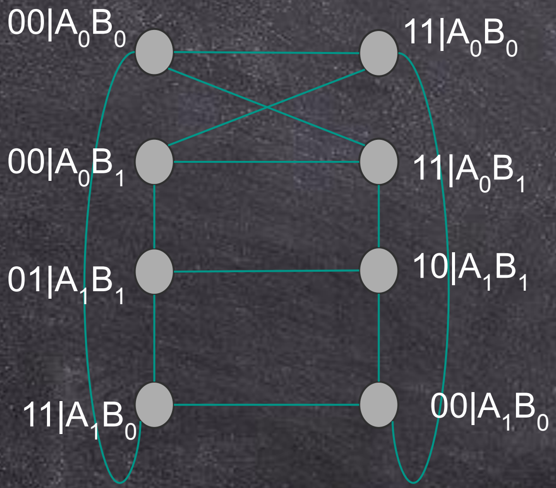


How to study activation effects?

Exclusivity graph : E_1 (Lisbon)

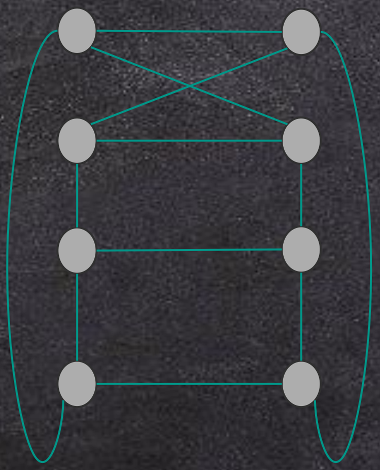


Exclusivity graph : E_2 (Braga)

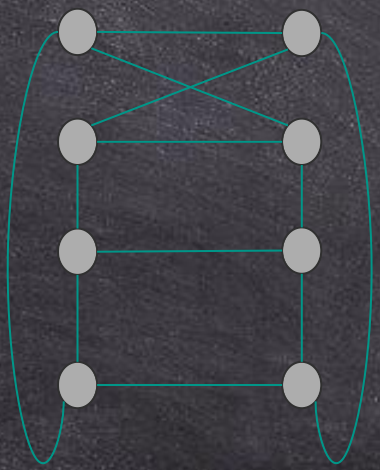


How to study activation effects?

Exclusivity graph : E_1 (Lisbon)



Exclusivity graph : E_2 (Braga)

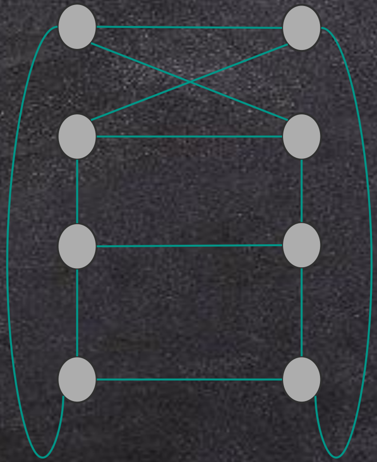


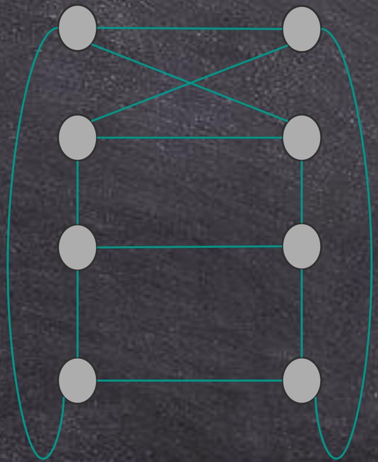
Construct a joint event graph from E_1, E_2

How to study activation effects?

Exclusivity graph : E_1 (Lisbon)

Exclusivity graph : E_2 (Braga)





$= G^2$

Construct a joint event graph from E_1, E_2

How to study activation effects?

$$G = E_1 \otimes E_2$$

How to study activation effects?

$$G = E_1 \otimes E_2$$

$$V(G) = \{(v, w) : v \in V(E_1) \land w \in V(E_2)\}$$

E.g. $(00 \,|\, A_0 B_0,\ 11 \,|\, A_1' B_0')$

Event from Lisbon

Event from Braga

How to study activation effects?

$$G = E_1 \otimes E_2$$ Joint exclusivity graph

$$V(G) = \{(v, w) : v \in V(E_1) \wedge w \in V(E_2)\}$$

We draw an edge between (v,w) & (u,t):

v is exclusive to u AND/OR w is exclusive to t

How to study activation effects?

Edge between two nodes exists:

There exists at least one city for which the events are exclusive

How to study activation effects?

Edge between two nodes exists:

There exists at least one local site for which the events are exclusive

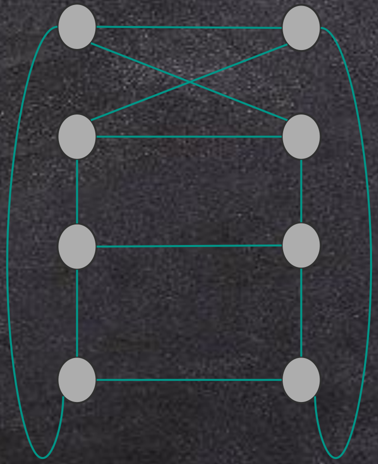
$$(00 \mid A_0 B_0,\ 11 \mid A_1' B_0') \text{ ------ } (11 \mid A_0 B_0, 11 \mid A_1' B_0')$$

$$(00 \mid A_0 B_0,\ 11 \mid A_1' B_0') \text{ ------ } (01 \mid A_0 B_0, 00 \mid A_1' B_0')$$

$$(00 \mid A_0 B_0,\ 11 \mid A_1' B_0') \qquad\qquad (00 \mid A_0 B_0, 11 \mid A_0' B_1')$$

How to study activation effects?

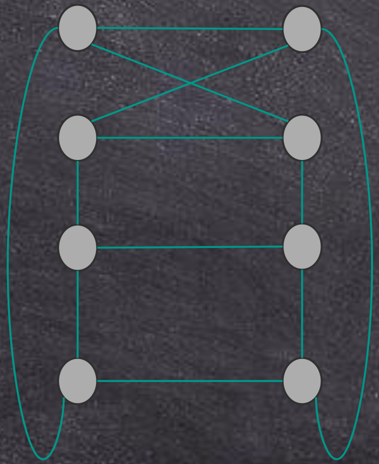
Exclusivity graph : E_1 (Lisbon)



$\otimes$

OR product

Exclusivity graph : E_2 (Braga)



$= \ G^2$

Activation effect : Finding a clique inside G^2 that violates the E-principle

How to study activation effects?

	00	01	10	11
A_0B_0	1/2	0	0	1/2
A_0B_1	1/2	0	0	1/2
A_1B_0	1/2	0	0	1/2
A_1B_1	0	1/2	1/2	0

Lisbon

$\otimes$

	00	01	10	11
A_0B_0	1/2	0	0	1/2
A_0B_1	1/2	0	0	1/2
A_1B_0	1/2	0	0	1/2
A_1B_1	0	1/2	1/2	0

Braga

Each possible joint event in G^2 occurs with probability 1/4

How to study activation effects?

Activation effect : Finding a clique inside G that violates the E-principle

Condition of independence: Each joint event occurs with probability 1/4

If a clique of size L, exists inside graph G, then for violation of E-principle:

$$L/4 > 1$$

$$L > 4$$

The bigger L is, the bigger is the violation

How to study activation effects?

Activation effect : Finding a clique inside G that violates the E-principle

Condition of independence: Each joint event occurs with probability 1/4

If a clique of size L, exists inside graph G, then for violation of E-principle:

$$L/4 > 1$$

$$L > 4$$

Finding a clique of size > 4 inside G^2 suffices for the violation of E-principle

How to study activation effects?

How to find cliques inside the joint exclusivity graph?

How to find cliques inside G?

Brute Force?

Clique decision problem is NP-complete

Brute Force

Article | Published: 16 August 2013

Local orthogonality as a multipartite principle for quantum correlations

T. Fritz, A.B. Sainz, R. Augusiak, J Bohr Brask, R. Chaves, A. Leverrier & A. Acín

Nature Communications **4**, Article number: 2263 (2013) | Cite this article

Brute Force

Article | Published: 16 August 2013

Local orthogonality as a multipartite principle for quantum correlations

T. Fritz, A.B. Sainz, R. Augusiak, J Bohr Brask, R. Chaves, A. Leverrier & A. Acín

Nature Communications **4**, Article number: 2263 (2013) | Cite this article

Found a K_5 for two independent copies of PR boxes; no bigger clique though

How to find cliques inside G?

Can we use Ramsey theory to address clique-finding inside G?

What is Ramsey theory?

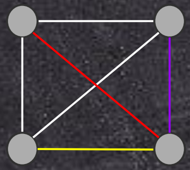
Ramsey theory is about the necessary existence of certain substructures inside sufficiently bigger structures. Popularly described as “complete disorder is impossible”.

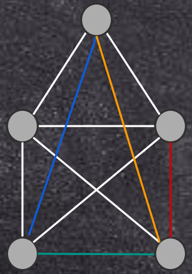
What is Ramsey theory?

Sufficiently big cliques with colored edges must contain certain monochromatic substructures.

What is Ramsey theory?

Sufficiently big cliques with colored edges must contain certain monochromatic substructures.





What is Ramsey theory?

Edge-colored clique must contain certain monochromatic substructures.

$$R(C_3, C_3) = 6$$

In **ANY** clique of size 6 each of whose edges is either red or blue:

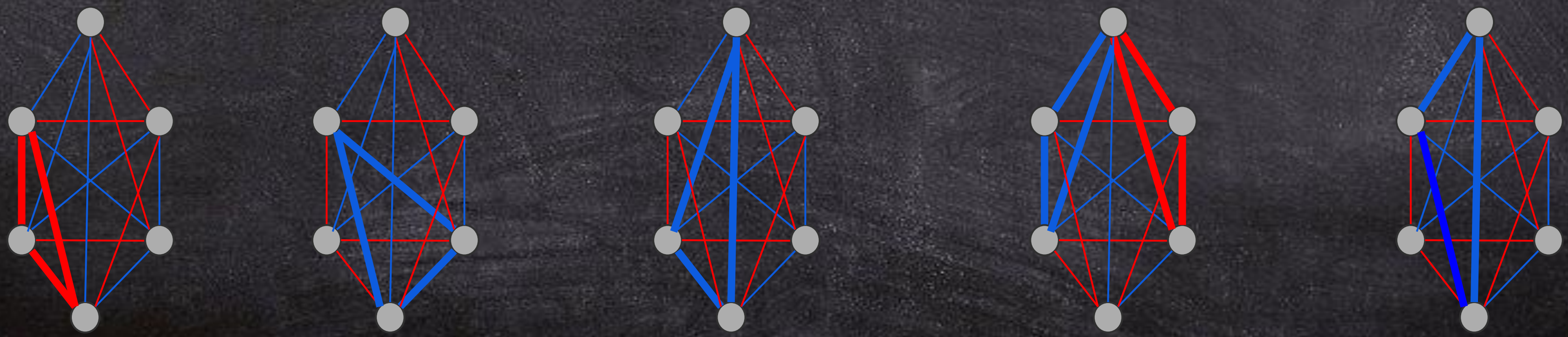
There will at least be a C_3 in one of the two colors

What is Ramsey theory?

Edge multi-colored clique must contain certain monochromatic substructures.

$R(C_3, C_3) = 6$

In **ANY** clique of size 6 each of whose edges is either red or blue:



What is Ramsey theory?

Edge multi-colored clique must contain certain monochromatic substructures.

$$R(S_1,S_2,S_3,\dots,S_k) = ?$$

What is the size of a k edge-colored clique such that at least one of the colors contains the correspondingly assigned structure?

How to find cliques inside G?

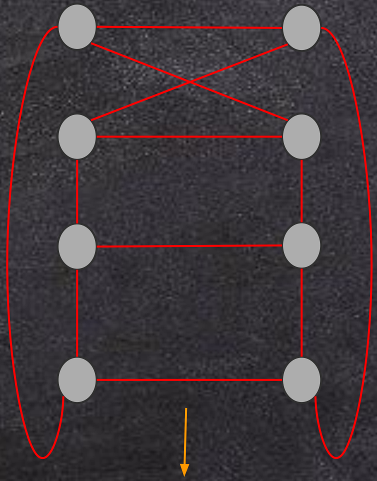
Can we use Ramsey theory to address clique-finding inside G?

Yes

Refinement of the earlier case?

assign a unique color to each city's E-graph

Exclusivity graph : E_1

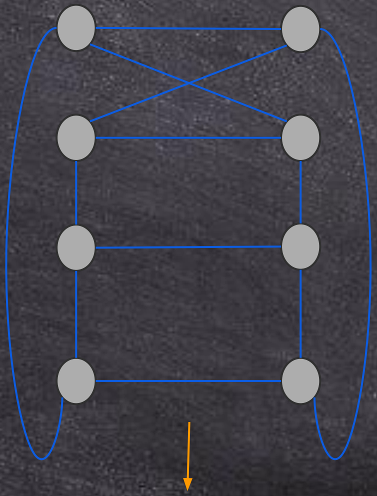


PR box in Lisbon

$\otimes$

OR product

Exclusivity graph : E_2



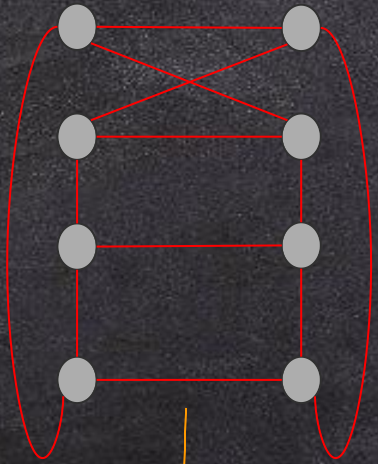
PR box in Braga

$= G^2$

Refinement of the earlier case?

assign a unique color to each city's E-graph

Exclusivity graph : E_1

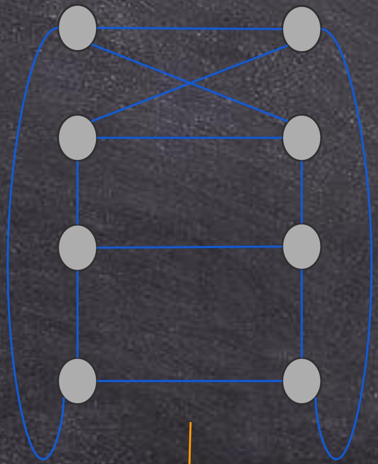


PR box in Lisbon

$\otimes$

OR product

Exclusivity graph : E_2



PR box in Braga

$= G^2$

edge bi-colored graph

Refined graph G^2?

$$G = E_1^{c_1} \otimes E_2^{c_2}$$ Joint edge-colored exclusivity graph

$$V(G) = \{(v, w) : v \in V(E_1) \wedge w \in V(E_2)\}$$

I[th] colored edge between two nodes exists:

If the orthogonality due to the I[th] city events exists across two joint events

$$(00 \mid A_0 B_0,\ 11 \mid A_1' B_0') \qquad (11 \mid A_0 B_0, 11 \mid A_1' B_0')$$

$$(00 \mid A_0 B_0,\ 11 \mid A_1' B_0') \qquad (01 \mid A_0 B_0, 00 \mid A_1' B_0')$$

$$(00 \mid A_0 B_0,\ 11 \mid A_1' B_0') \qquad (00 \mid A_0 B_0, 11 \mid A_0' B_1')$$

Violation inside G^2?

Condition of independence: Each joint event occurs with probability 1/4

If a clique of size L, exists inside graph G, then for violation of E-principle:

$$L/4 > 1$$

$$L > 4$$

Finding a clique of size more than 4 inside G^2 suffices for the violation of E-principle

The bigger L is, the bigger is the violation

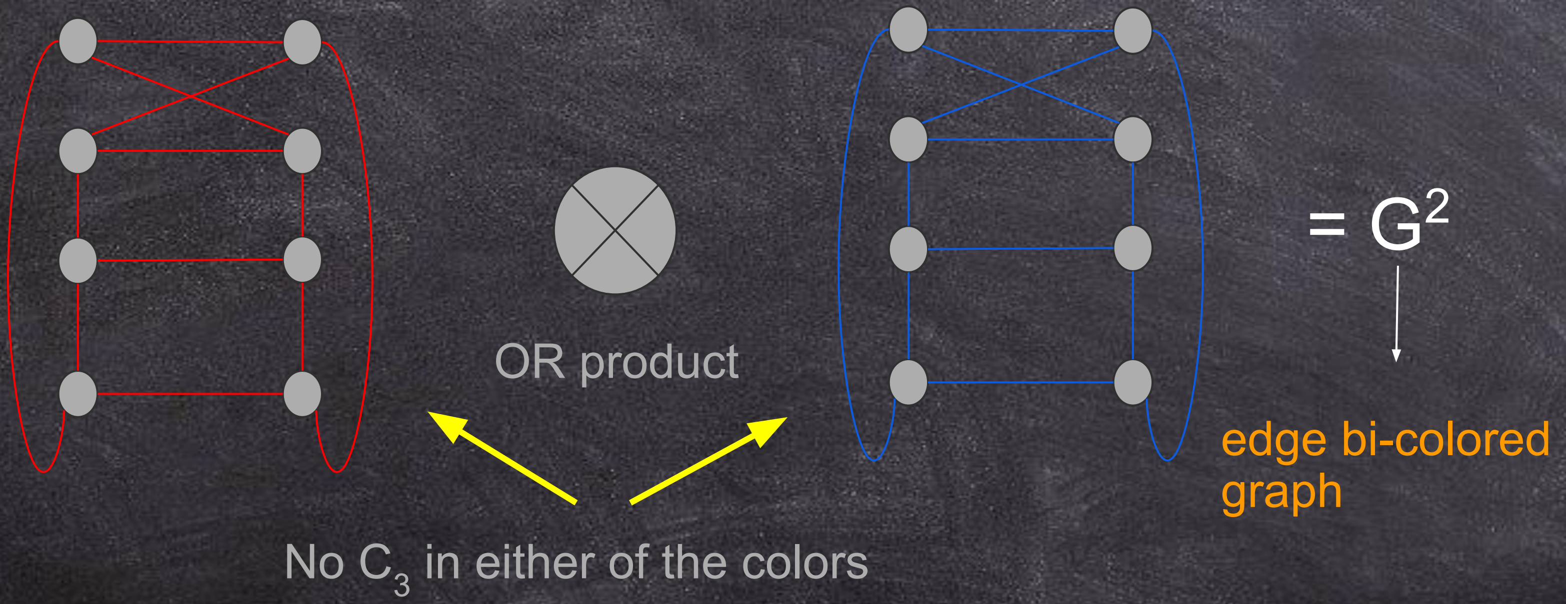
Clique finding and Ramsey theory?

$R(C_3, C_3) = 6$

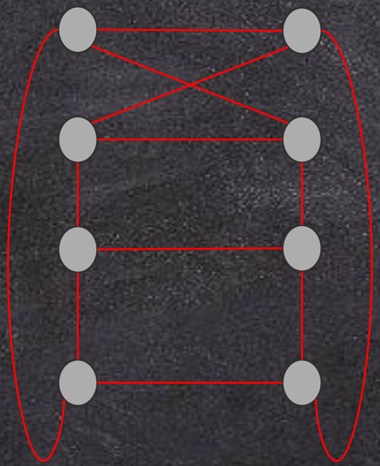
For K_6 to exist inside G, there must be at least a red C_3 or a blue C_3

There must be a C_3 in E_1 AND/OR a C_3 in E_2

Clique finding and Ramsey theory?

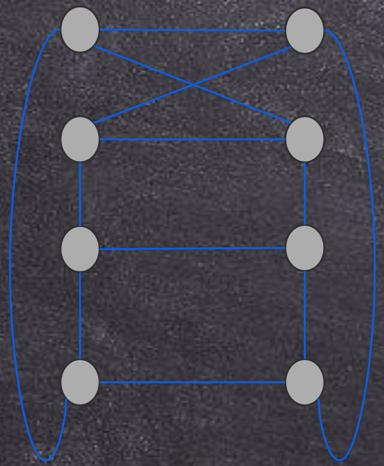


Clique finding and Ramsey theory?



$\otimes$

OR product



$= G^2$

doesn't have a K_6

No C_3 in either of the colors

Ramsey theory is already telling us that you may only minimally violate the E-principle in this case i.e. you can have a K_5 at best.

Clique finding and Ramsey theory?

1. Using our refinement we preserve the information about where the exclusivity is coming from within G.

2. Ramsey numbers tell us what structures need to exist inside different city’s E-graph to have the desired clique inside G.

Violation inside G^2?

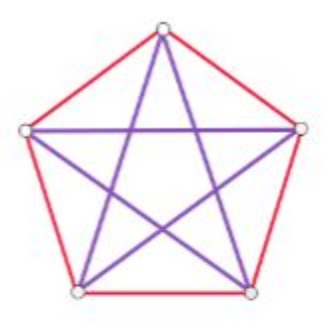
K_6 can't exist due to $R(C_3, C_3) = 6$, but what about K_5?

Violation inside G^2?

K_6 can't exist due to $R(C_3, C_3) = 6$, but what about K_5?

No non-trivial Ramsey number exists for K_5 but good news!

We still don't need to go brute force to look for K_5 inside G^2



Violation inside G^k?

For a given k, which n-cycle contextual boxes violate the E-principle?

Condition of independence: Each joint event occurs with probability $1/2^k$

If a clique of size L, exists inside graph G, then for violation of E-principle:

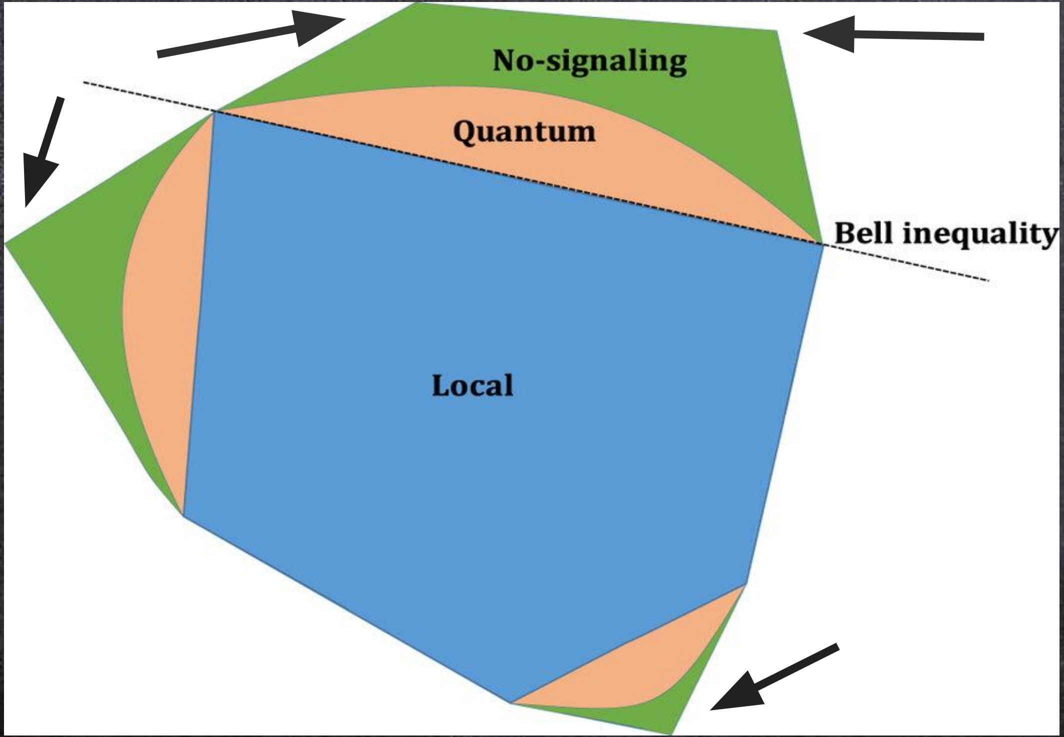
$$L/2^k > 1$$

$$L > 2^k$$

For k copies finding a clique of size $2^k + 1$ inside G^k suffices for violation.

Violation inside G^k?

(n-cycle contextual extremal boxes)



Violation inside G^k?

(n-cycle contextual extremal boxes)

Utilizing known results in Ramsey theory to identify which n-cycle scenarios can witness the violation of E-principle?

For k copies: finding a clique of size $2^k + 1$ inside G^k, irrespective of n, suffices for violation

Multicolour Ramsey numbers of odd cycles

Journal of Combinatorial Theory, Series B

Volume 124, May 2017, Pages 56-63

A. Nicholas Day, J. Robert Johnson

length of the shortest monochromatic odd cycle found in $\mathscr{C}$. It is a simple exercise to see that it is possible to k-colour the complete graph K_{2^k} such that each colour comprises a bipartite graph. Moreover, such colourings only exist for K_n if $n \leqslant 2^k$. Indeed, consider labelling each vertex of K_n with a binary vector of length k, where the ith coordinate of the label given to a vertex is determined by which side of the bipartition of colour i the vertex lies in. All vertices of K_n must receive distinct labels, and so $n \leqslant 2^k$. It follows that any k-colouring of K_{2^k+1} must contain a monochromatic odd cycle. Based on this observation, Erdős and Graham [4] asked the following question:

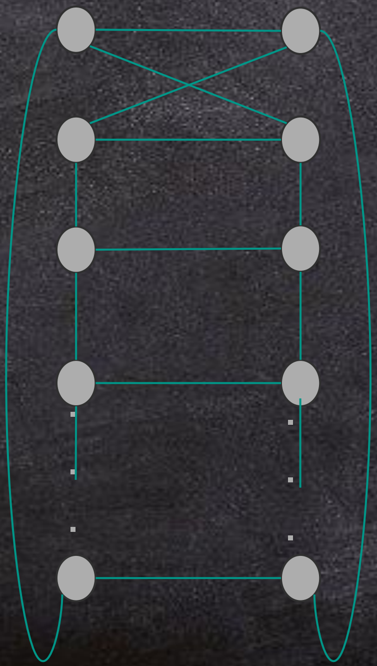
G^k

In simple words

Each of the k city E-graphs must have an odd cycle for a clique of size $2^k + 1$ to exist inside G^k.

n-cycle extremal boxes

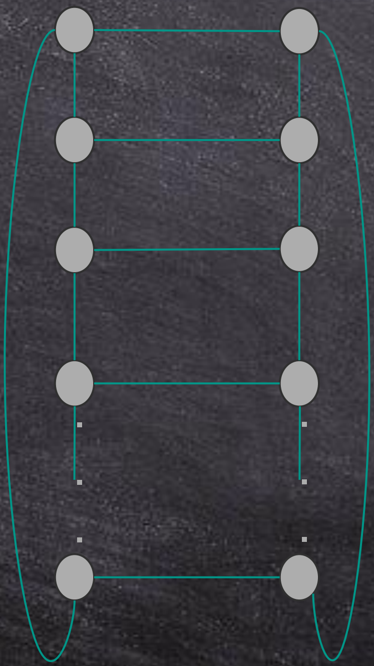
Exclusivity graph : n = even



Mobius Ladder

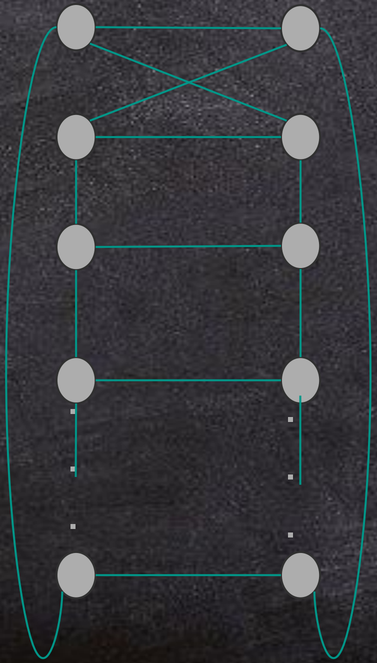
Exclusivity graph : n = odd

Ladder



n-cycle extremal boxes

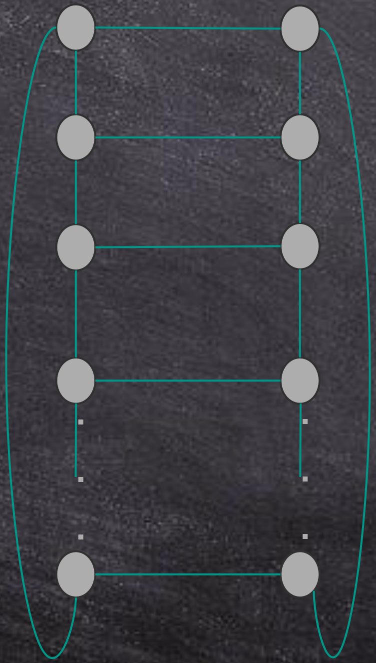
Exclusivity graph : n = even



Mobius Ladder

Smallest odd-cycle:
n + 1

Exclusivity graph : n = odd



Ladder

Smallest odd-cycle:
n

n-cycle extremal boxes

For violation with k = 2 copies, we need a clique of size 5, irrespective of n

n-cycle scenario (n)	Smallest odd-cycle (n+1)
4	5
5	5
6	7

More nodes than in K_5

For n ≥ 6 extremal boxes, k = 2 isn't enough to witness violation of E-principle i.e. no K_5 exists for these boxes. Although it does exist for n=4,5.

n-cycle extremal boxes

For violation with $k = 3$ copies, we need a clique of size 9, irrespective of n

n-cycle scenario (n)	Smallest odd-cycle (n+1)
4	5
5	5
6	7
7	7
8	9
9	9
10	11 → More nodes than K_9

For $n \geq 10$ cycle extremal boxes, $k = 3$ isn't enough to witness violation of E-principle i.e. no K_9 exists for these boxes. It does exist for $n = 4,5$.

n-cycle extremal boxes

For violation with k copies, we need a clique of size $2^k + 1$, irrespective of n

For k copies, no violation exists for extremal boxes from $n \geq 2^k + 2$ cycle scenarios, while we don't have a proof for what happens to the violation for $6 \leq n \leq 2^k + 1$ cycle scenario extremal boxes.

Except for n= 4,5 where we know it for all $k \geq 2$.

n-cycle extremal boxes

For violation with k = 3 copies, we need a clique of size 9, irrespective of n

n-cycle scenario (n)	Smallest odd-cycle (n+1)
4	5
5	5
6	7
7	7
8	9
9	9
10	11

?

More nodes than K_9

For $n \geq 10$ cycle extremal boxes, k = 3 isn't enough to witness violation of E-principle i.e. no K_9 exists for these boxes. It does exist for n = 4,5.

n-cycle extremal boxes

For violation with k = 3 copies, we need a clique of size 9, irrespective of n

ELSEVIER

Journal of Combinatorial Theory, Series B

Volume 20, Issue 3, June 1976, Pages 250-264

Generalized Ramsey theory for multiple colors

P Erdös, R.J Faudree, C.C Rousseau, R.H Schelp

n-cycle extremal boxes

For violation with k = 3 copies, we need a clique of size 9, irrespective of n

ELSEVIER

Journal of Combinatorial Theory, Series B

Volume 20, Issue 3, June 1976, Pages 250-264

Generalized Ramsey theory for multiple colors

P Erdös, R.J Faudree, C.C Rousseau, R.H Schelp

First paper ever to discuss cliques that have more than two colors coloring the edges

n-cycle extremal boxes

For violation with k = 3 copies, we need a clique of size 9, irrespective of n

$$
\begin{aligned}
r(\leqslant(C_7, C_7, C_7)) &= r(\leqslant(C_7, C_7, C_5)) = r(\leqslant(C_7, C_7, C_3)) \\
&= r(\leqslant(C_7, C_5, C_5)) = \boxed{r(\leqslant(C_5, C_5, C_5)) = 9,} \\
r(\leqslant(C_7, C_5, C_3)) &= r(\leqslant(C_5, C_5, C_3)) = 10, \\
r(\leqslant(C_9, C_3, C_3)) &= r(\leqslant(C_7, C_3, C_3)) = r(\leqslant(C_5, C_3, C_3)) = 12,
\end{aligned}
$$

For an edge-colored K_9 with three colors to exist, at least one of the colors must have an odd cycle of size 5 or less

n-cycle extremal boxes

For violation with k = 3 copies, we need a clique of size 9, irrespective of n

$$
\begin{aligned}
r(\leqslant(C_7, C_7, C_7)) &= r(\leqslant(C_7, C_7, C_5)) = r(\leqslant(C_7, C_7, C_3)) \\
&= r(\leqslant(C_7, C_5, C_5)) = \boxed{r(\leqslant(C_5, C_5, C_5)) = 9,} \\
r(\leqslant(C_7, C_5, C_3)) &= r(\leqslant(C_5, C_5, C_3)) = 10, \\
r(\leqslant(C_9, C_3, C_3)) &= r(\leqslant(C_7, C_3, C_3)) = r(\leqslant(C_5, C_3, C_3)) = 12,
\end{aligned}
$$

For an edge-colored K_9 with three colors to exist, at least one of the colors must have an odd cycle of size 5 or less which is impossible to exist for n ≥ 6, since the smallest odd cycle then has size 7.

n-cycle extremal boxes

1. For all $k \geq 2$, all extremal boxes for $n = 4,5$ violate the E-principle.
2. For $k = 3$, no violation exists for any extremal box of $n \geq 6$ cycle scenarios.
3. For $k \geq 4$, violation exists for all extremal boxes of $n \geq 2^k + 2$ cycle scenarios while the same can't be said when $6 \leq n \leq 2^k + 1$.

Physical significance of activation effects?

1. To identify non-quantum behaviors: Violation of E-principle means that the correlation is outside the quantum set.

2. Any good principle - aiming to single out quantum theory - must reject non-classical extremal boxes as non-quantum: activation effects might help benchmark ‘goodness’ of physical principles.

Hardness of finding Ramsey numbers?

Computational complexity [edit]

Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, he believes, we should attempt to destroy the aliens.[6]

Mathematicians have only been able to exactly compute a handful of the smallest Ramsey numbers. They proved that $r(4, 5) = 25$ in 1995. But nobody knows the value of $r(4, 6)$. Similarly, in the early 1980s, they showed that $r(3, 9) = 36$, but $r(3, 10)$ remains an open problem. (The symmetric case is just as difficult: $r(4) = 18$, but the value of $r(5)$ is not known.)

And so mathematicians instead try to estimate Ramsey numbers — coming up with upper and lower bounds on their values.

Ramsey Theory in the Work of Paul Erdős

R. L. Graham[1] and J. Nešetřil[2]

[1] AT&T Bell Laboratories, Murray Hill, NJ 07974
[2] Charles University, Praha, The Czech Republic

Summary. Ramsey's theorem was not discovered by P. Erdős. But perhaps one could say that Ramsey theory was created largely by him. This paper will attempt to demonstrate this claim.

Logical Structure

Violation of E-principle

Refinement

Finding edge-colored clique C inside the joint exclusivity graph G^k

Ramsey theory

Existence of monochromatic structures inside C

Our proof

Same structures must exist inside individual city E-graphs

Our proof

Capturing range of n-cycle scenario extremal boxes that don't have such structures

contradiction: hence no violation

Thank you

Questions?

Say about the Ramsey theory being one of the hardest problems to solve

These numbers measure the size that graphs must reach before inevitably containing objects called cliques.

any finite coloring of a large enough system contains a monochromatic subsystem of higher degree of organization than the system itself, Proving violation is harder compared to guaranteeing no violation using Ramsey numbers.