U.PORTO
FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

A practical introduction to ZX-calculus

Group: Quantum and Linear Optical Computation (INL)
PhD Project: Optimizing models of hybrid quantum/classical computation

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Supervisor: Professor Ernesto Galvão
Co-supervisor: Professor João Lopes dos Santos

Outline

1. Working with ZX-calculus and ZX-diagrams

2. Application in circuit compilation

3. Application in classical simulation

https://zxcalculus.com/publications.html

B. Coecke and R. Duncan, A graphical calculus for quantum observables (2007)
Preprint: http://www.cs.ox.ac.uk/people/bob.coecke/GreenRed.pdf

J. van de Wetering, ZX-calculus for the working quantum computer scientist (2020). arXiv:2012.13966.

What is ZX-calculus?

A diagrammatic language for reasoning about quantum computations

What is a ZX-diagram?

A graphical depiction of a linear map between qubits

Elements of ZX-diagrams

Wires

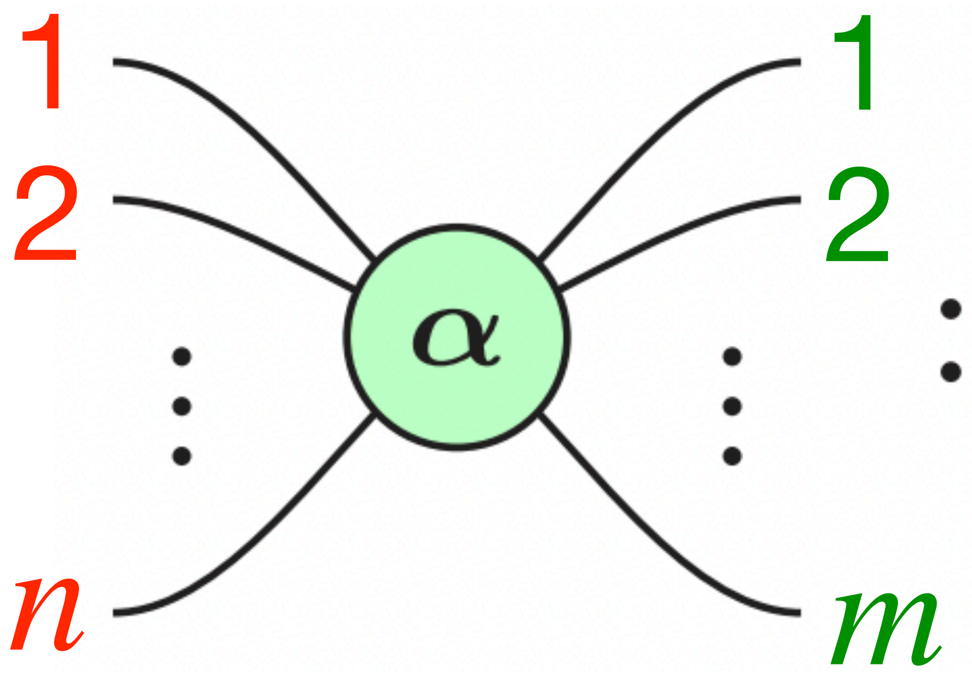
- Wires entering from the left $\rightarrow$ inputs
- Wires exiting tho the right $\rightarrow$ outputs

Spiders

- Linear operations (maps) with any number of input and output wires

$|0\rangle^{\otimes n}$

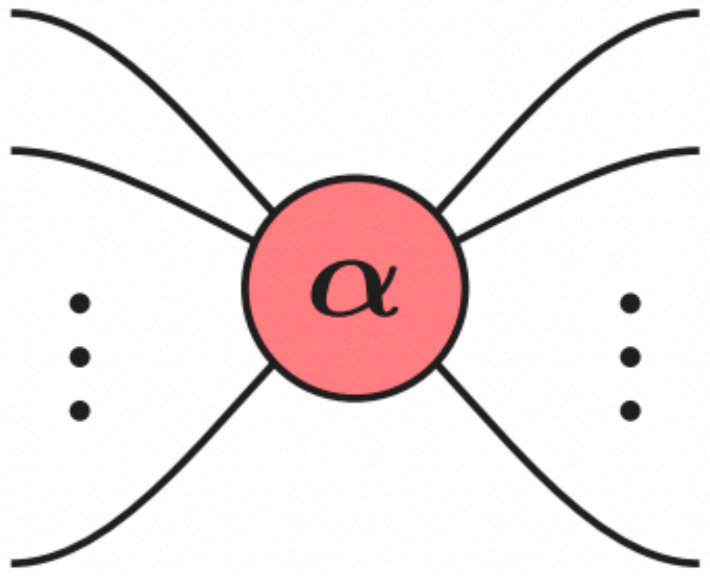
Z spider:



$$:= |0...0\rangle\langle 0...0| + e^{i\alpha}\,|1...1\rangle\langle 1...1|$$

$|0\rangle^{\otimes m}$

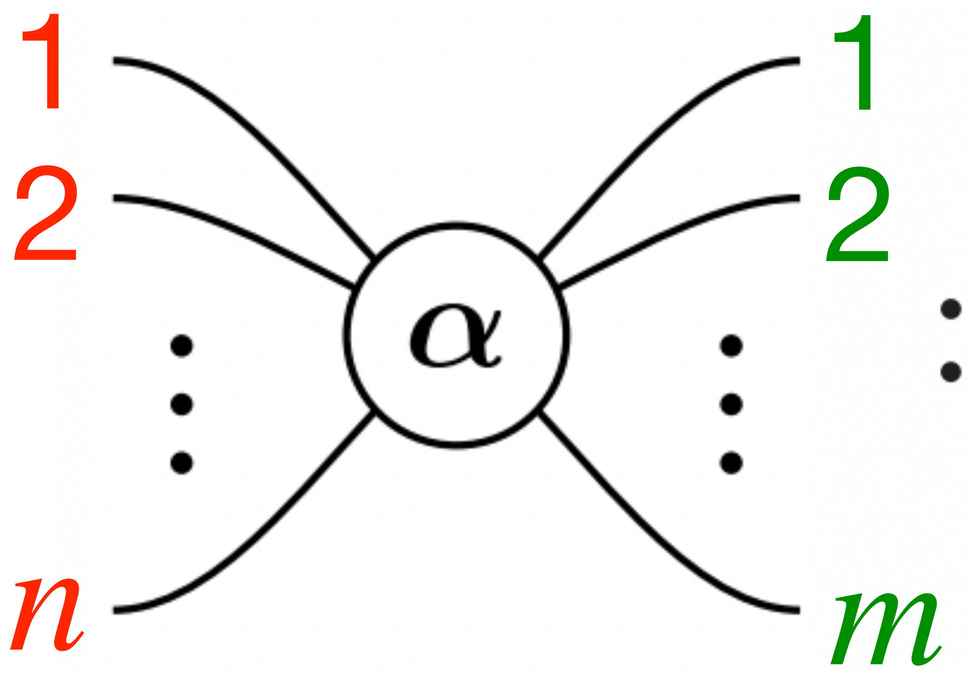
X spider:



$$:= |+...+\rangle\langle +...+| + e^{i\alpha}\,|-...-\rangle\langle -...-|$$

$$|0\rangle^{\otimes n}$$

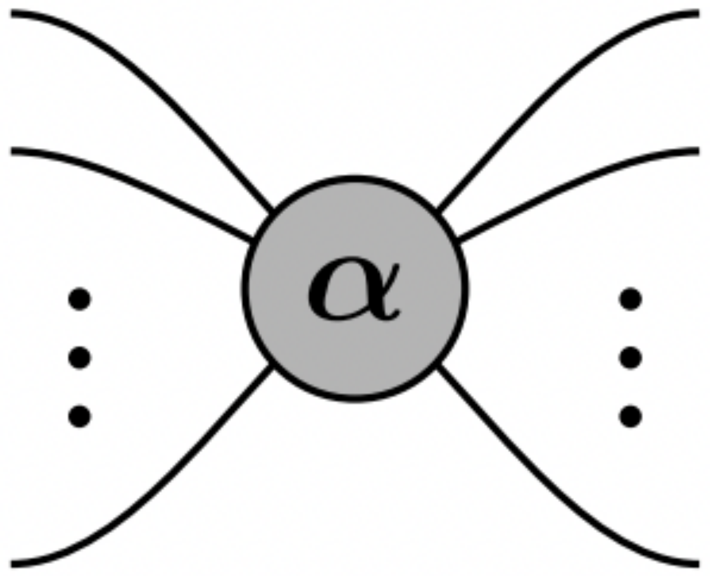
Z spider:



$$:= |0...0\rangle\langle 0...0| + e^{i\alpha}|1...1\rangle\langle 1...1|$$

$$|0\rangle^{\otimes m}$$

X spider:



$$:= |+...+\rangle\langle +...+| + e^{i\alpha}|-...-\rangle\langle -...-|$$

 $= |0\rangle + |1\rangle = \sqrt{2}|+\rangle$

 $= |+\rangle + |-\rangle = \sqrt{2}|0\rangle$

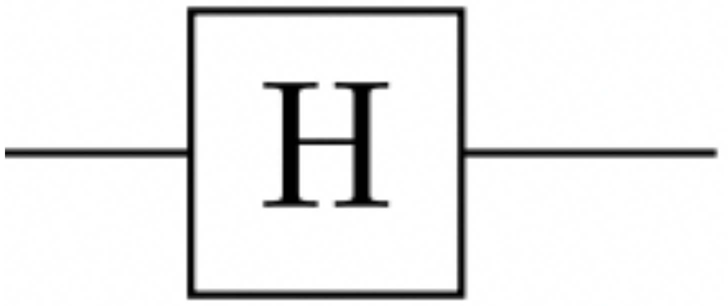
$= \left|A\right\rangle \propto \left|0\right\rangle + e^{i\pi/4}\left|1\right\rangle$

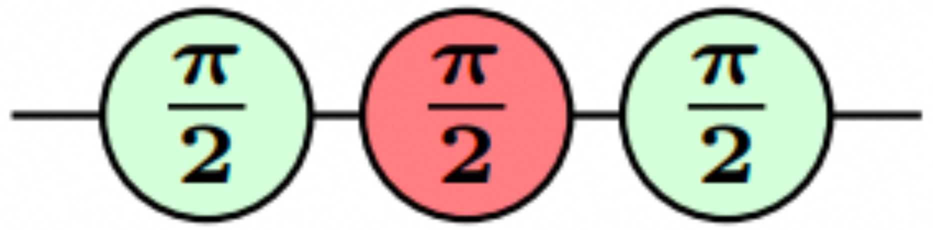
$= |0\rangle\langle 0| + e^{i\alpha}|1\rangle\langle 1| = Z_\alpha$

$= Z$

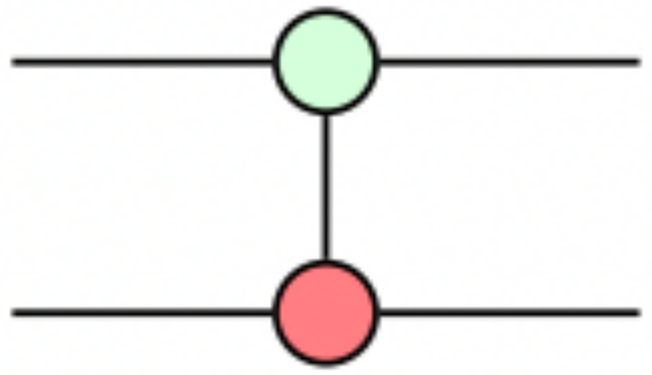
$= |+\rangle\langle +| + e^{i\alpha}|-\rangle\langle -| = X_\alpha$

$= X$



$= e^{-i\frac{\pi}{4}}$


$=:$

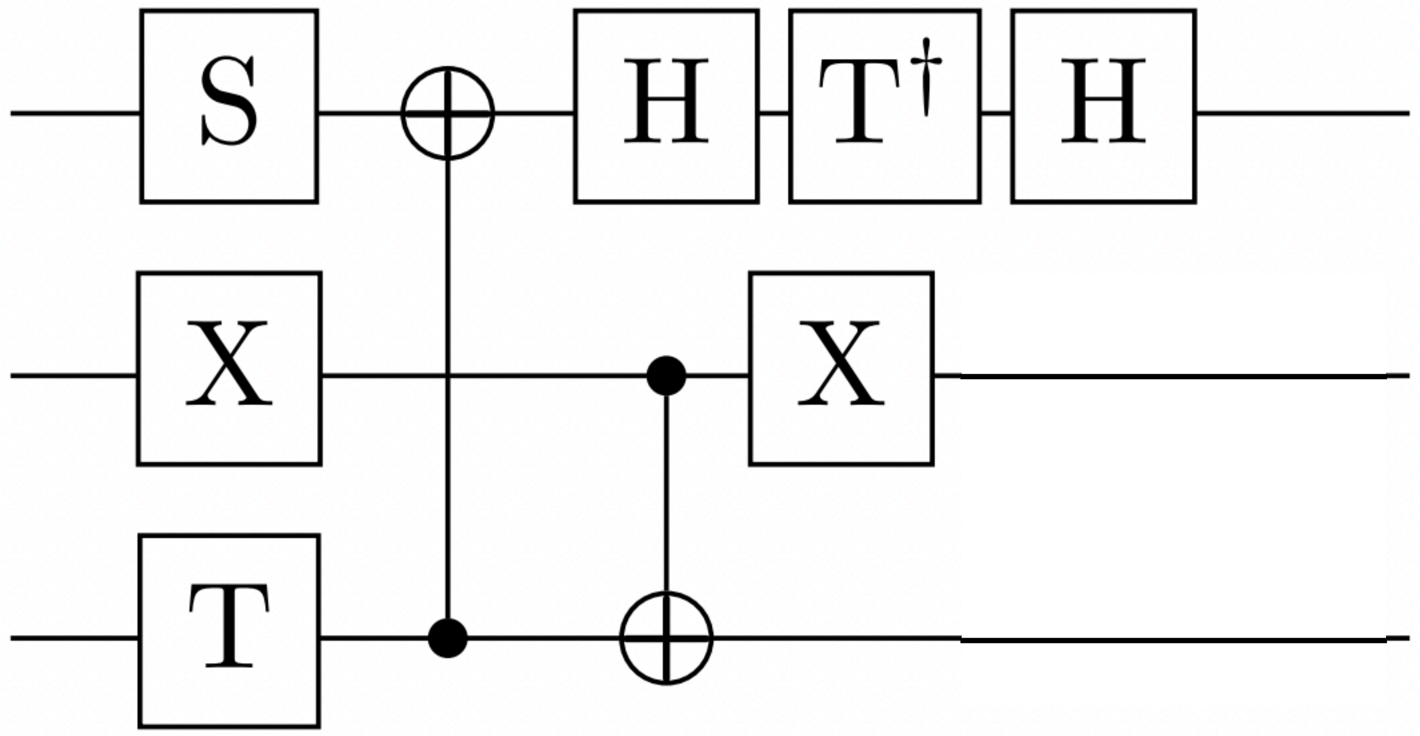

$\mathrm{CNOT} = \sqrt{2}$


Hadamard edge:

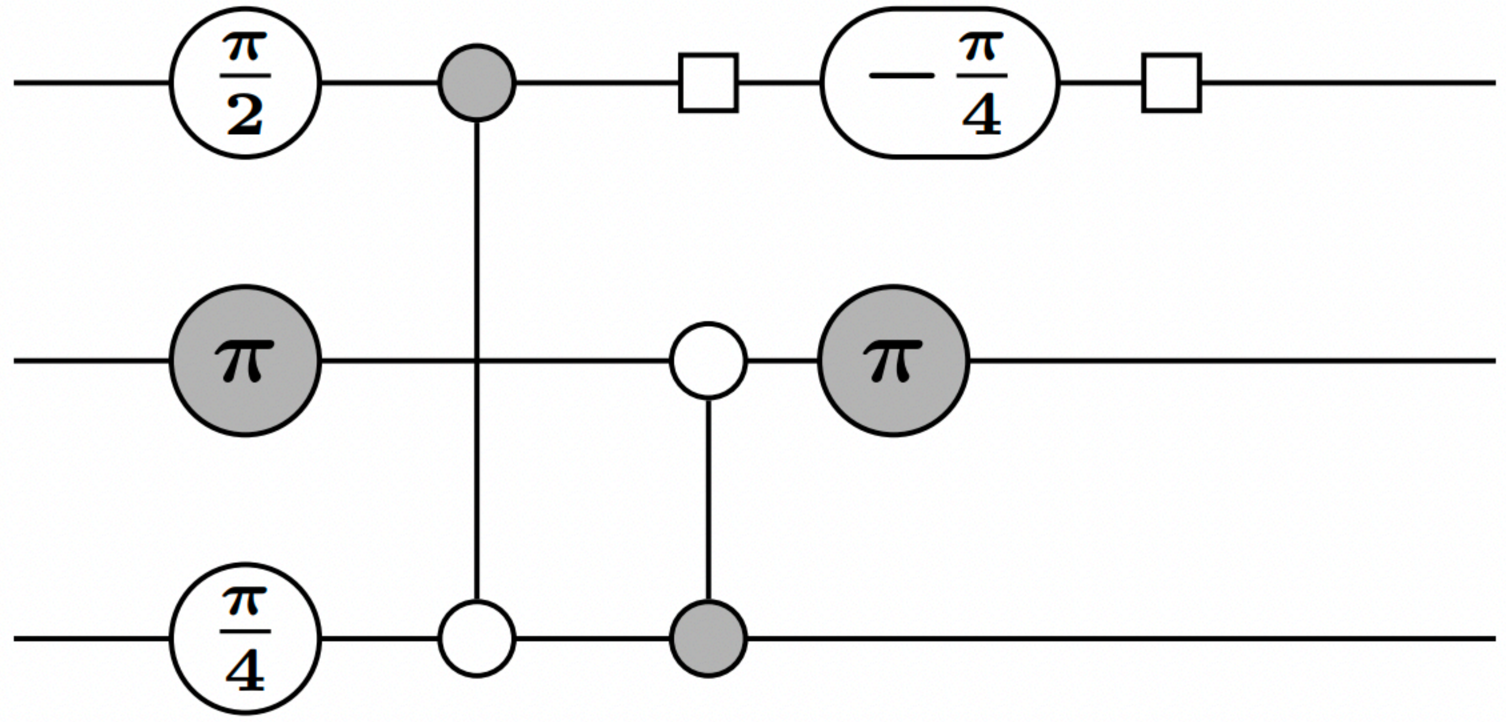
exercise 1:

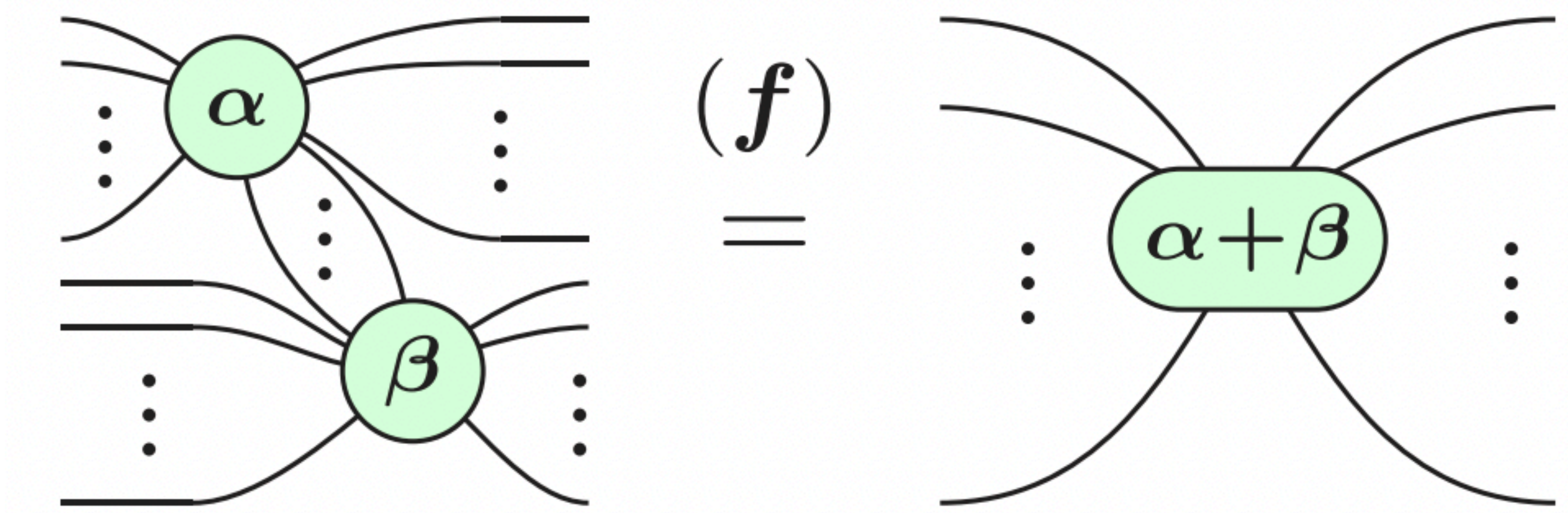
$\rightsquigarrow$

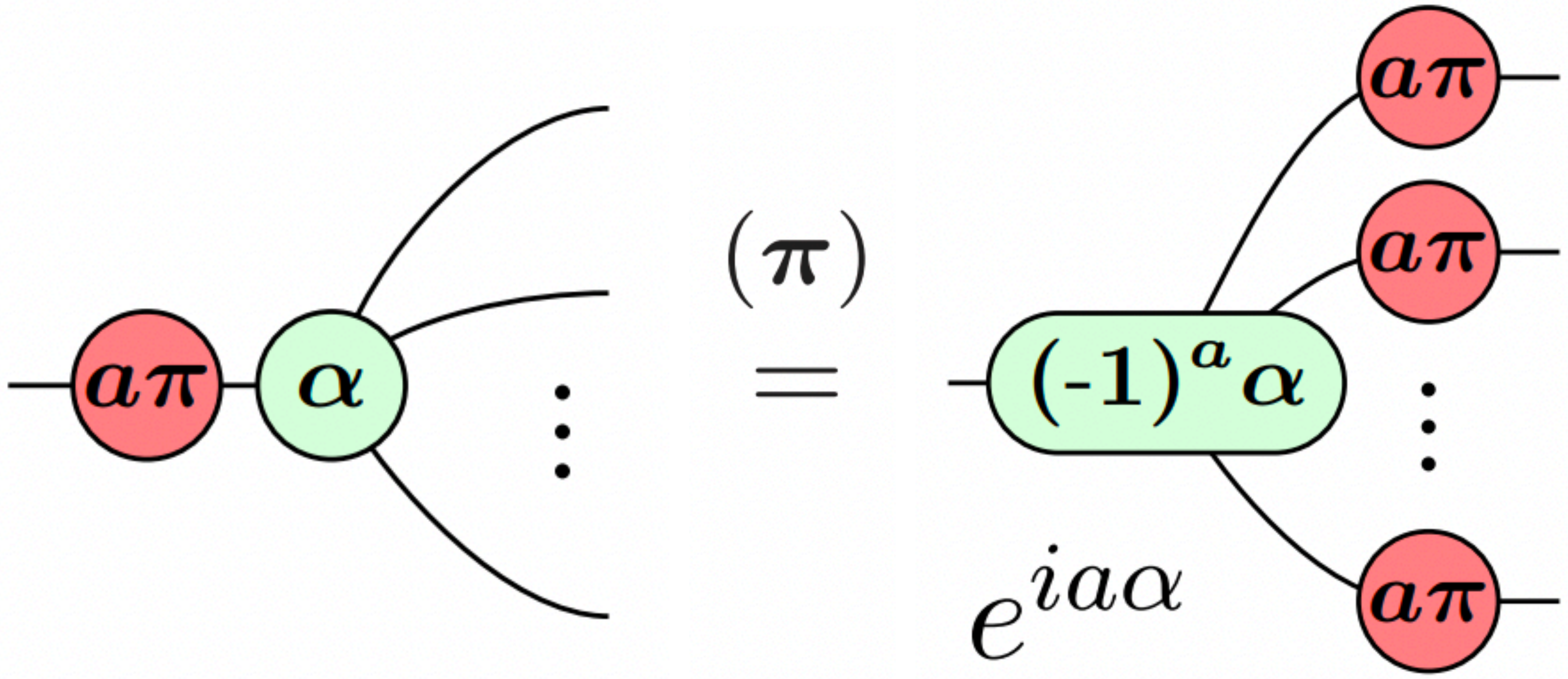
exercise 2:



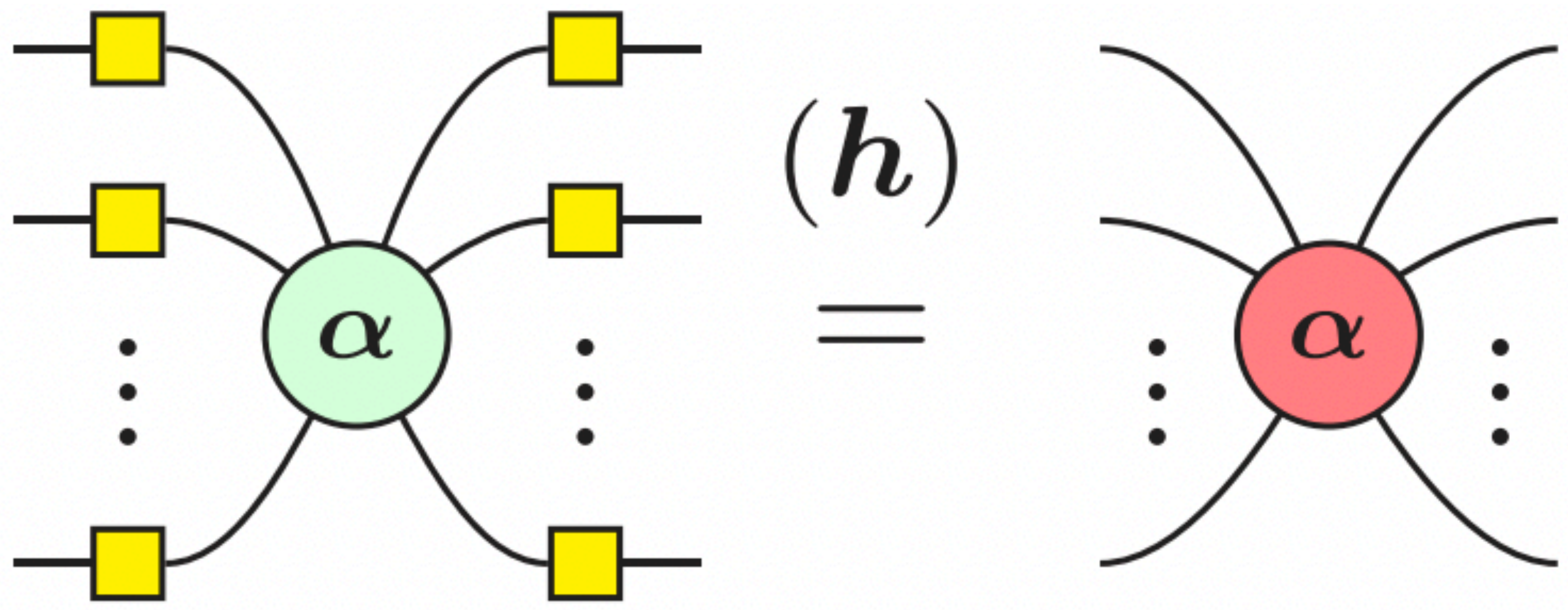
$\rightsquigarrow$

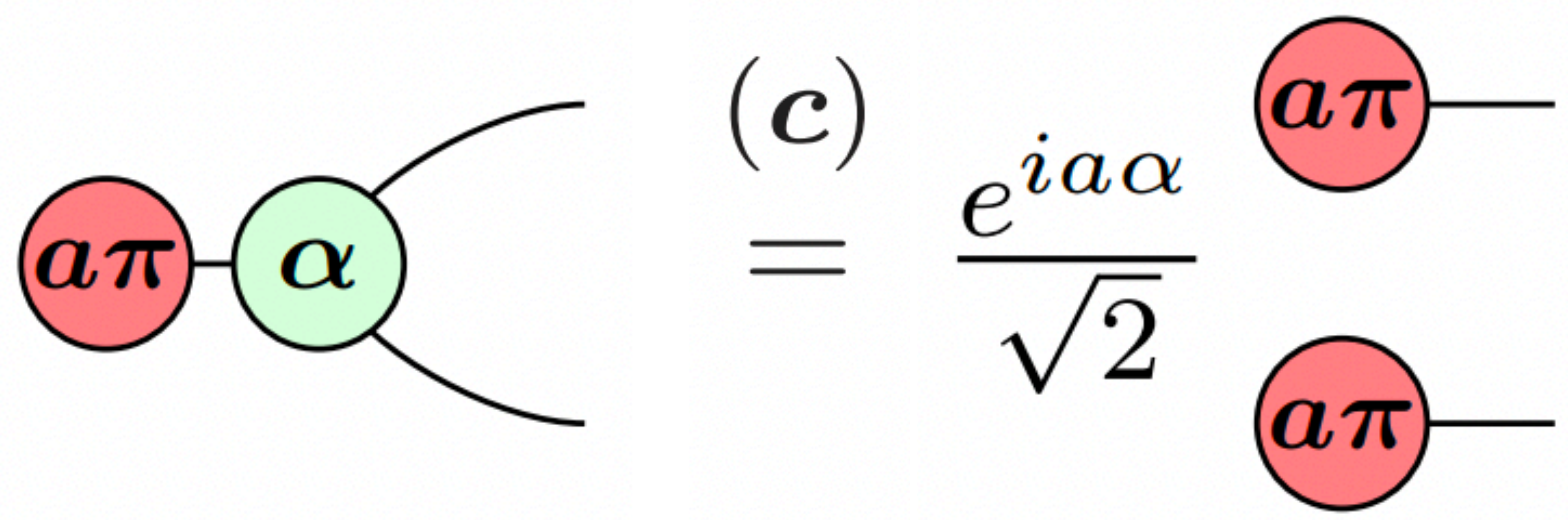


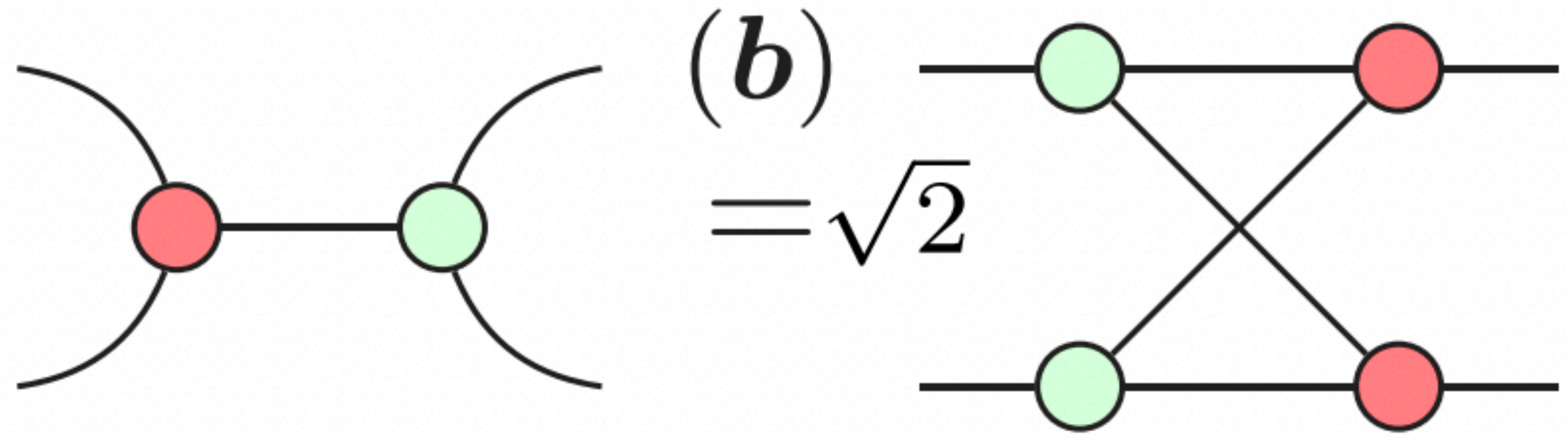

α
β
(f)
=
α+β

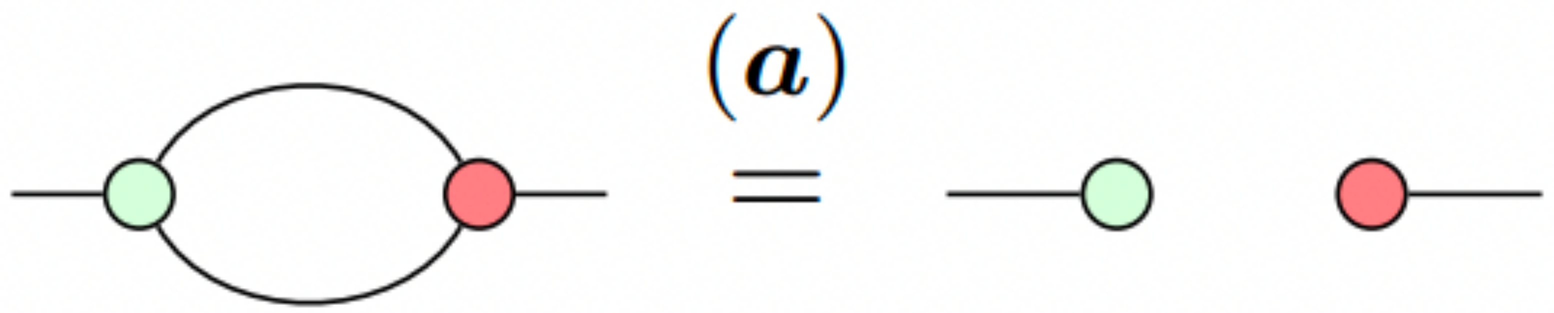

aπ
α
(π)
=
(-1)^a α
aπ
aπ
aπ
e^{iaα}

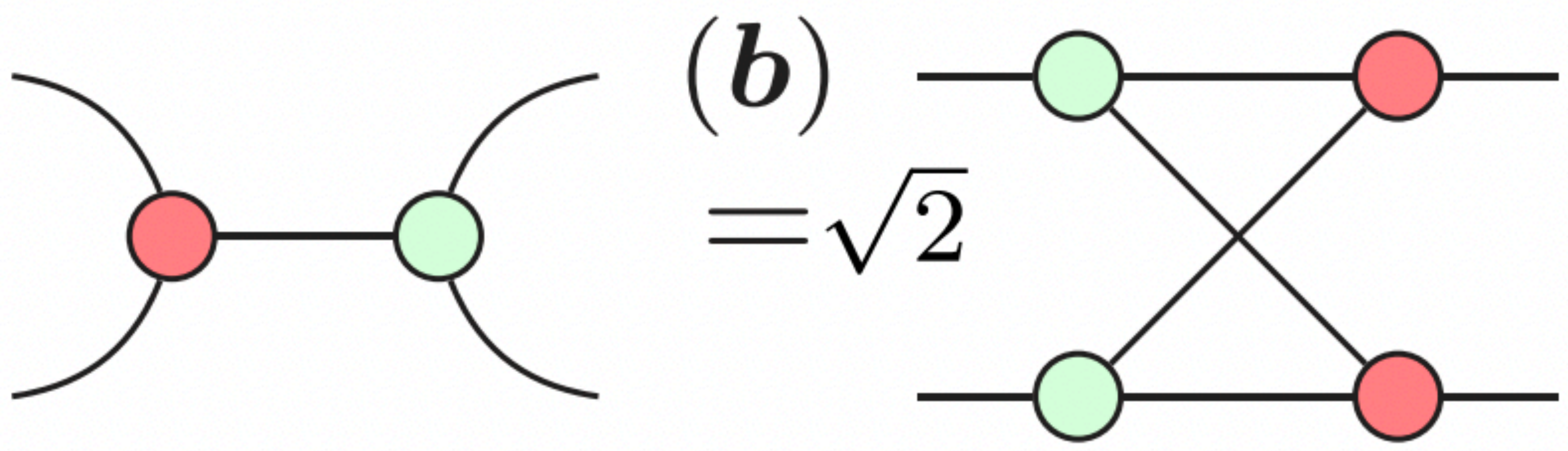

(i1)
=
(i2)
=

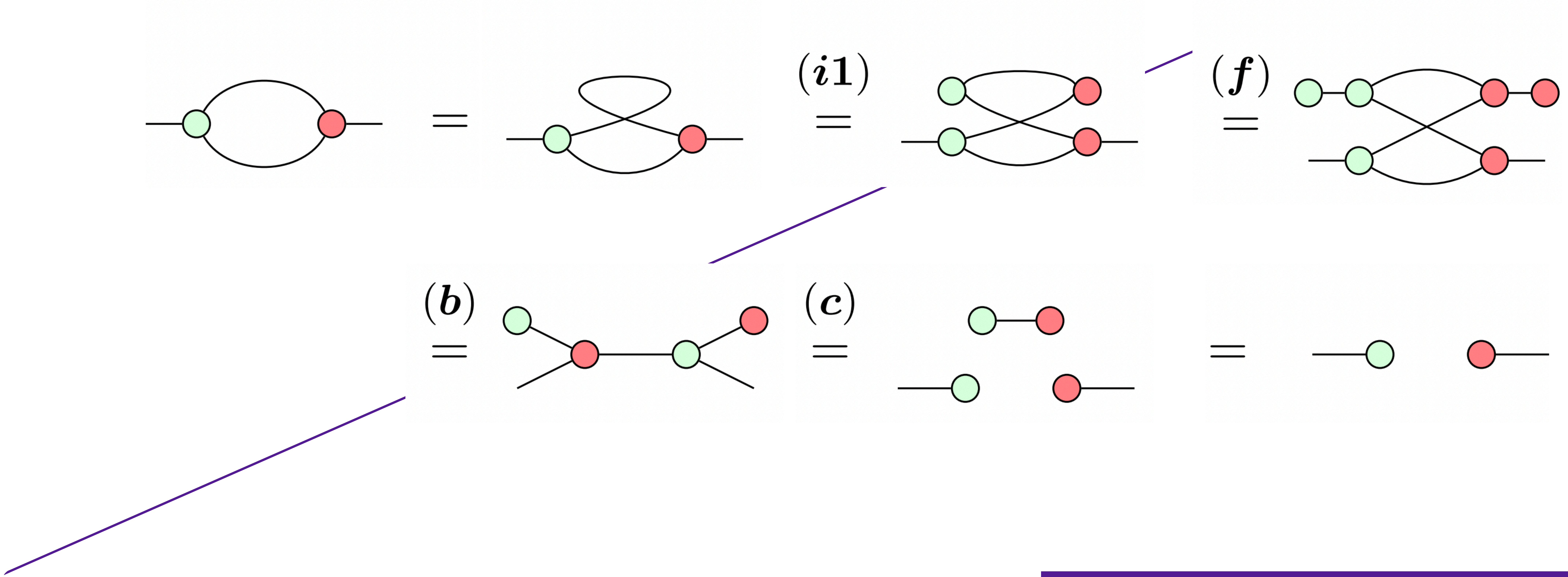

α
(h)
=
α


aπ
α
(c)
=
e^{iaα}/√2
aπ
aπ


(b)
=√2


(a)
=


(b)
=√2


=
(i1)
=
(f)
=
(b)
=
(c)
=
=

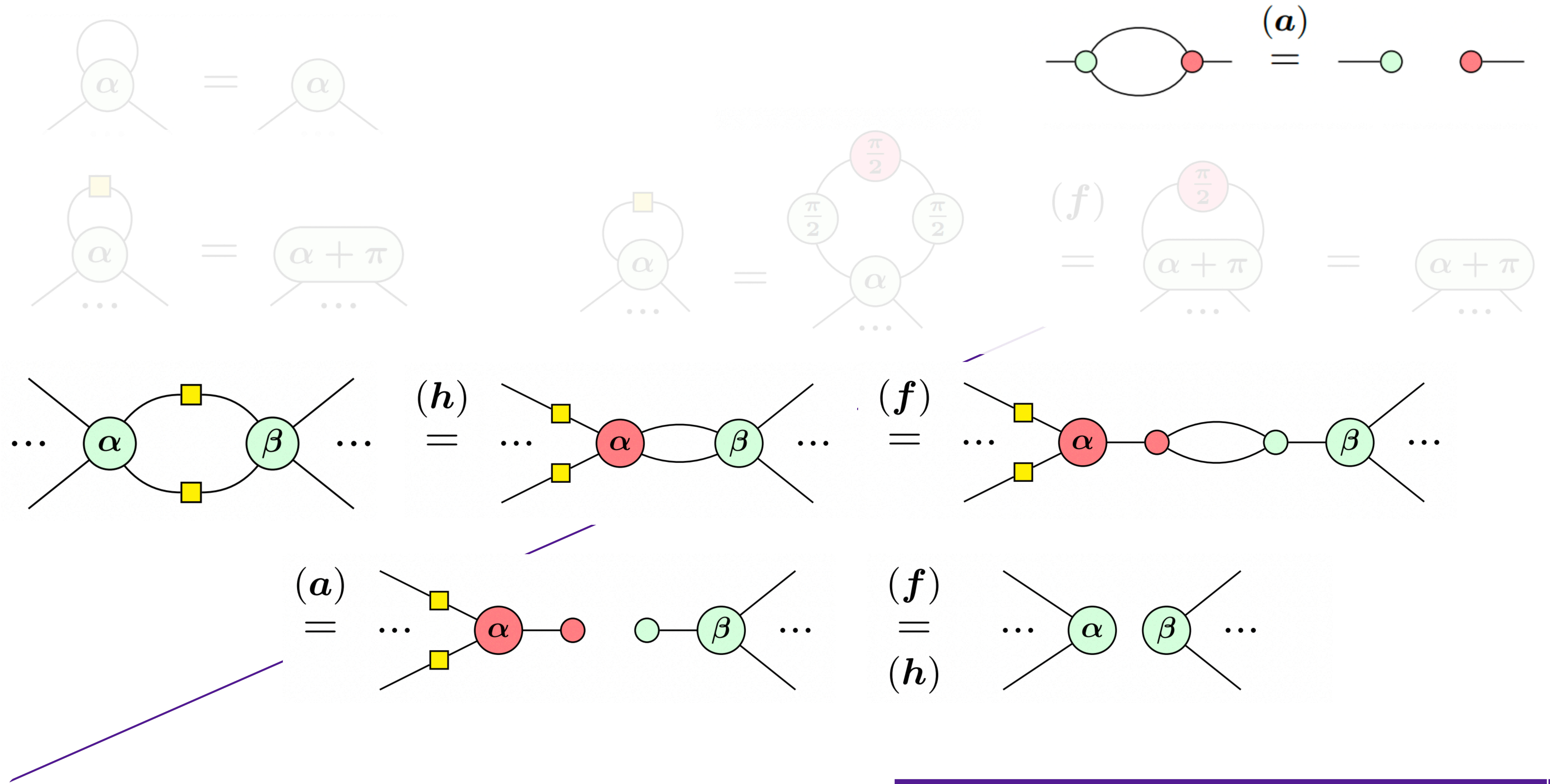
Definition: A ZX-diagram is ***graph-like*** when:

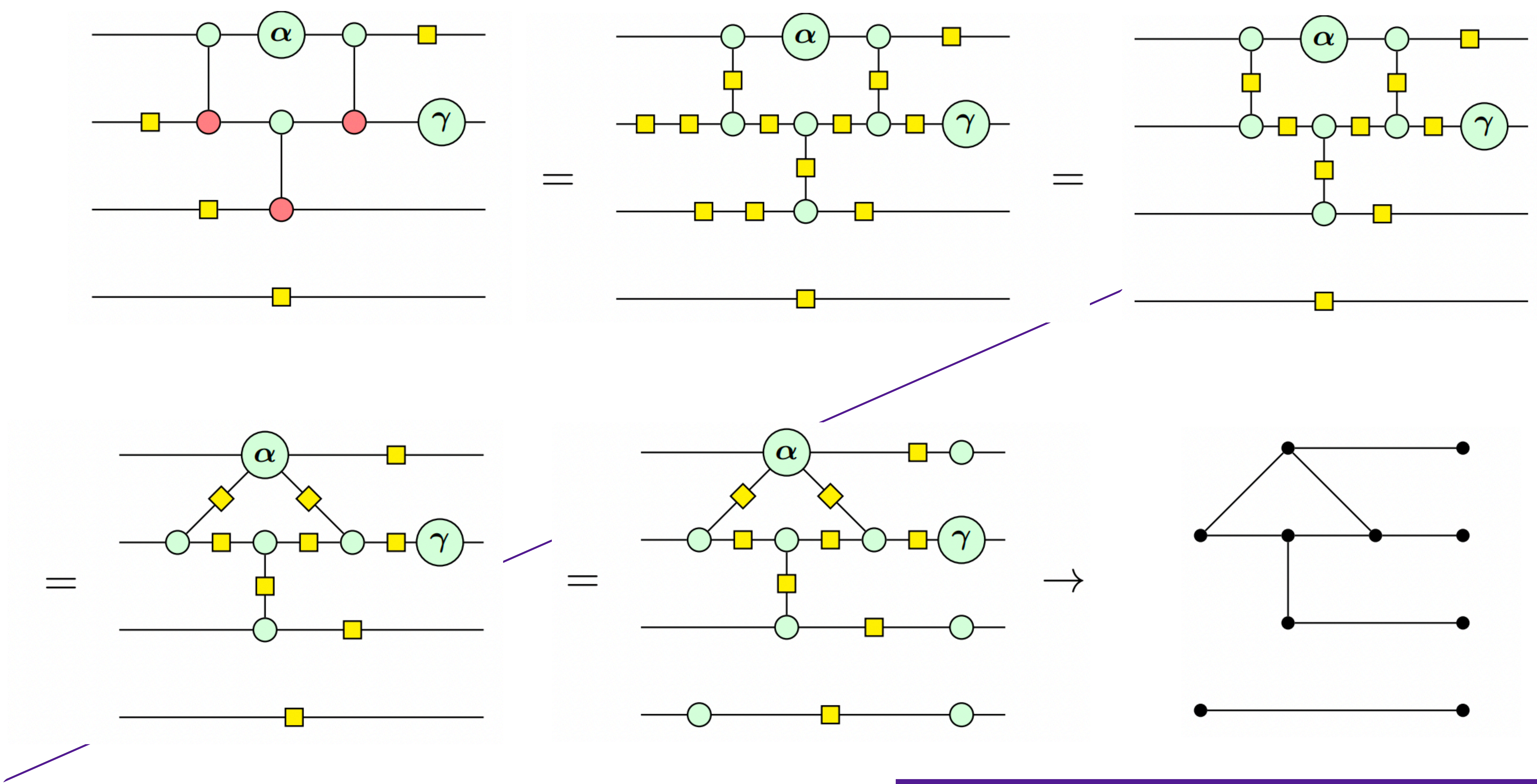
1. All spiders are Z-spiders;

2. Z-spiders are only connected through Hadamard edges;

3. There are no parallel Hadamard edges or self-loops;

4. Every input or output is connected to a Z-spider and every Z-spider is connected to at most one input or output.

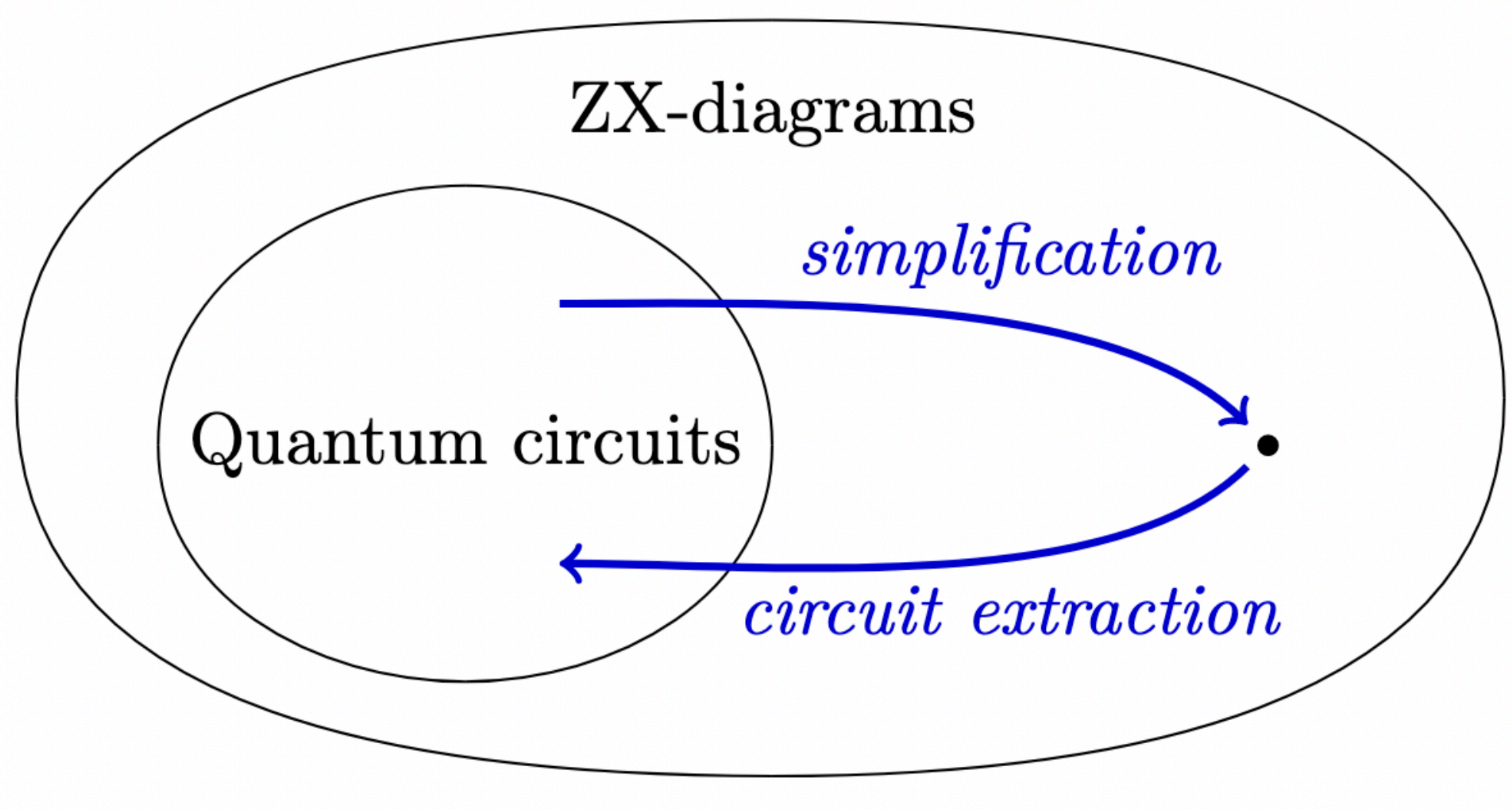


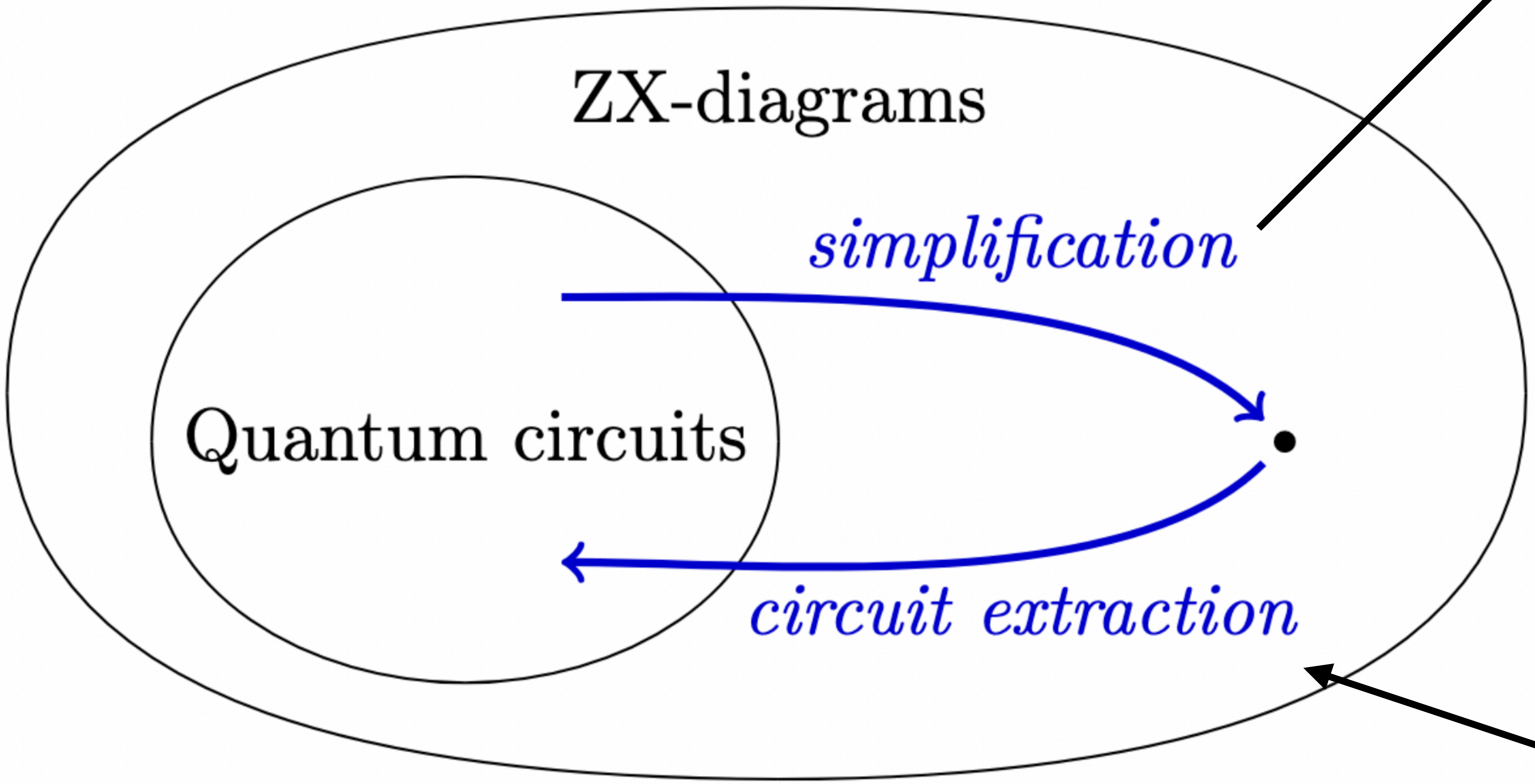


https://zxcalculus.com/publications.html

R. Duncan, A. Kissinger, S. Pedrix, and J. van de Wetering, Graph-theoretic simplification of quantum circuits with the ZX calculus, Quantum **4**, 279 (2020).

A. Kissinger and J. van de Wetering. Reducing the number of non-Clifford gates in quantum circuits. Physical Review A, **102**, 022406 (2020).

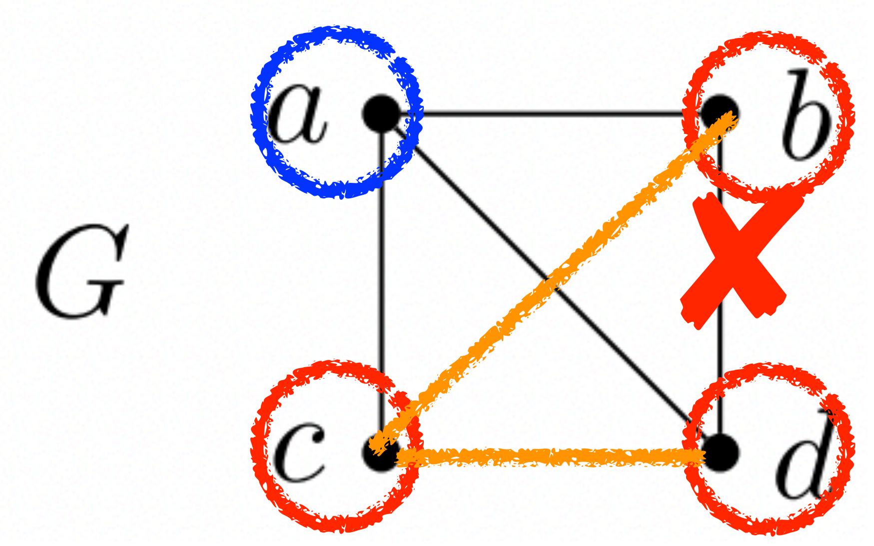

ZX-diagrams
Quantum circuits
simplification
circuit extraction

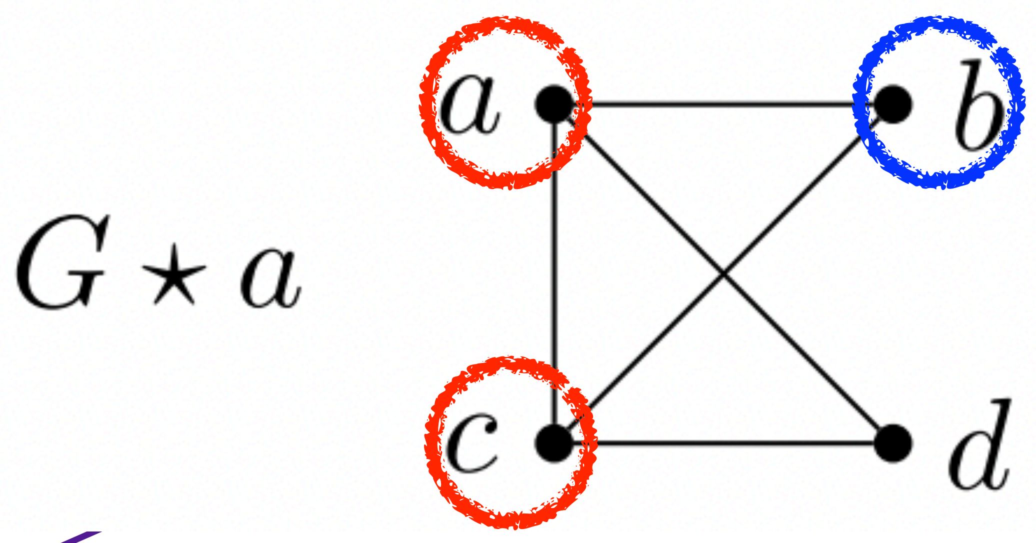


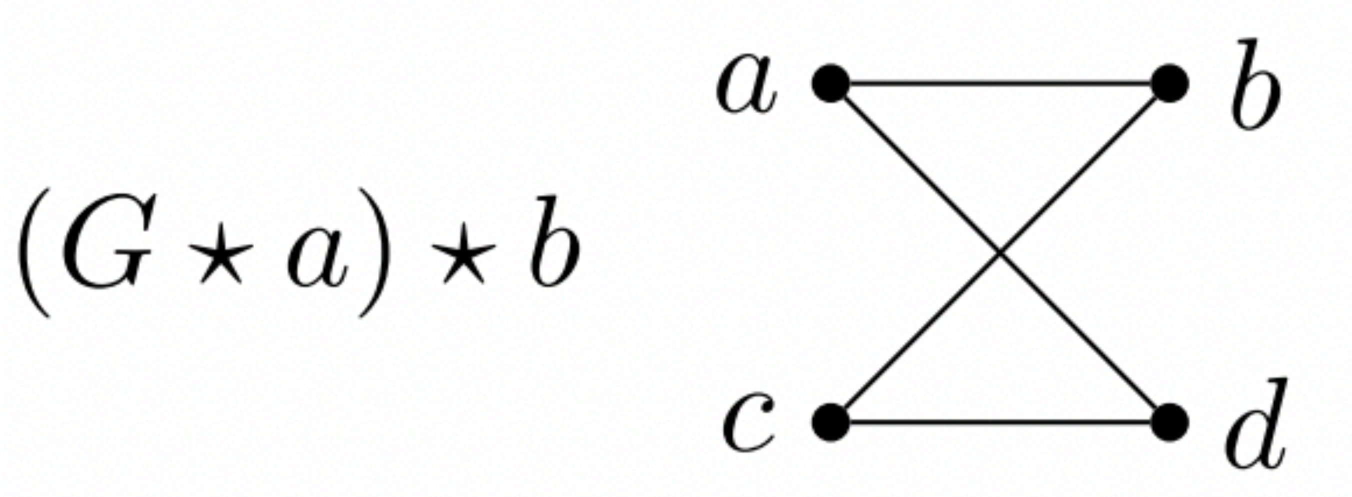
Key simplification steps involve the graph-theoretic transformations of **local complementation** and **pivoting**

Strategy preserves the *focused gFlow*

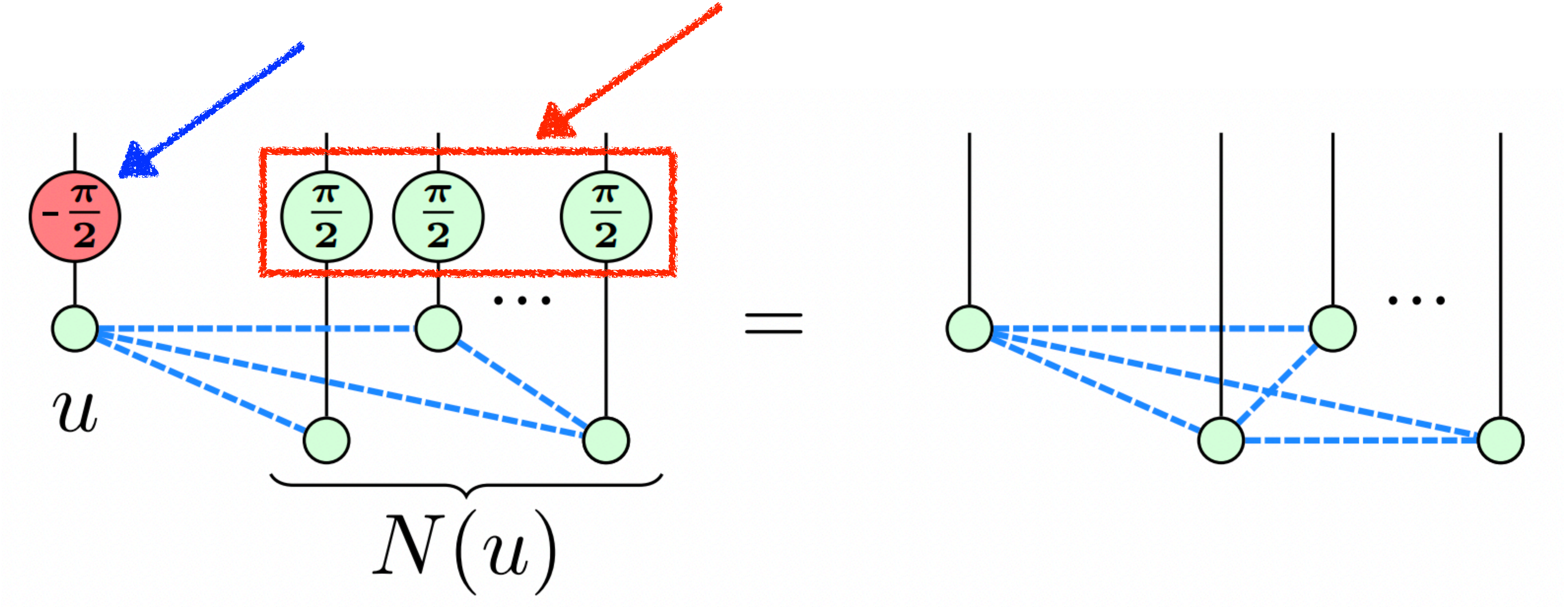
Local complementation:



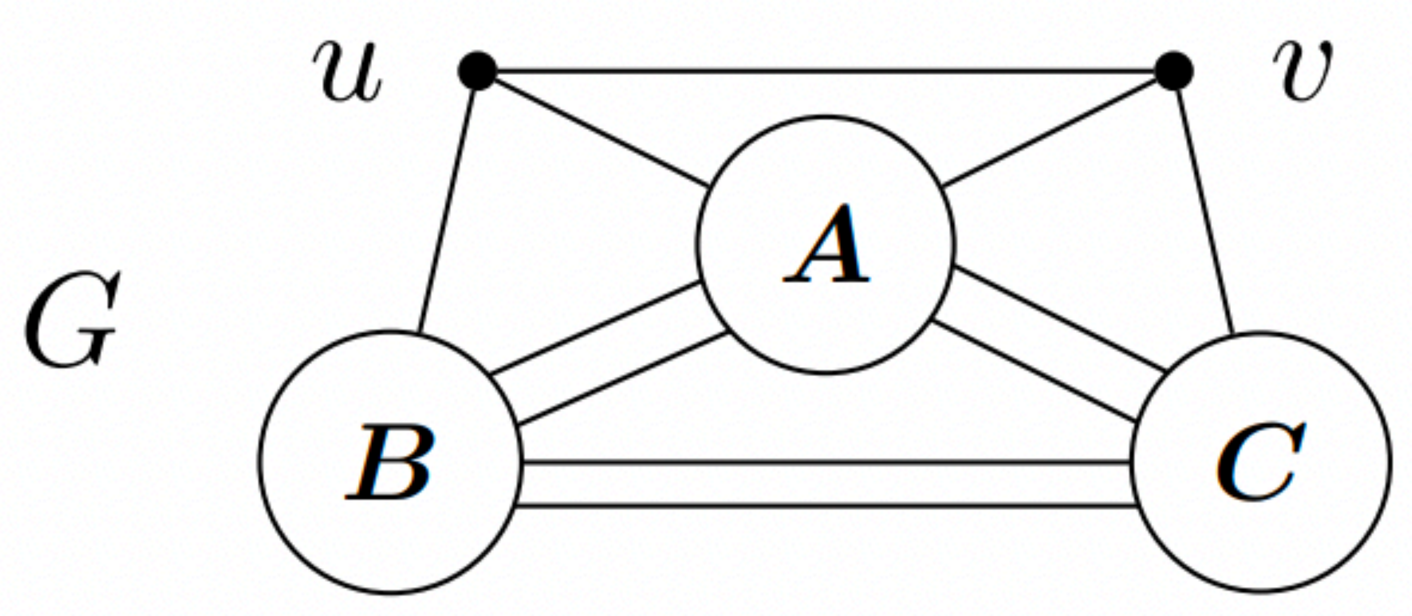


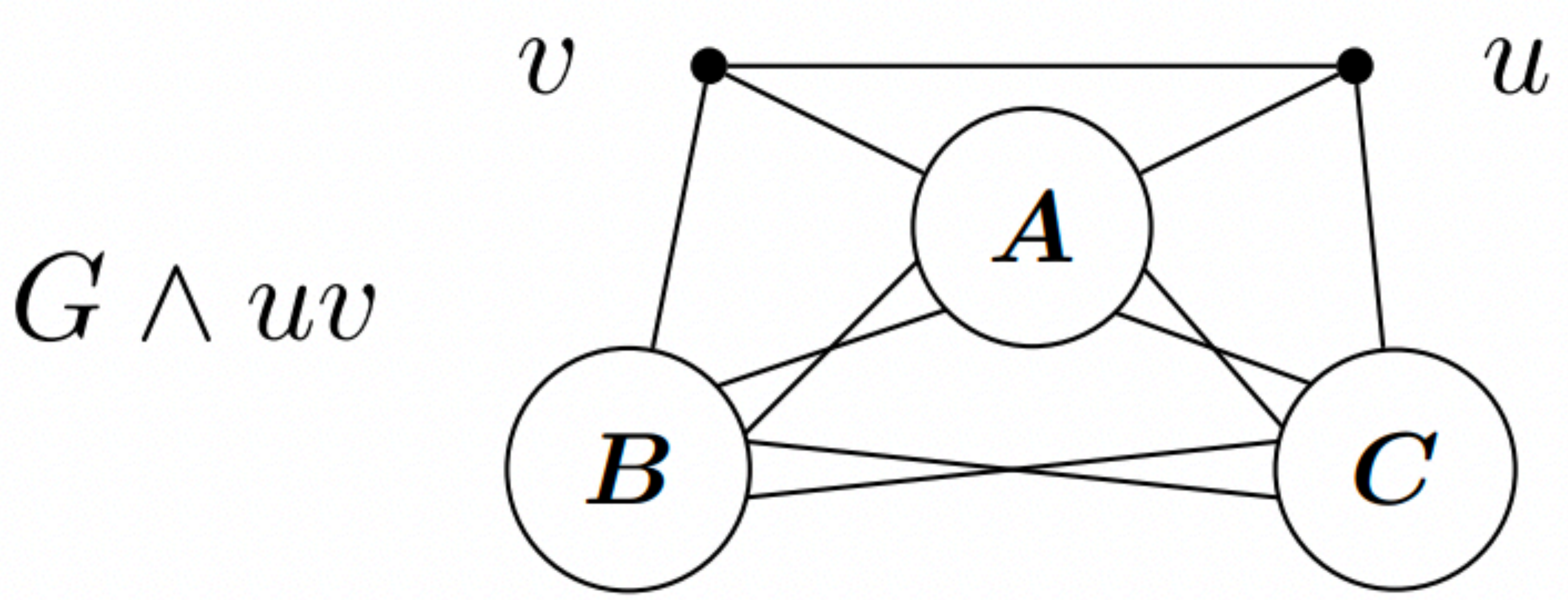


In the special case where the ZX-diagram is graph-like (i.e., represents a graph-state)…

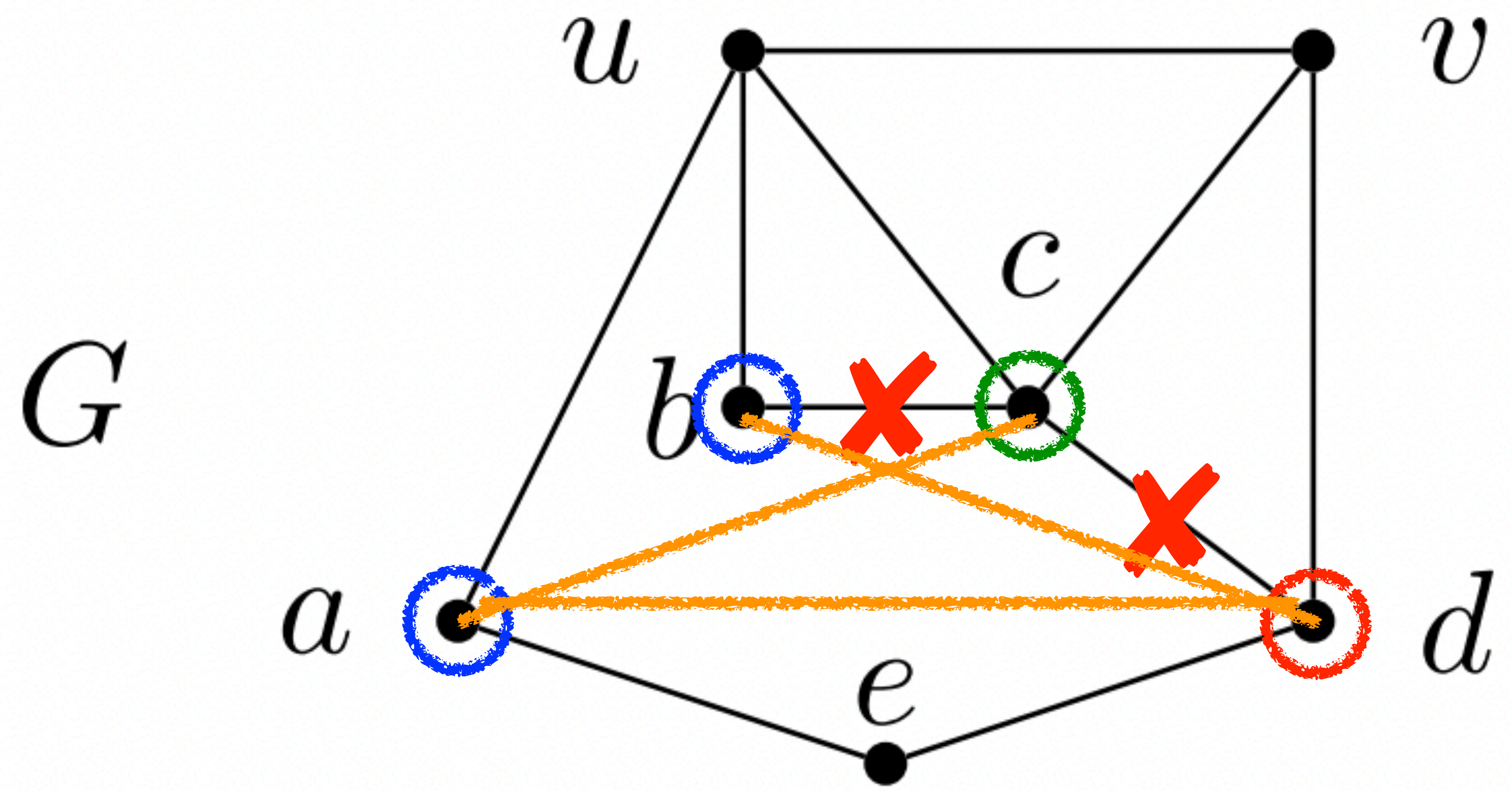


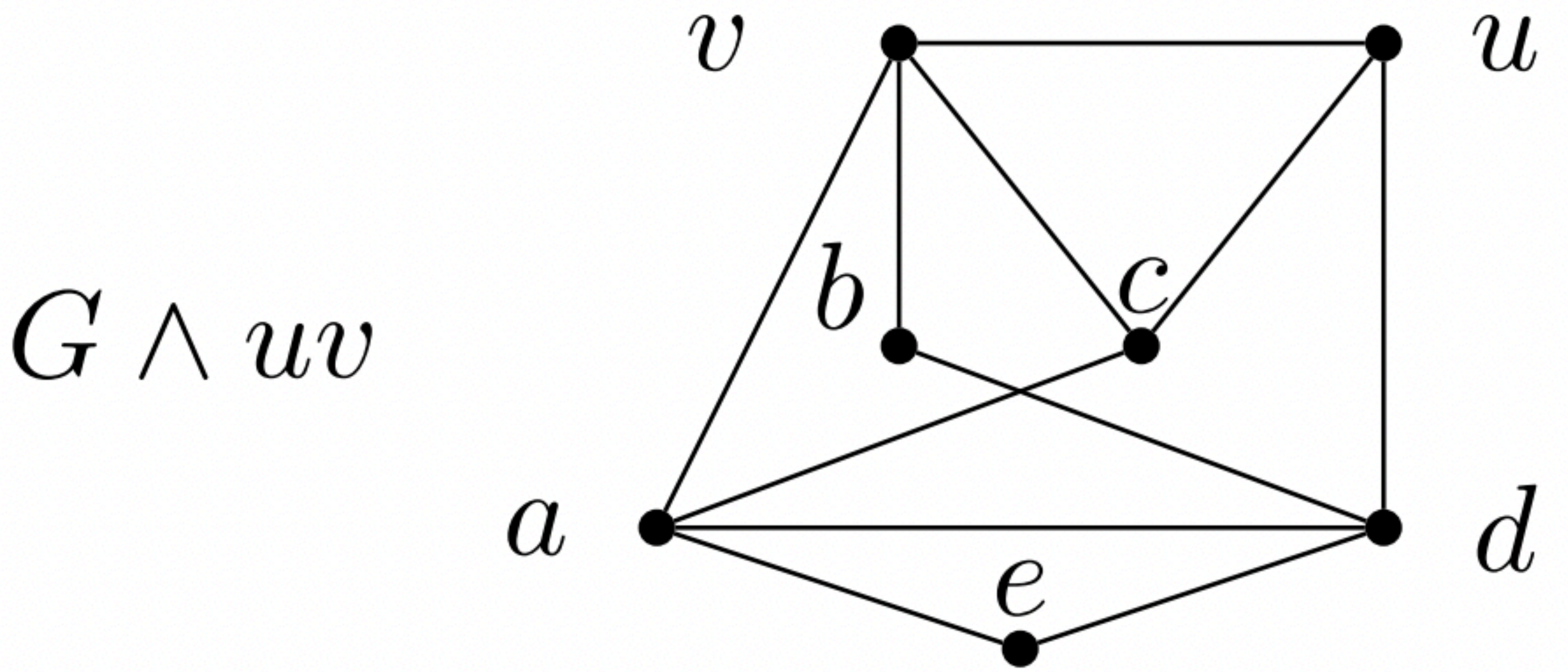
Pivoting:



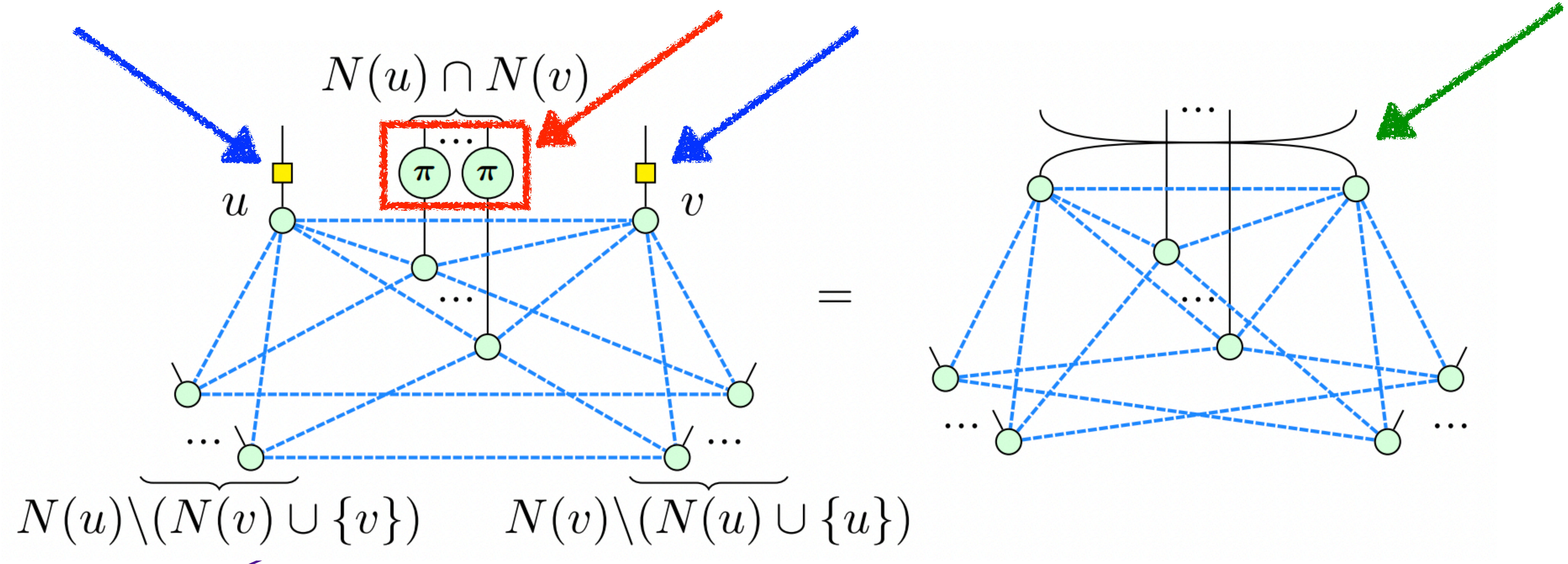


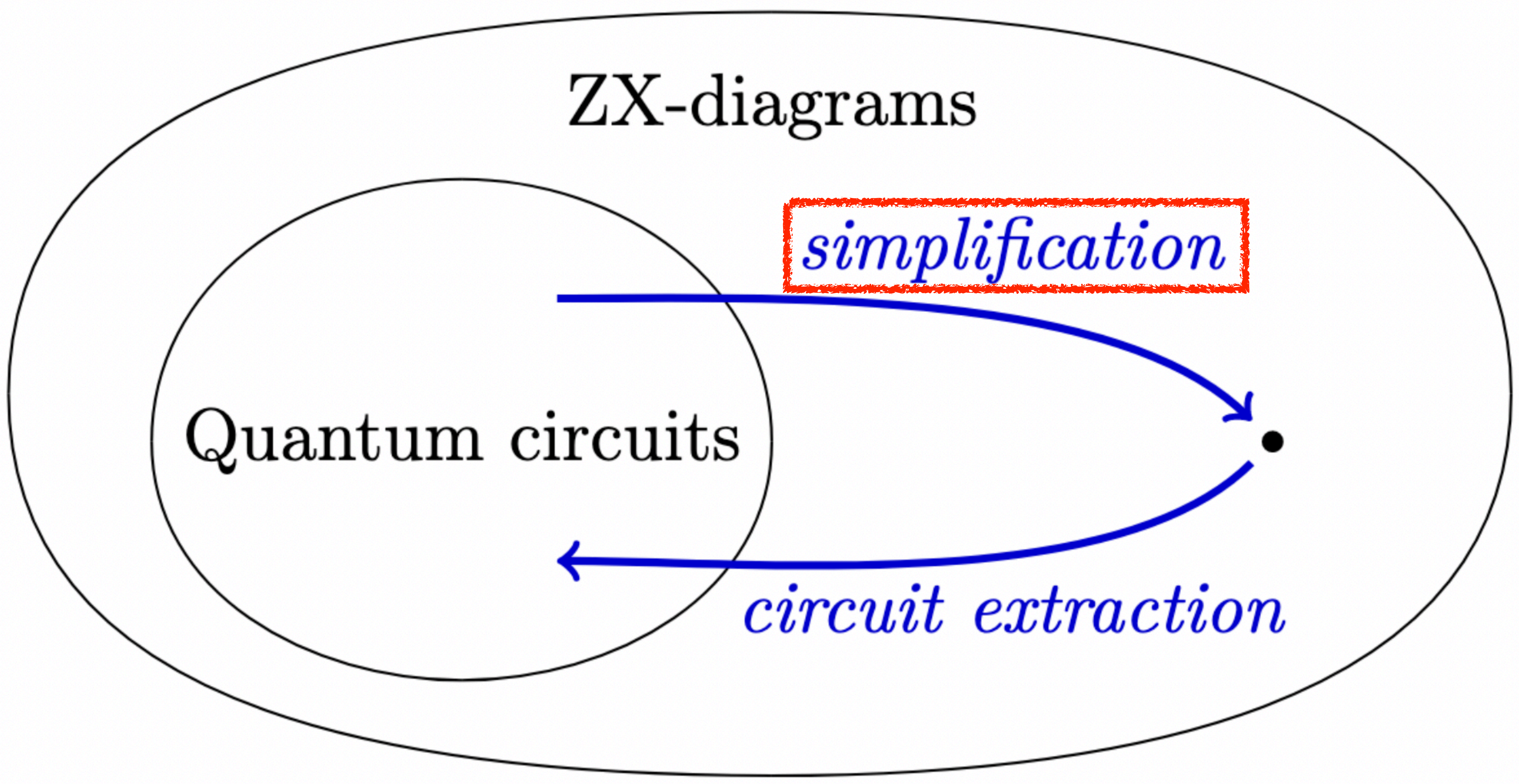
example:





In the special case where the ZX-diagram is graph-like (i.e., represents a graph-state)…

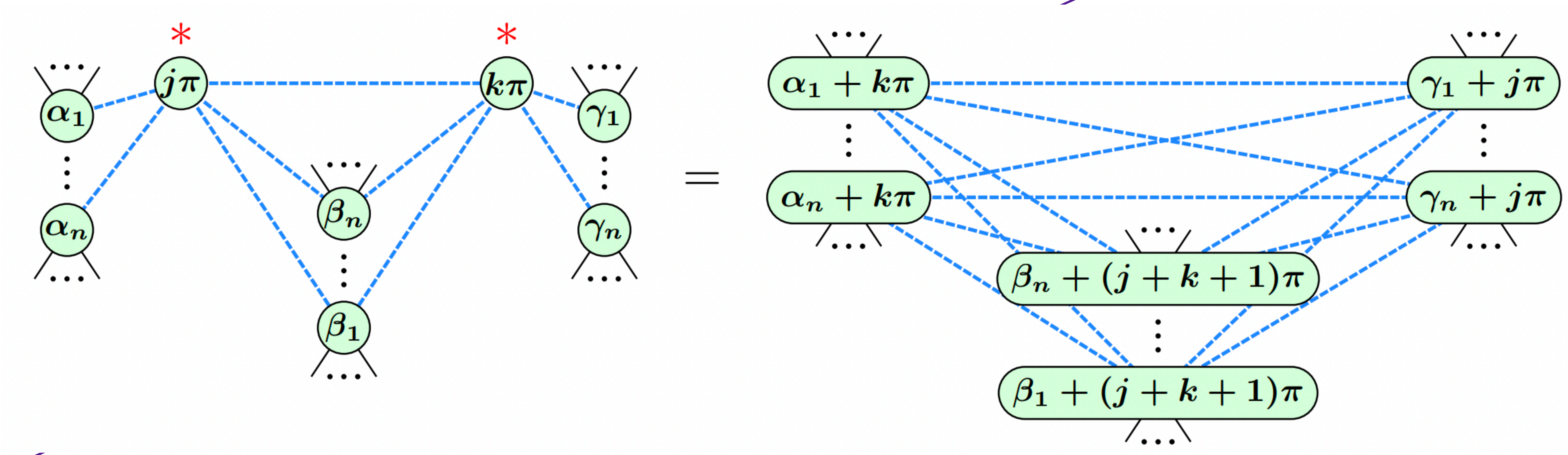



ZX-diagrams
simplification
Quantum circuits
circuit extraction

Procedure is able to eliminate:

1. all interior proper Clifford spiders;

2. and all Pauli spiders adjacent either to a boundary spider or any interior Pauli spider.

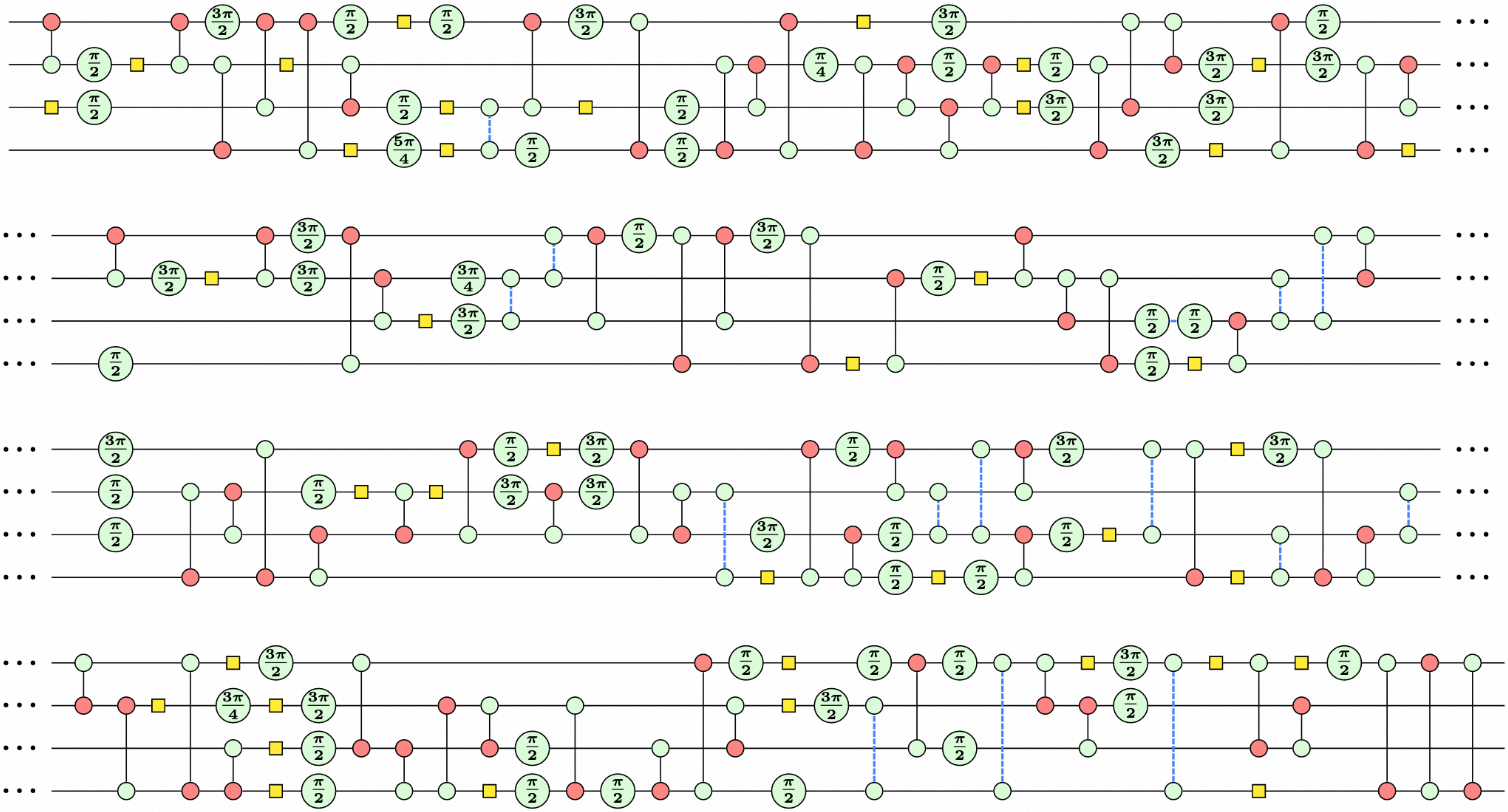


Theorem 5.4. There exists a terminating procedure which turns any graph-like ZX-diagram D into a graph-like ZX-diagram D' (up to single-qubit unitaries on inputs/outputs) which does not contain

1. interior proper Clifford spiders,
2. adjacent pairs of interior Pauli spiders,
3. and interior Pauli spiders adjacent to a boundary spider.

In particular, if D only contains Clifford spiders, then D' contains no interior spiders.

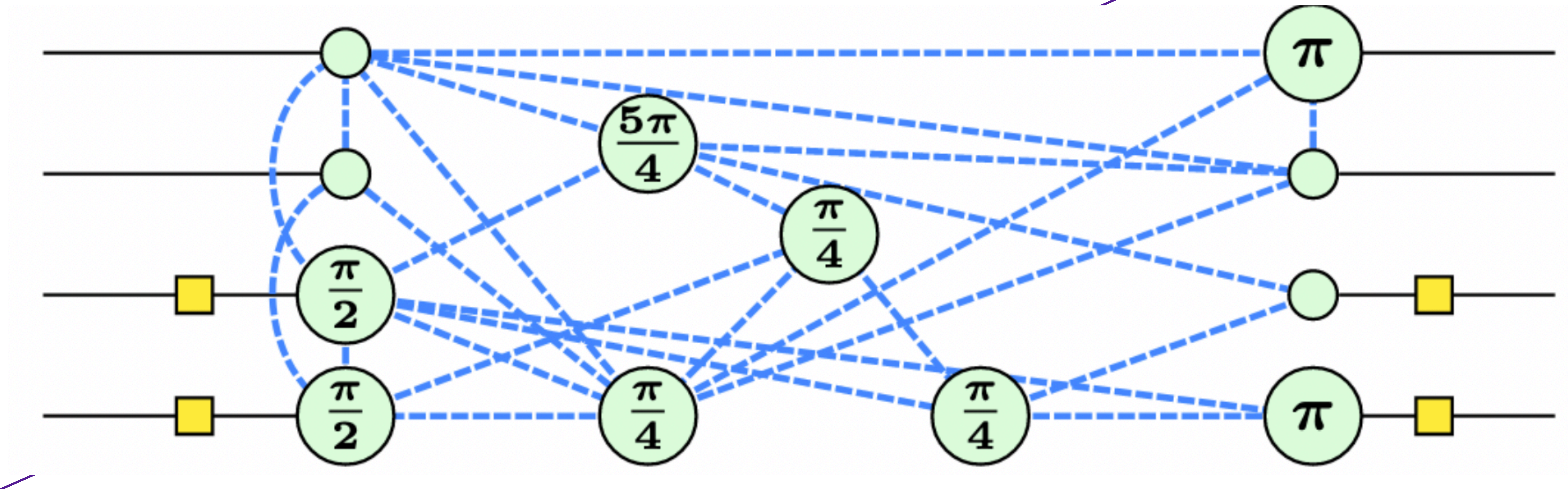
This procedure is very effective, especially for circuits with few non-Clifford gates.

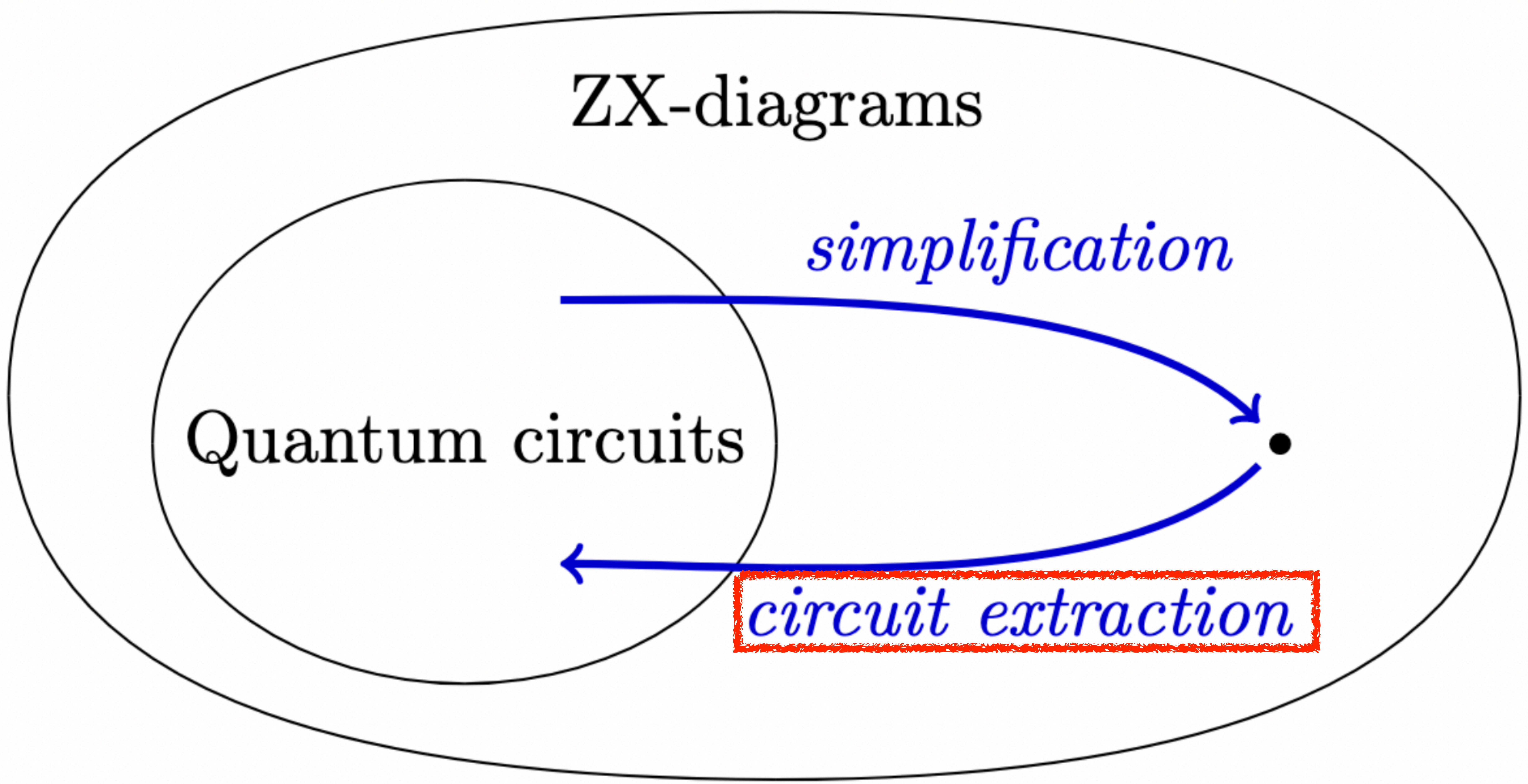


Original circuit has:

- 4 qubits;
- 195 Clifford spiders;
- 4 non-Clifford spiders.

Worst case complexity: $O(n^3)$




ZX-diagrams
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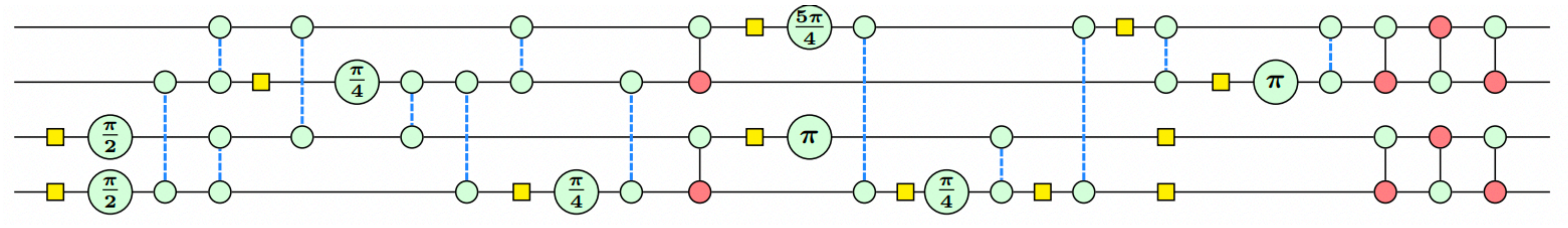
Theorem 4.5. Let (G, I, O) be an open graph that admits a focused gFlow, then (G', I, O) also admits a gFlow in the following two cases:

1. for $u \notin I \cup O$, setting $G' := (G \star u) \setminus \{u\}$
2. for adjacent $u, v \notin I \cup O$, setting $G' := (G \wedge uv) \setminus \{u, v\}$

where $G \setminus W$ is the graph obtained by deleting the vertices in W and any incident edges.



Summary:

1. Original circuit with 195 gates;
2. Intermediate ZX-diagram with 12 spiders;
3. Final circuit with 41 gates.

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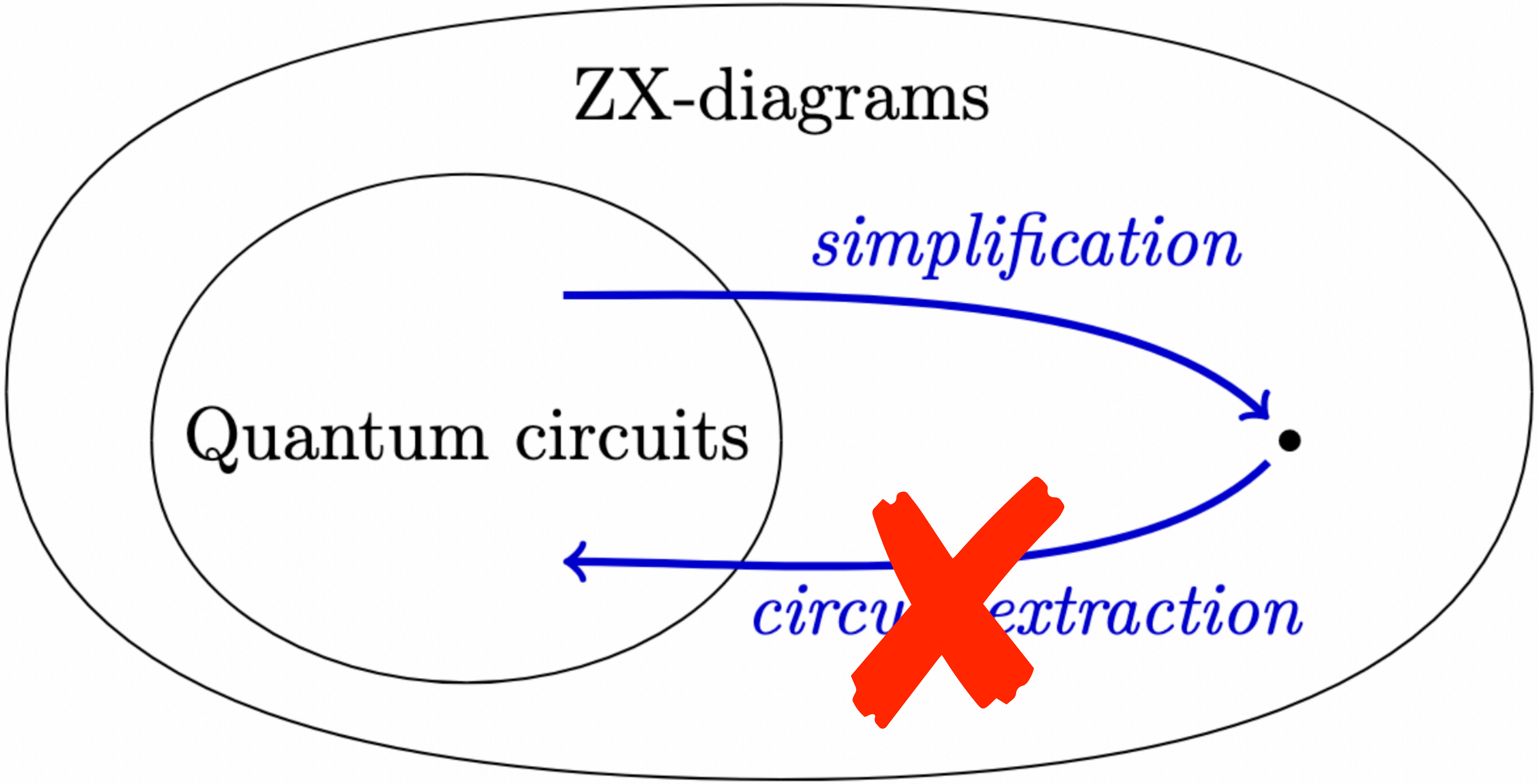
R. Duncan, A. Kissinger, S. Pedrix, and J. van de Wetering, Graph-theoretic simplification of quantum circuits with the ZX calculus, Quantum **4**, 279 (2020).

A. Kissinger and J. van de Wetering. Reducing the number of non-Clifford gates in quantum circuits. Physical Review A, **102**, 022406 (2020).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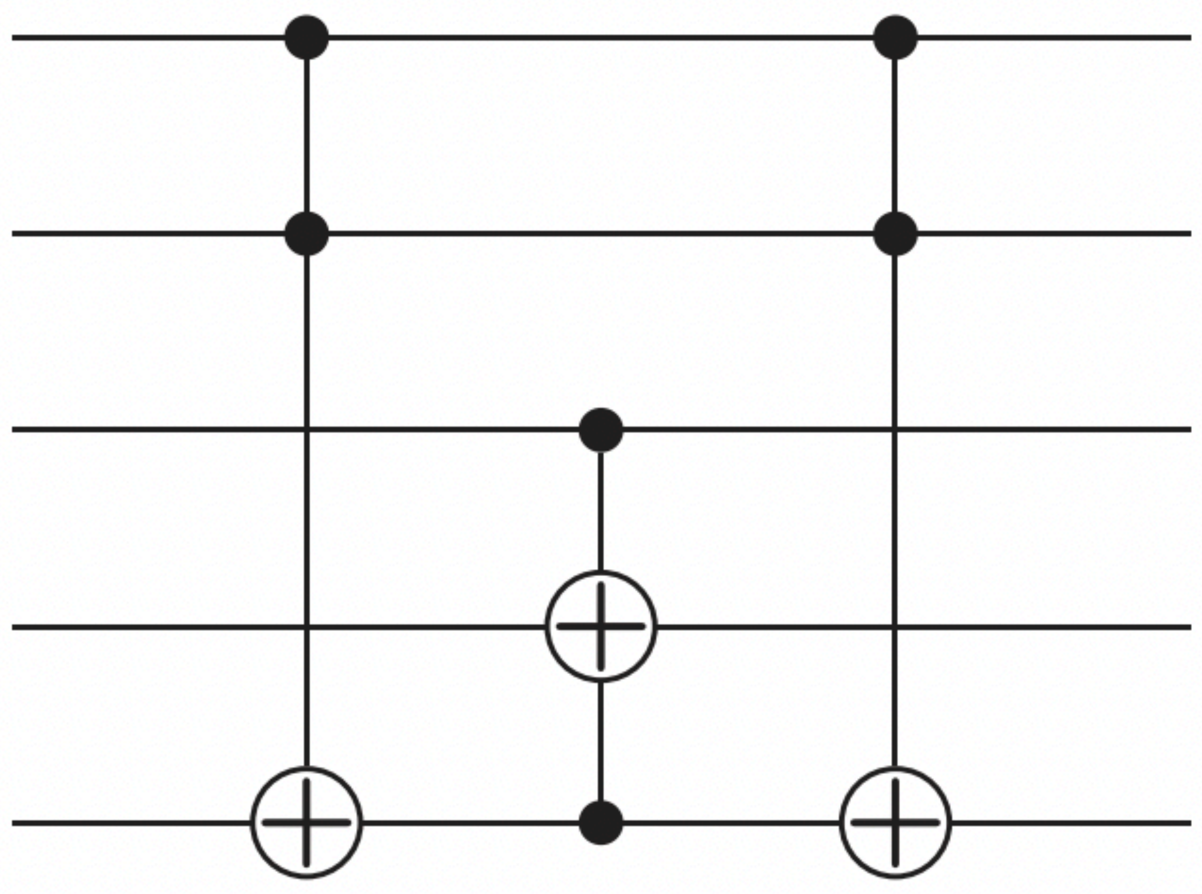
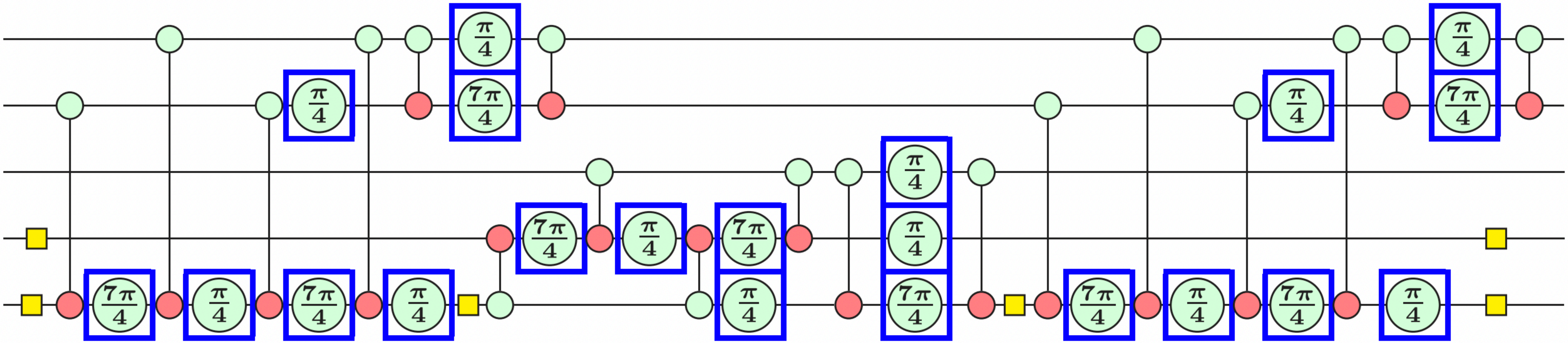
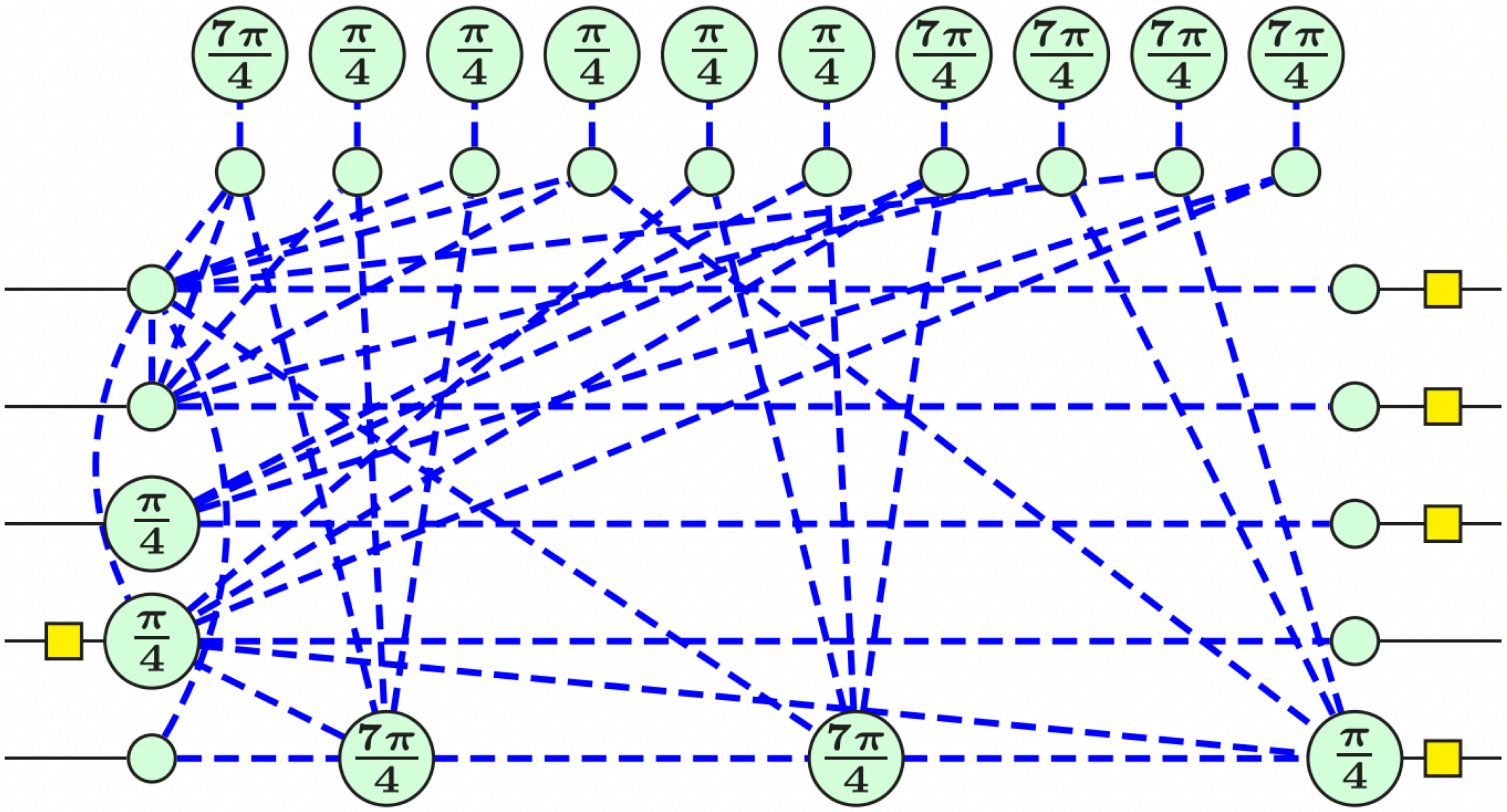


21 non-Clifford gates!





15 non-Clifford spiders!

π/4
7π/4
π/4
7π/4
π/4
7π/4
π/4
7π/4
π/4
7π/4
π/4
7π/4
π/4
π/4
π/4
7π/4
7π/4
π/4
7π/4
π/4
π/4
7π/4
π/4

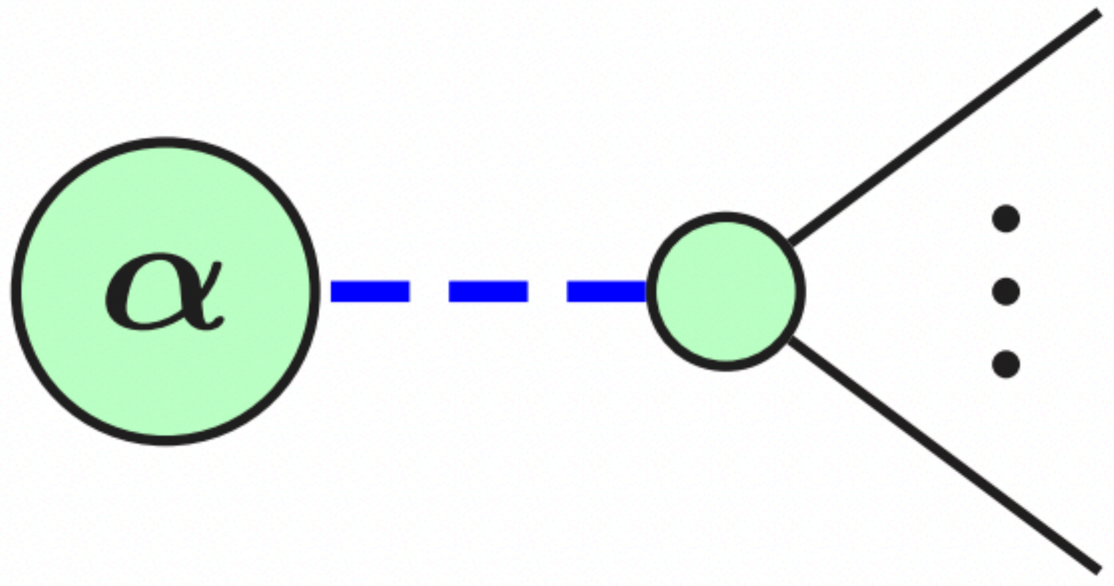
π/2
π/2
3π/2
7π/4
π/4
7π/4
π/4
7π/4
π/4
7π/4
π/4
π/4
π/4
7π/4
7π/4
π/4
7π/4
π/4
7π/4
π/4

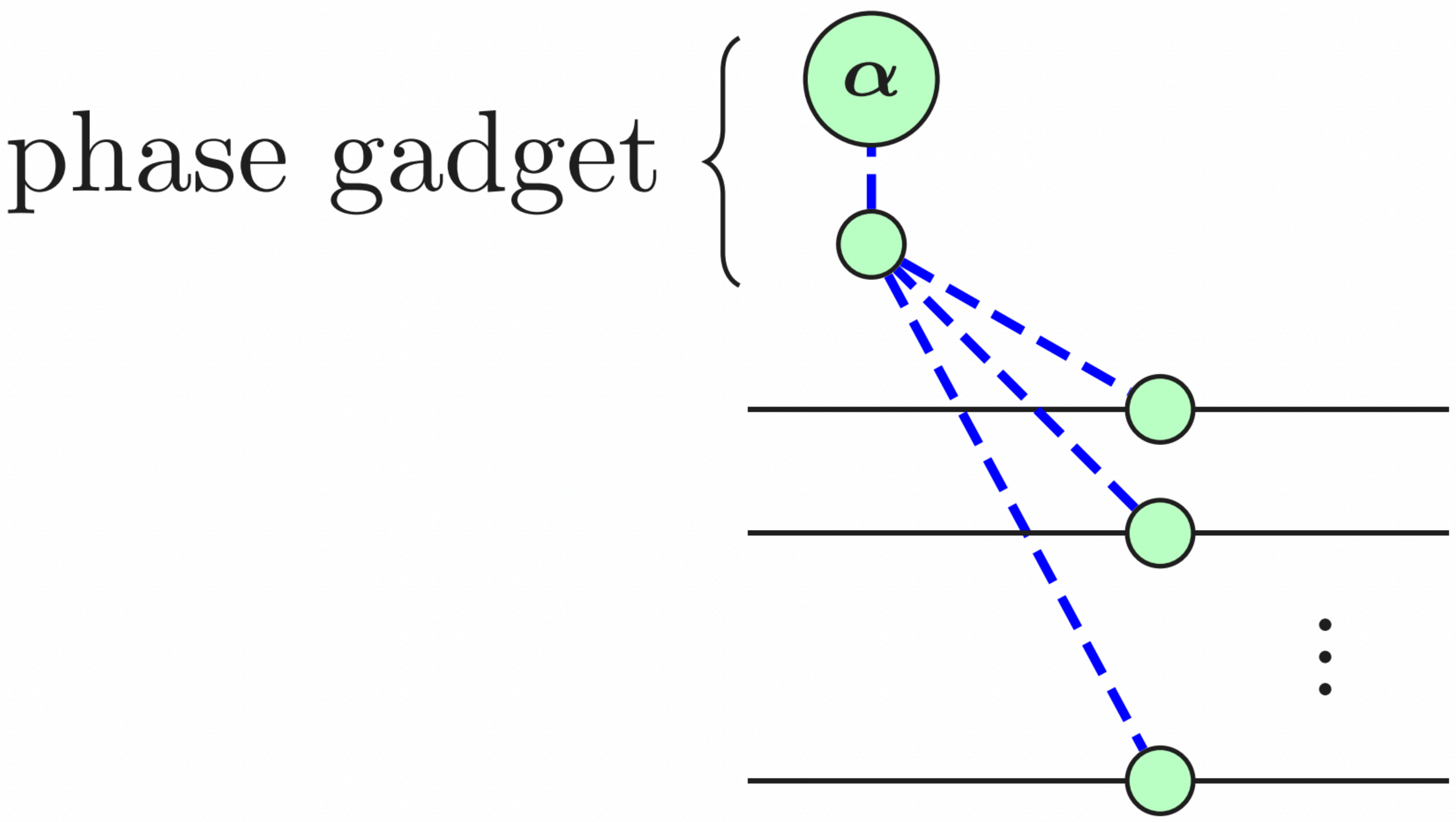
“(…) rather than re-extracting a circuit from a ZX-diagram, we use the diagram as a tool for discovering phases that can be shifted around nonlocally without changing the computed unitary. We call this technique phase teleportation.”

“A pleasant property of phase teleportation, as opposed to the simplify-and-extract method, is that it leaves the structure of the quantum circuit completely intact, only changing the parameters.”

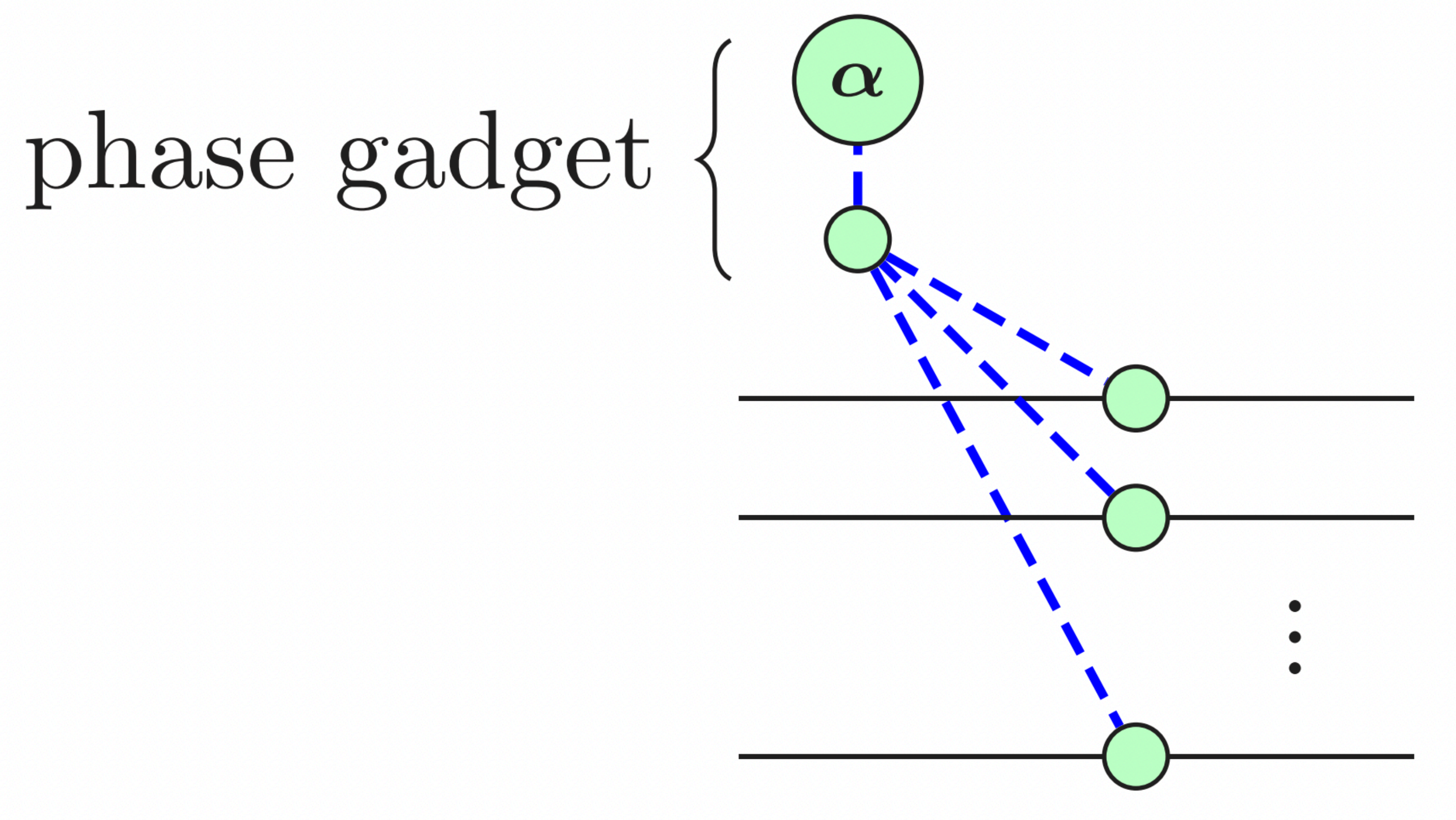
Phase gadgets

“A phase gadget is simply an arity-1 spider with angle α, connected via a Hadamard edge to a spider with no angle.”





Curiosity!

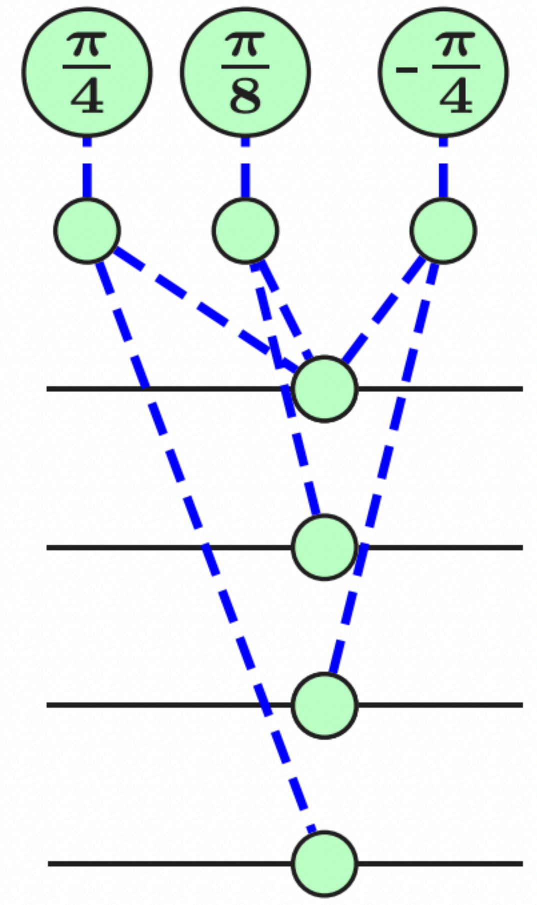


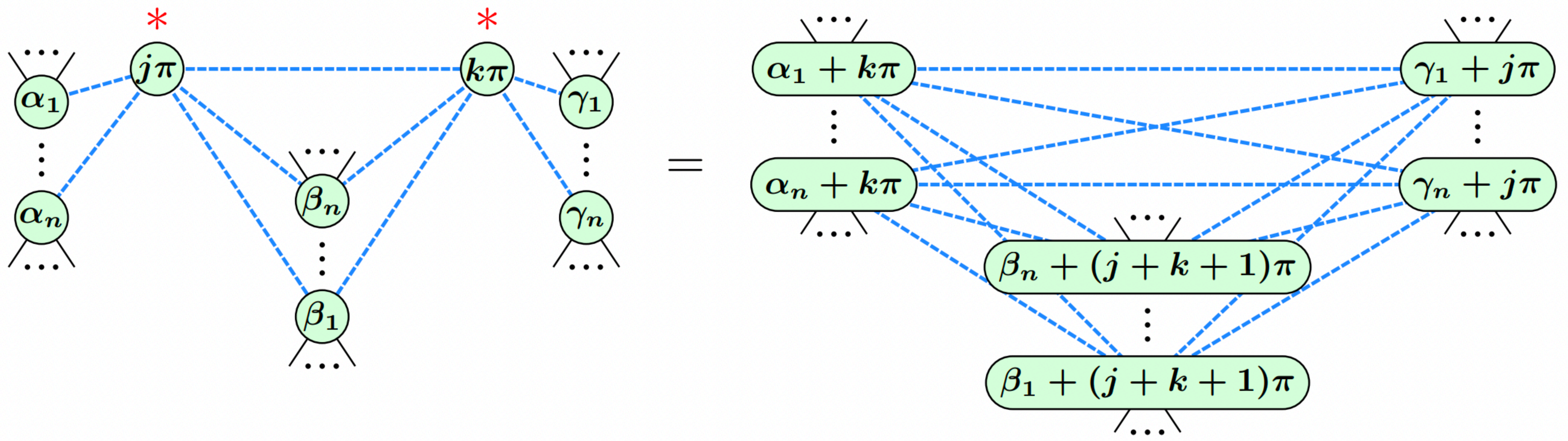
This corresponds to the following unitary:

$$U : \left|x_1, x_2, \ldots, x_n\right\rangle \rightarrow e^{i\alpha(x_1 \oplus x_2 \oplus \ldots \oplus x_n)} \left|x_1, x_2, \ldots, x_n\right\rangle$$

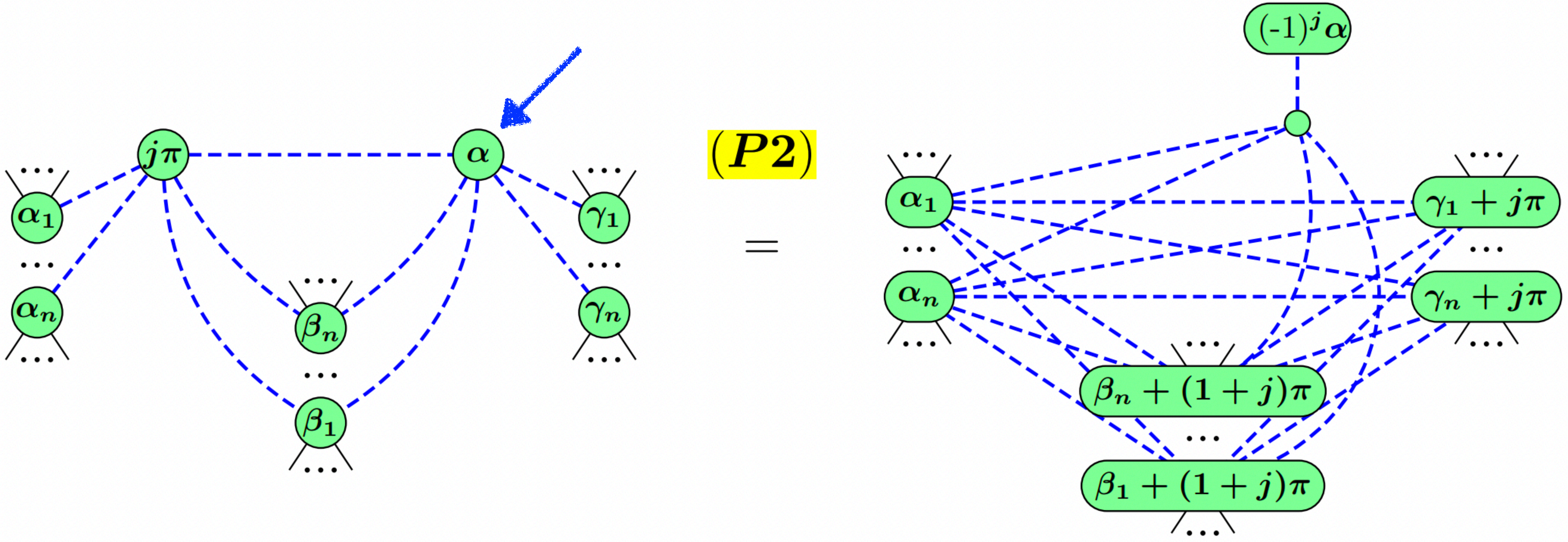
$$U : \left|x_1, x_2, x_3, x_4\right\rangle \rightarrow e^{if(x_1, x_2, x_3, x_4)} \left|x_1, x_2, x_3, x_4\right\rangle$$

$$f(x_1, x_2, x_3, x_4) =$$

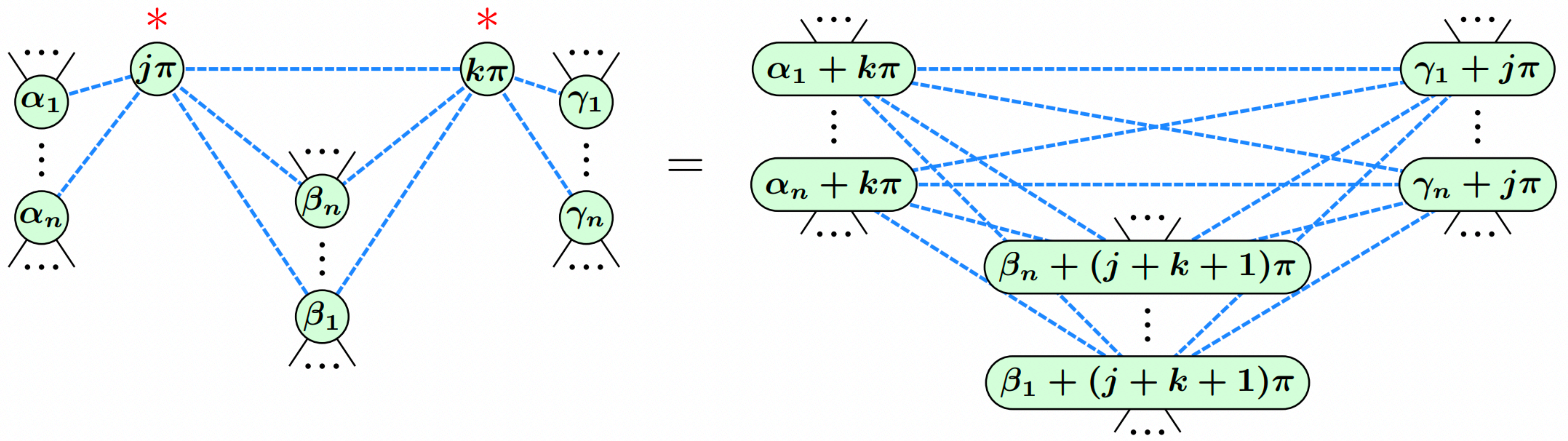


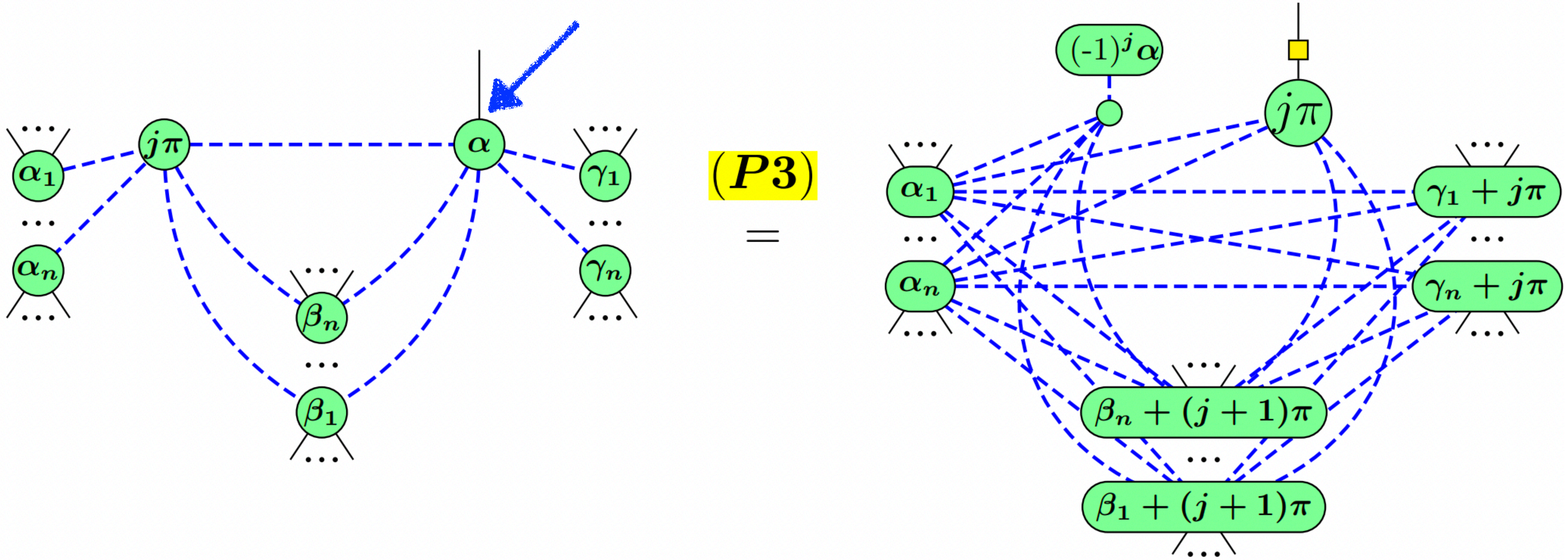


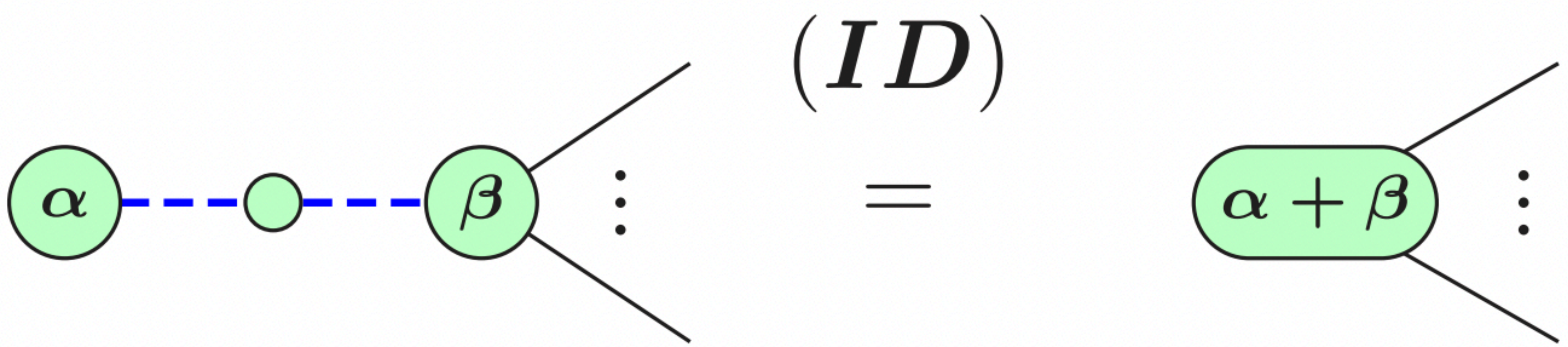
$j\pi$
$k\pi$
α_1
α_n
β_n
β_1
γ_1
γ_n
$=$
$\alpha_1 + k\pi$
$\alpha_n + k\pi$
$\gamma_1 + j\pi$
$\gamma_n + j\pi$
$\beta_n + (j+k+1)\pi$
$\beta_1 + (j+k+1)\pi$



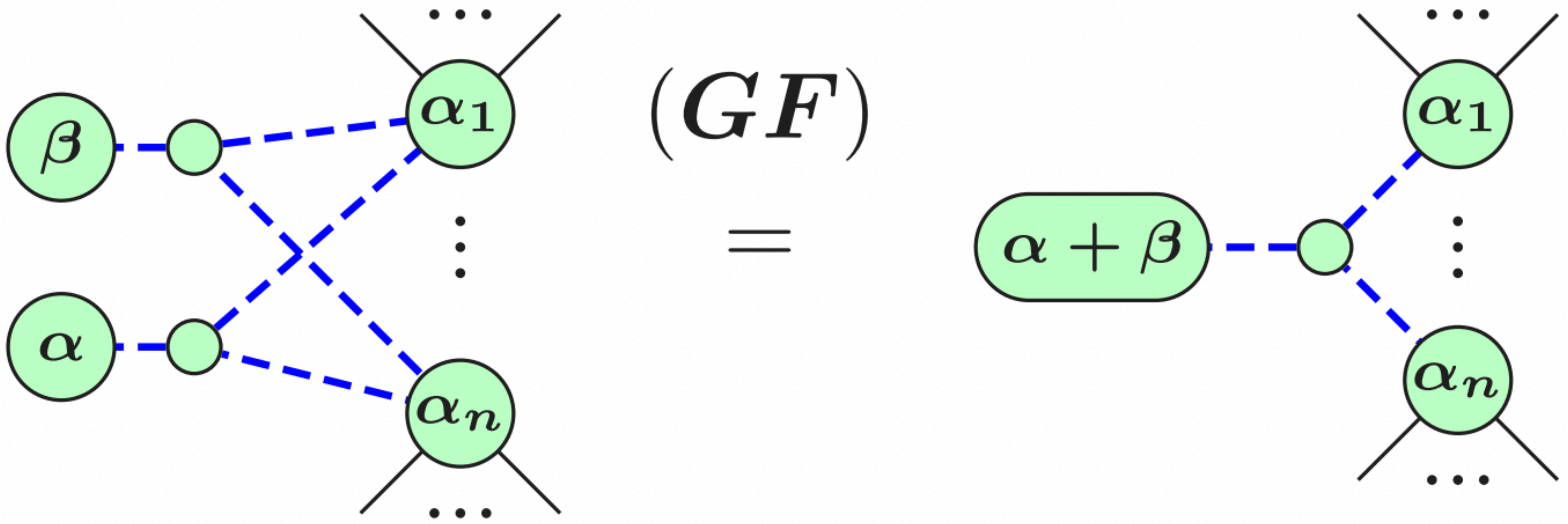
$j\pi$
α
α_1
α_n
β_n
β_1
γ_1
γ_n
$(P2)$
$=$
$(-1)^j\alpha$
α_1
α_n
$\gamma_1 + j\pi$
$\gamma_n + j\pi$
$\beta_n + (1+j)\pi$
$\beta_1 + (1+j)\pi$


$*$
$j\pi$
$*$
$k\pi$
α_1
α_n
β_n
β_1
γ_1
γ_n
$=$
$\alpha_1 + k\pi$
$\alpha_n + k\pi$
$\gamma_1 + j\pi$
$\gamma_n + j\pi$
$\beta_n + (j+k+1)\pi$
$\beta_1 + (j+k+1)\pi$


$j\pi$
α
α_1
α_n
β_n
β_1
γ_1
γ_n
$(P3)$
$=$
$(-1)^j\alpha$
$j\pi$
α_1
α_n
$\gamma_1 + j\pi$
$\gamma_n + j\pi$
$\beta_n + (j+1)\pi$
$\beta_1 + (j+1)\pi$



How to reduce the number of non-Clifford spiders



Simplification algorithm:

1. Apply (LC) until all interior proper Clifford vertices are removed;

2. Apply $(P1)$, $(P2)$, and $(P3)$ until all interior Pauli vertices are removed or transformed into phase gadgets;

3. Remove all Clifford phase gadgets using (LC) and $(P1)$.

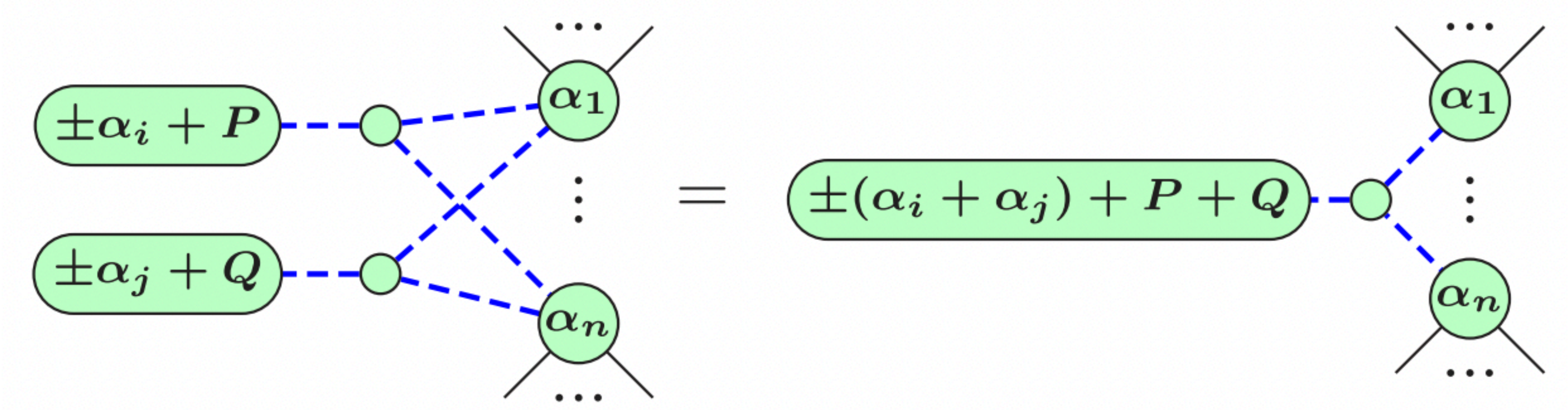
4. Apply (ID) and (GF) wherever possible. If any matches were found, go back to step 1, otherwise we are done.

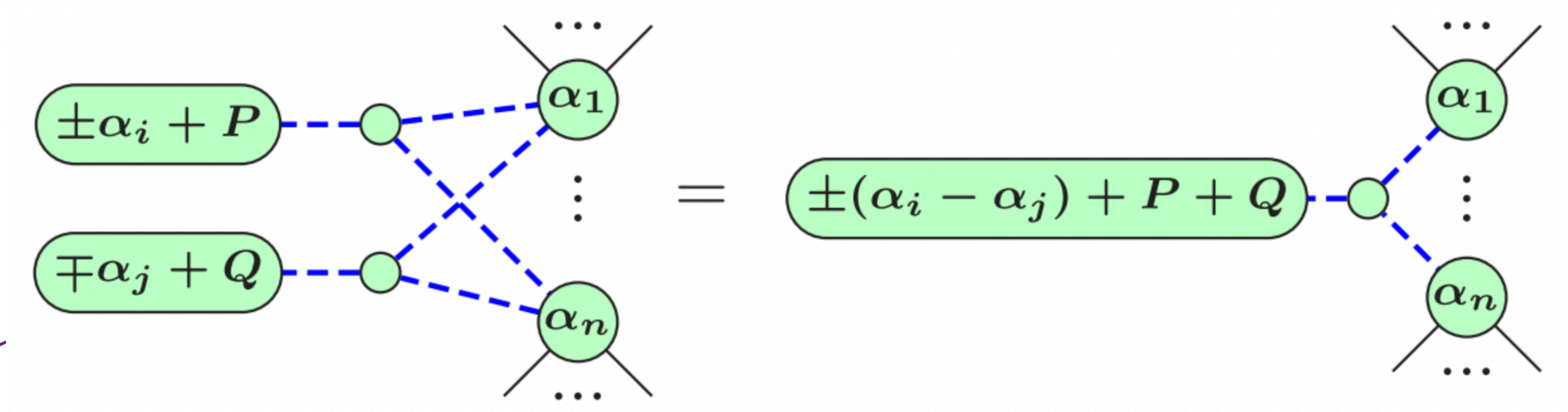
This procedure yields a ZX-digram that does not look like a circuit!

~~Extraction procedure~~

Phase teleportation

We can perform the simplification procedure symbolically.





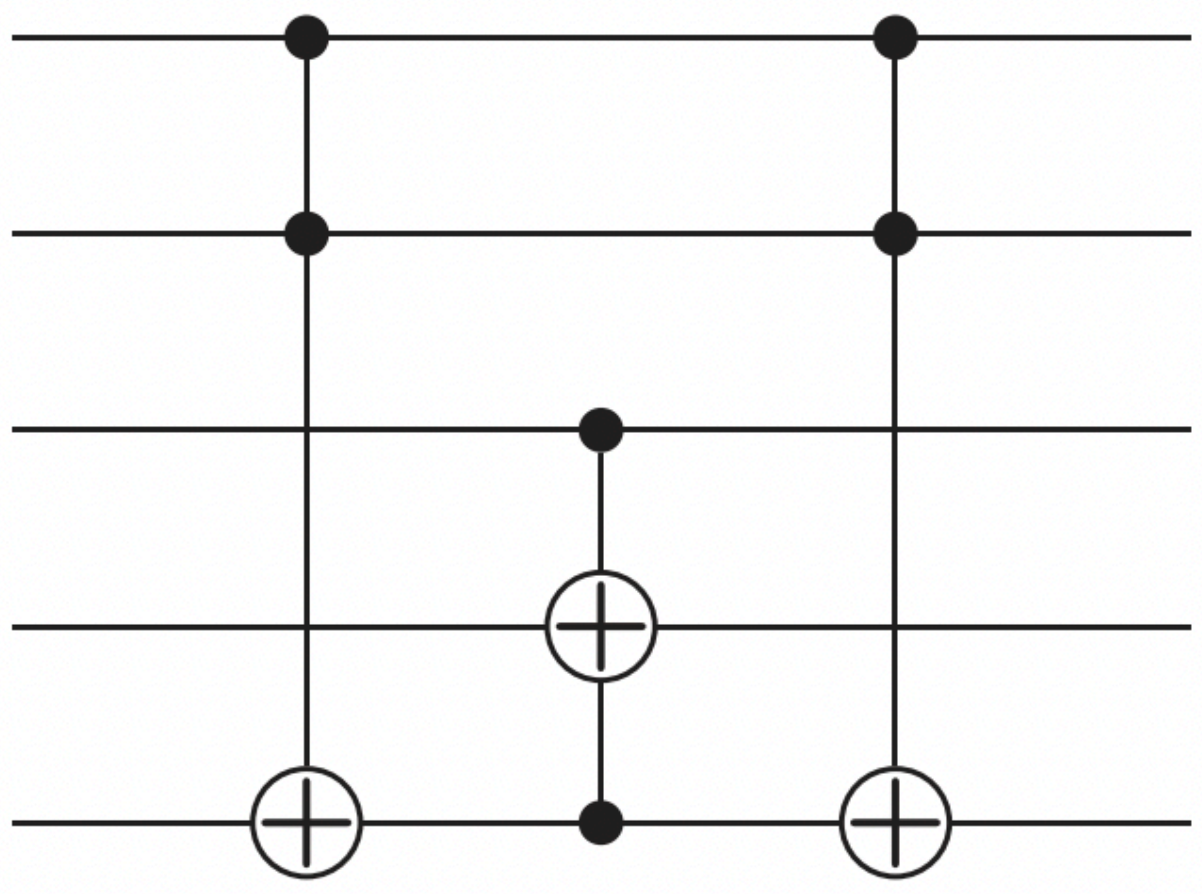
Phase teleportation algorithm: starting with the quantum circuit do the following:

1. Choose unique variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ for each non-Clifford phase and store the pair (C, τ), where C is the parametrized circuit and $\tau : \{1, \ldots, n\} \to \mathbb{R}$ assigns each variable to its phase;

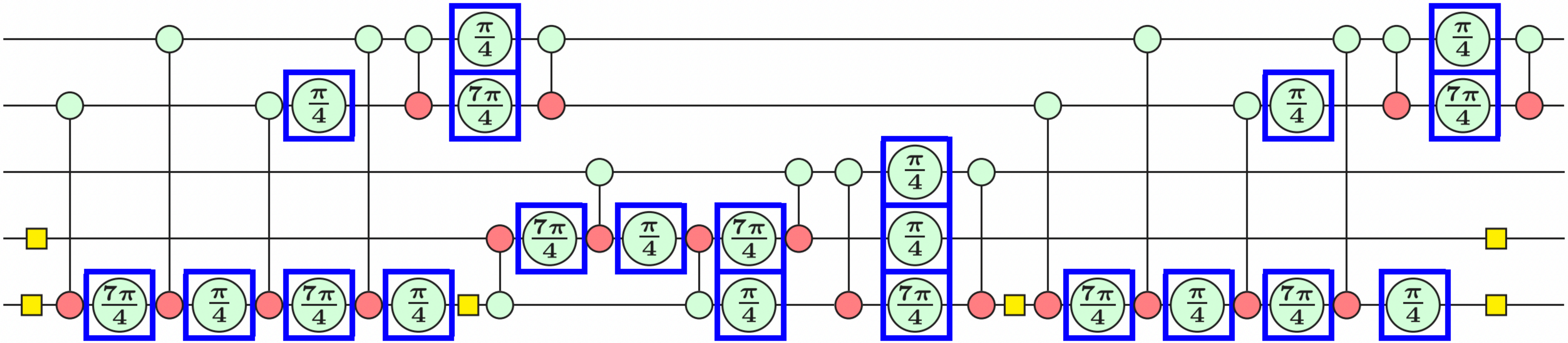
2. Interpret C as a ZX-diagram and run the simplification algorithm while doing the following:

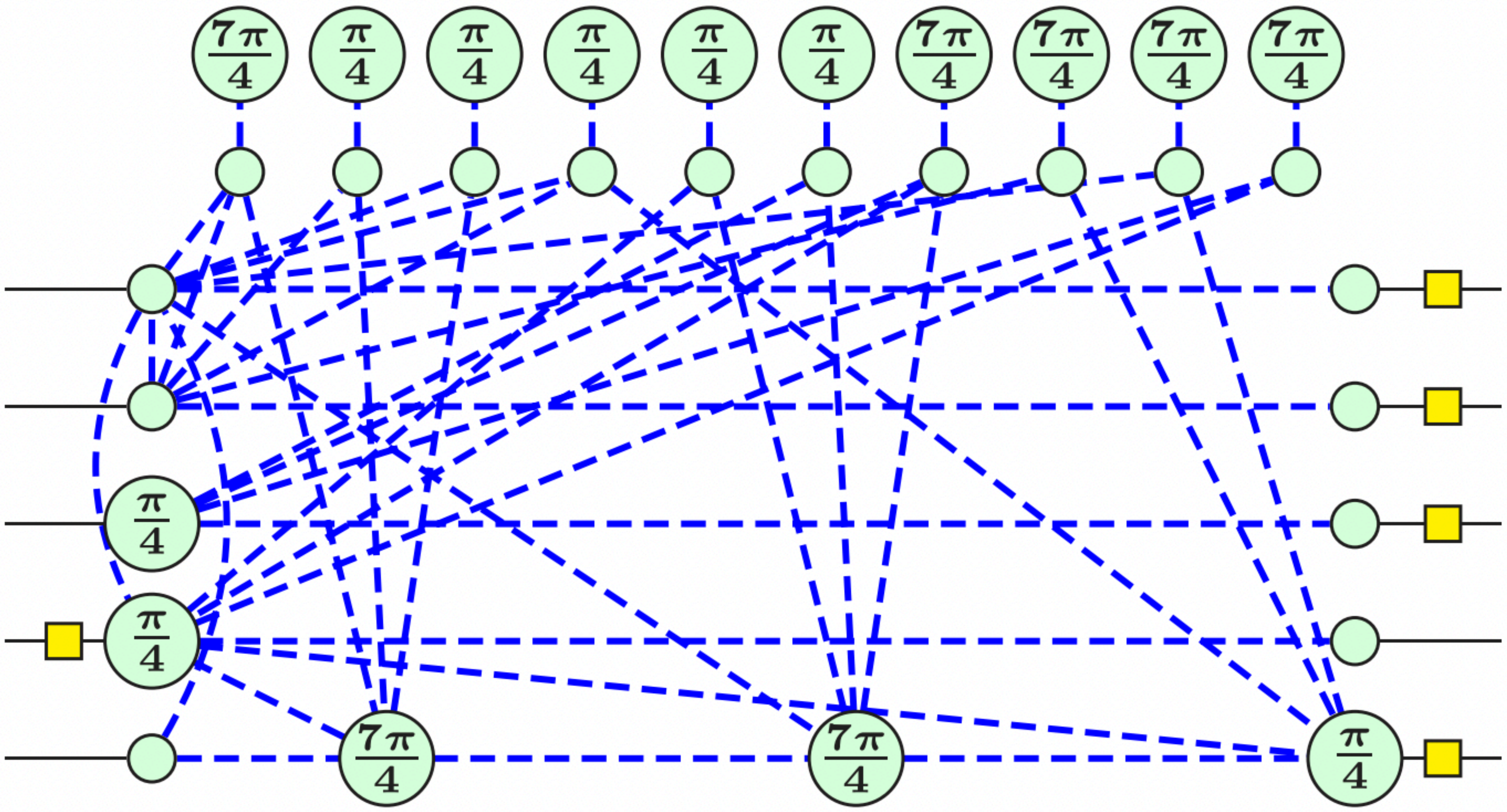
    - Whenever (ID) or (GF) are applied to a pair of vertices or phase gadgets containing variables α_i and α_j, update τ by replacing $\alpha_i := \alpha_i \pm \alpha_j$ and $\alpha_j := 0$;

3. When ZX-simplify finishes, the pair (C, τ) describes an equivalent circuit.



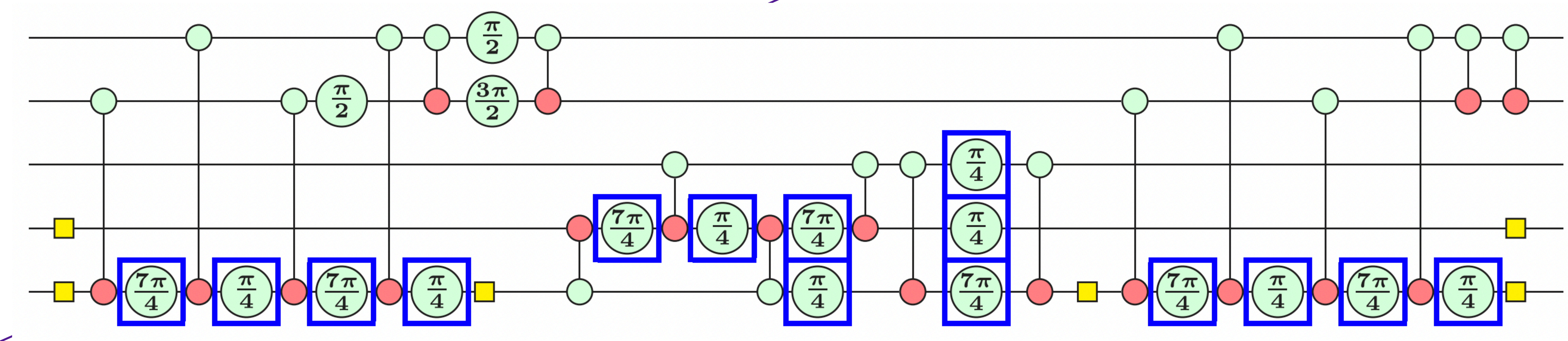
21 non-Clifford gates!





15 non-Clifford spiders!

$\frac{\pi}{4}$
$\frac{7\pi}{4}$



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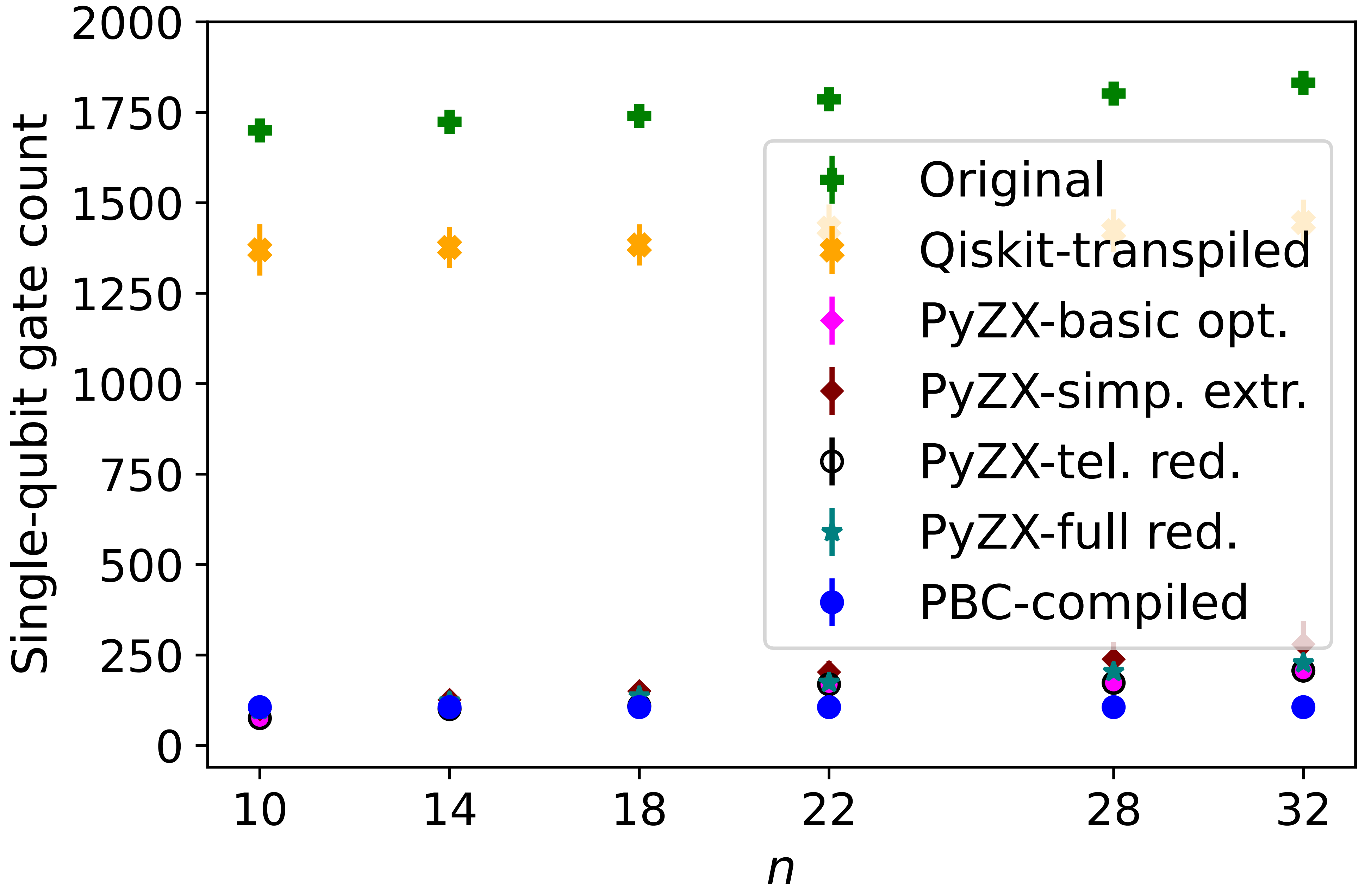
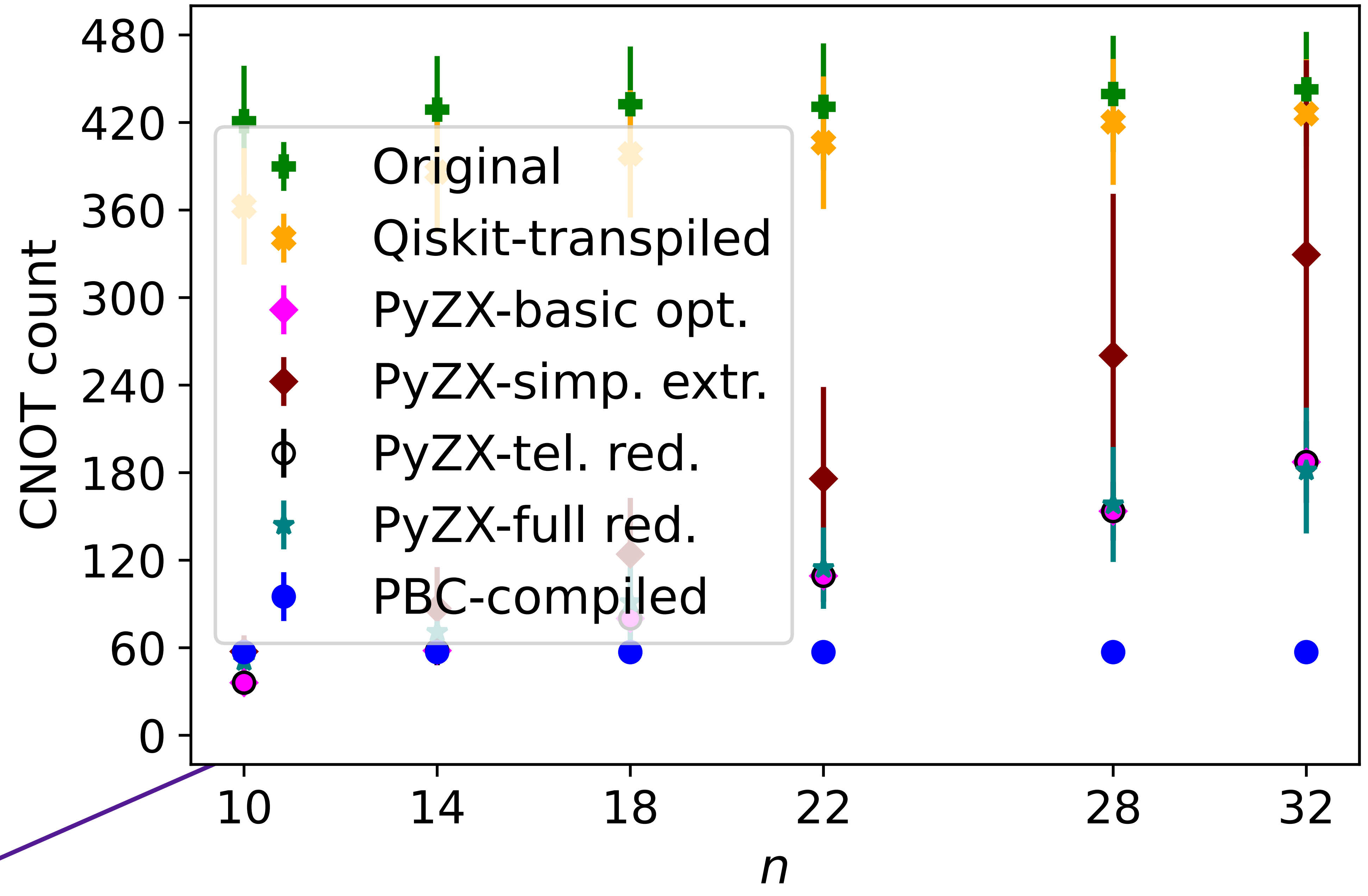
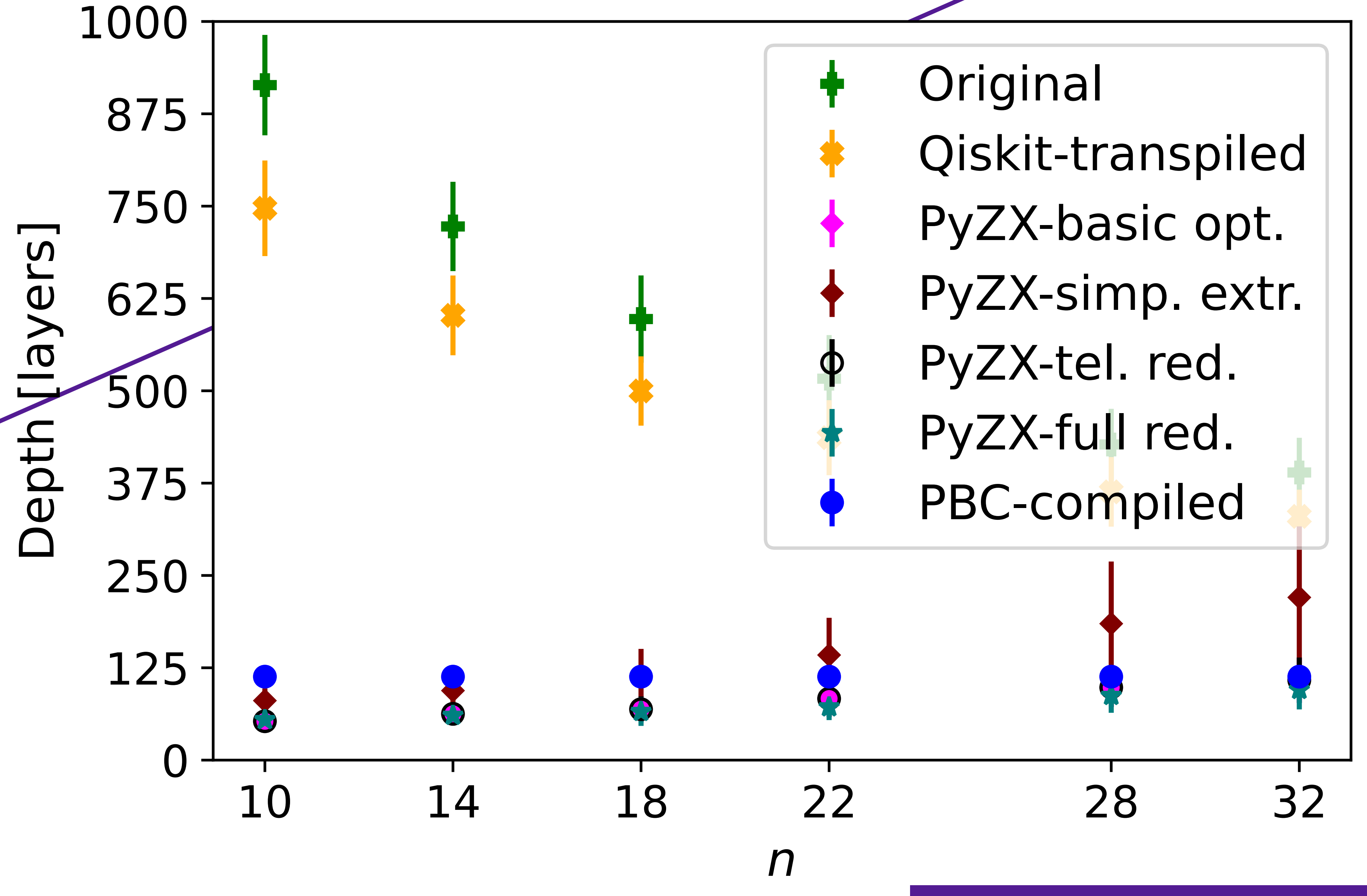
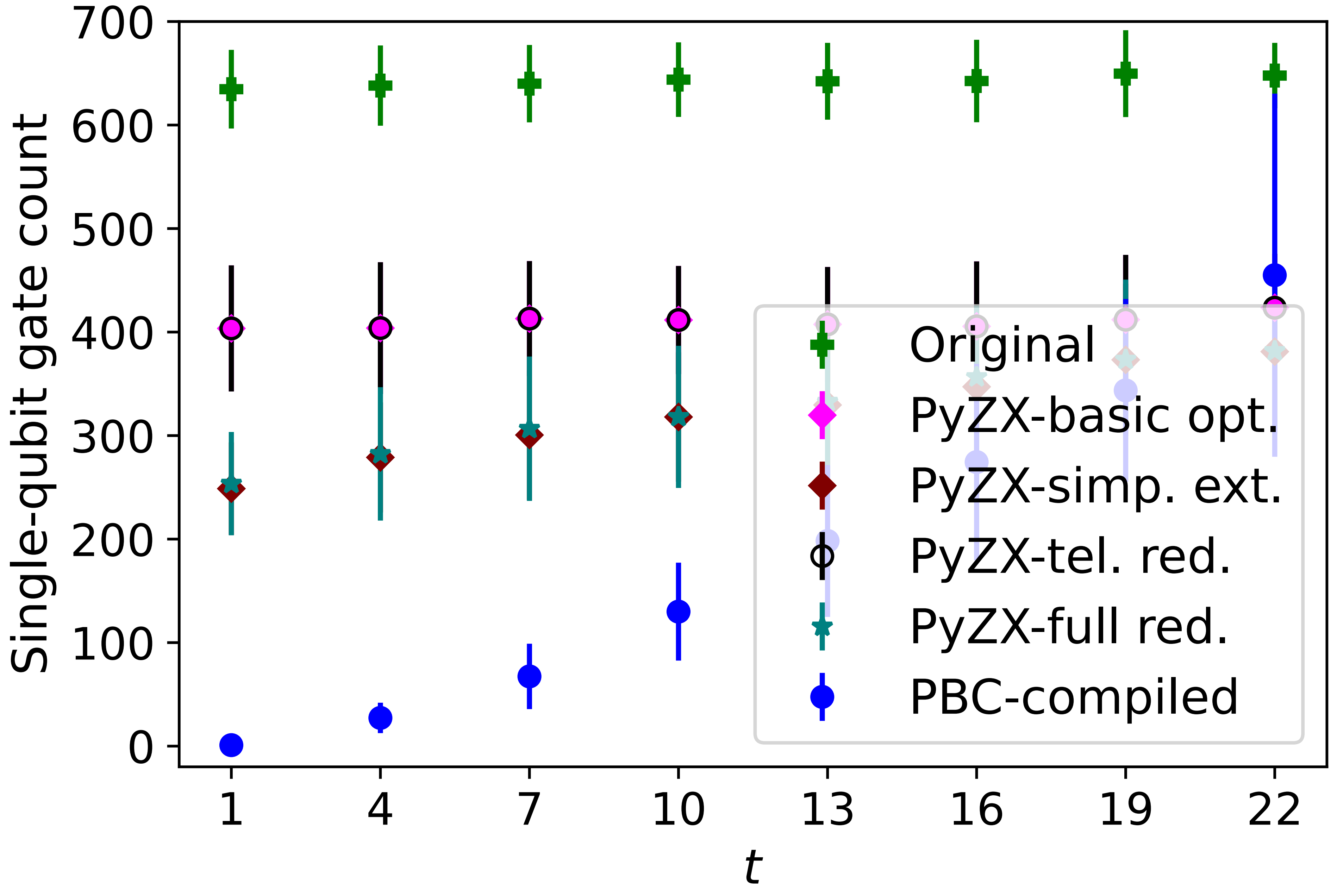


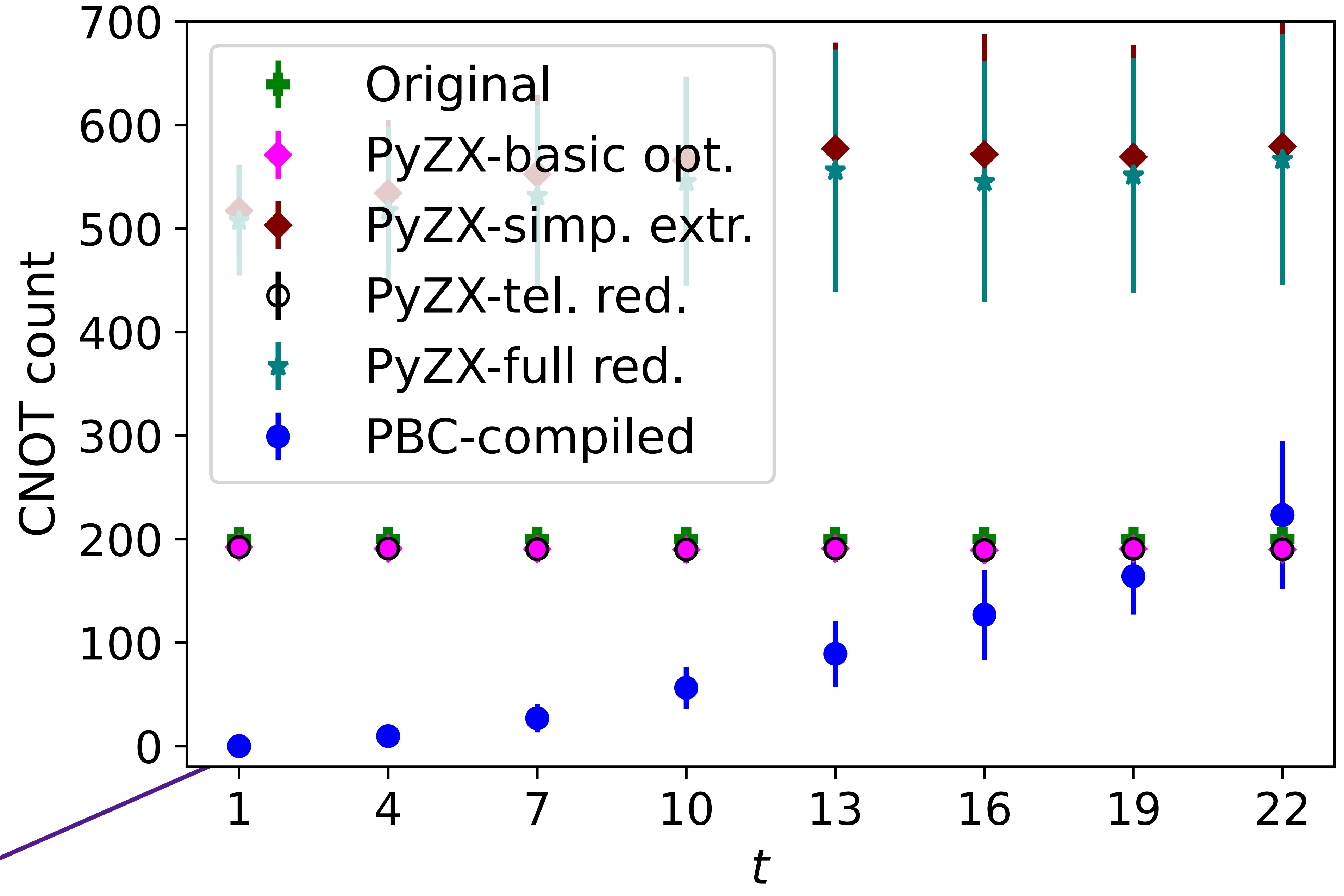
Hidden-shift circuits

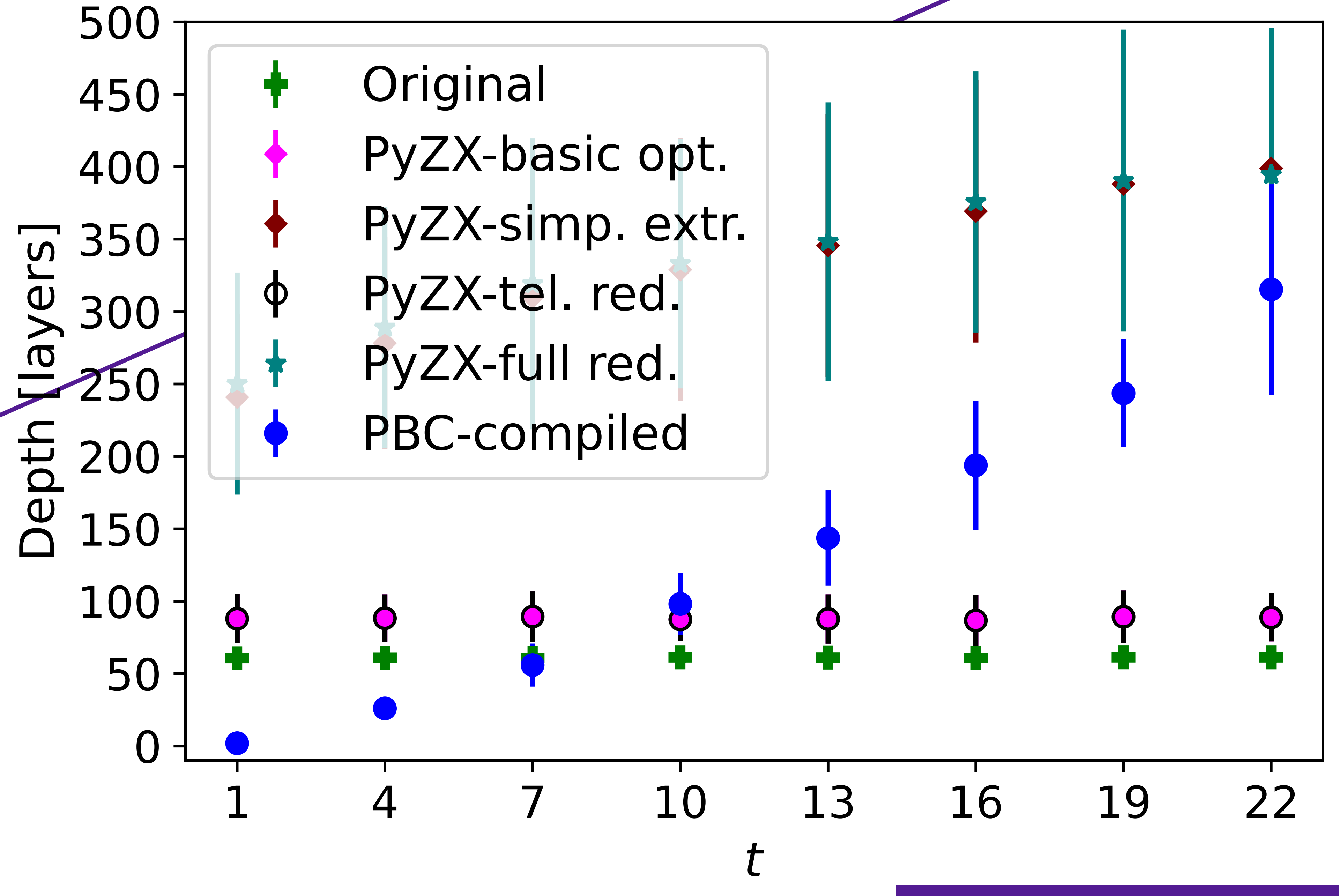






Random quantum circuits





https://zxcalculus.com/publications.html

A. Kissinger and J. van de Wetering. Simulating quantum circuits with ZX-calculus reduced stabiliser decompositions (2021). arXiv:2109.01076.

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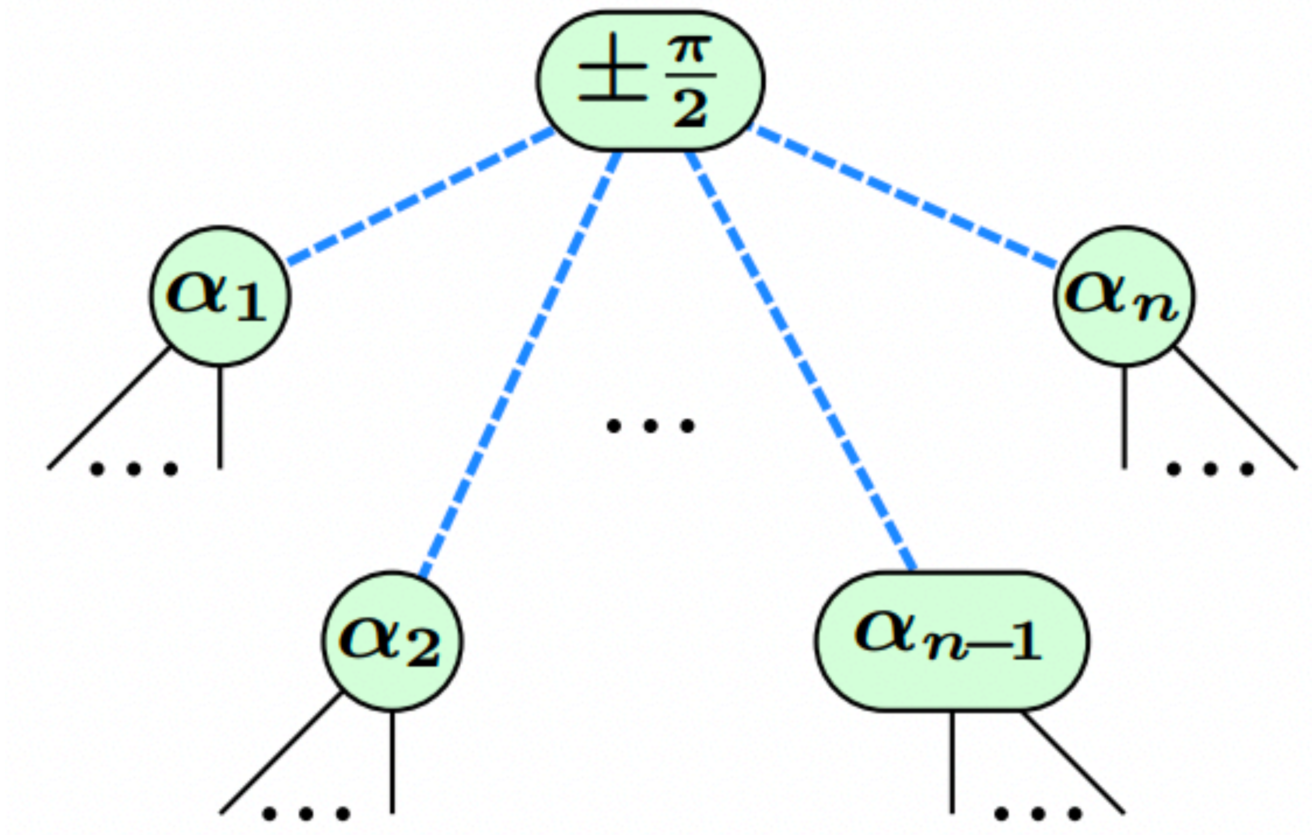
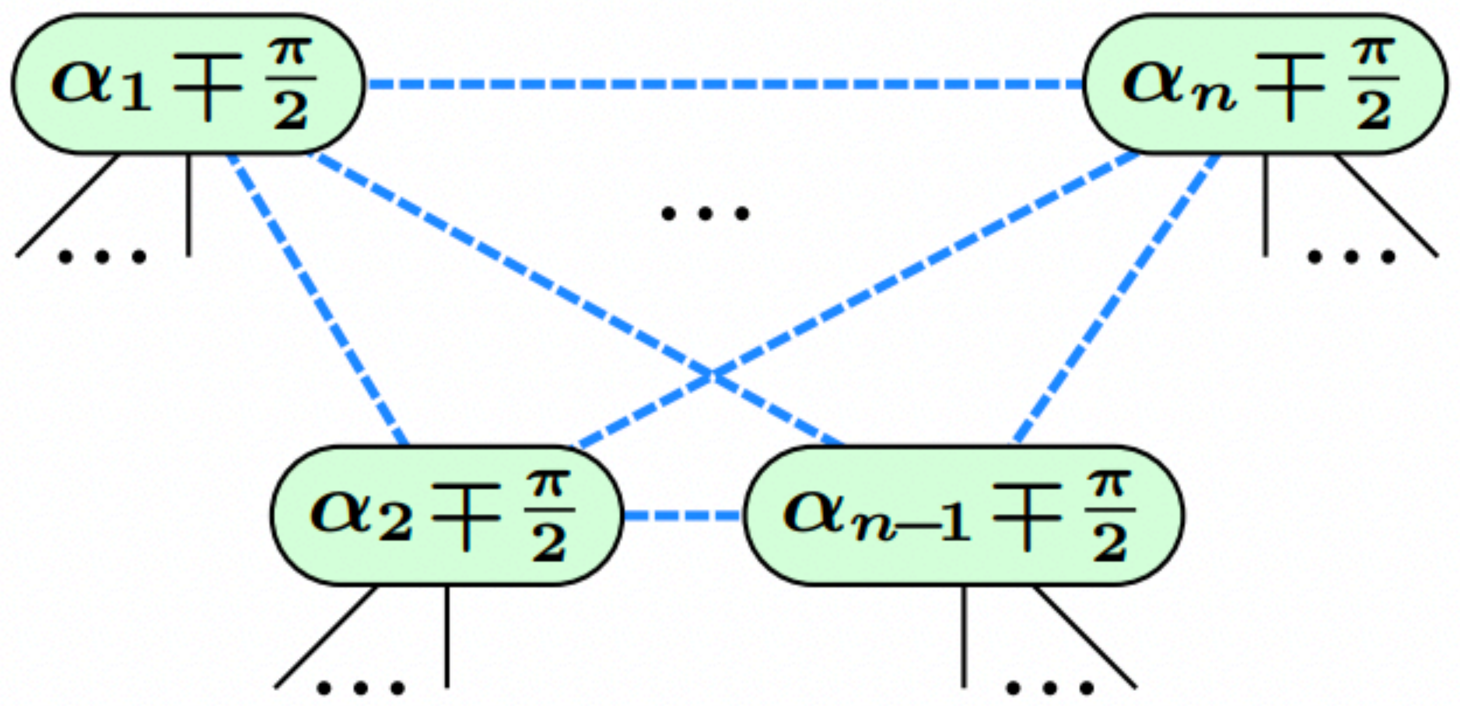
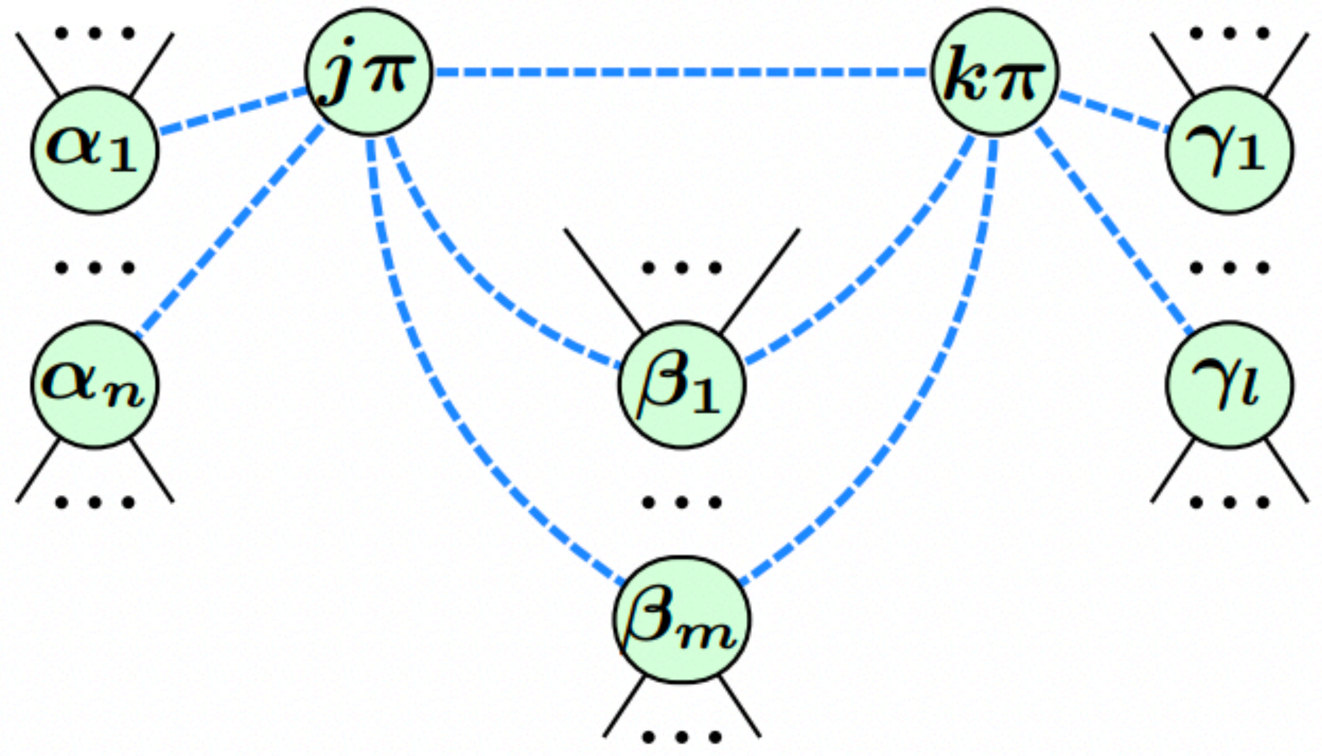
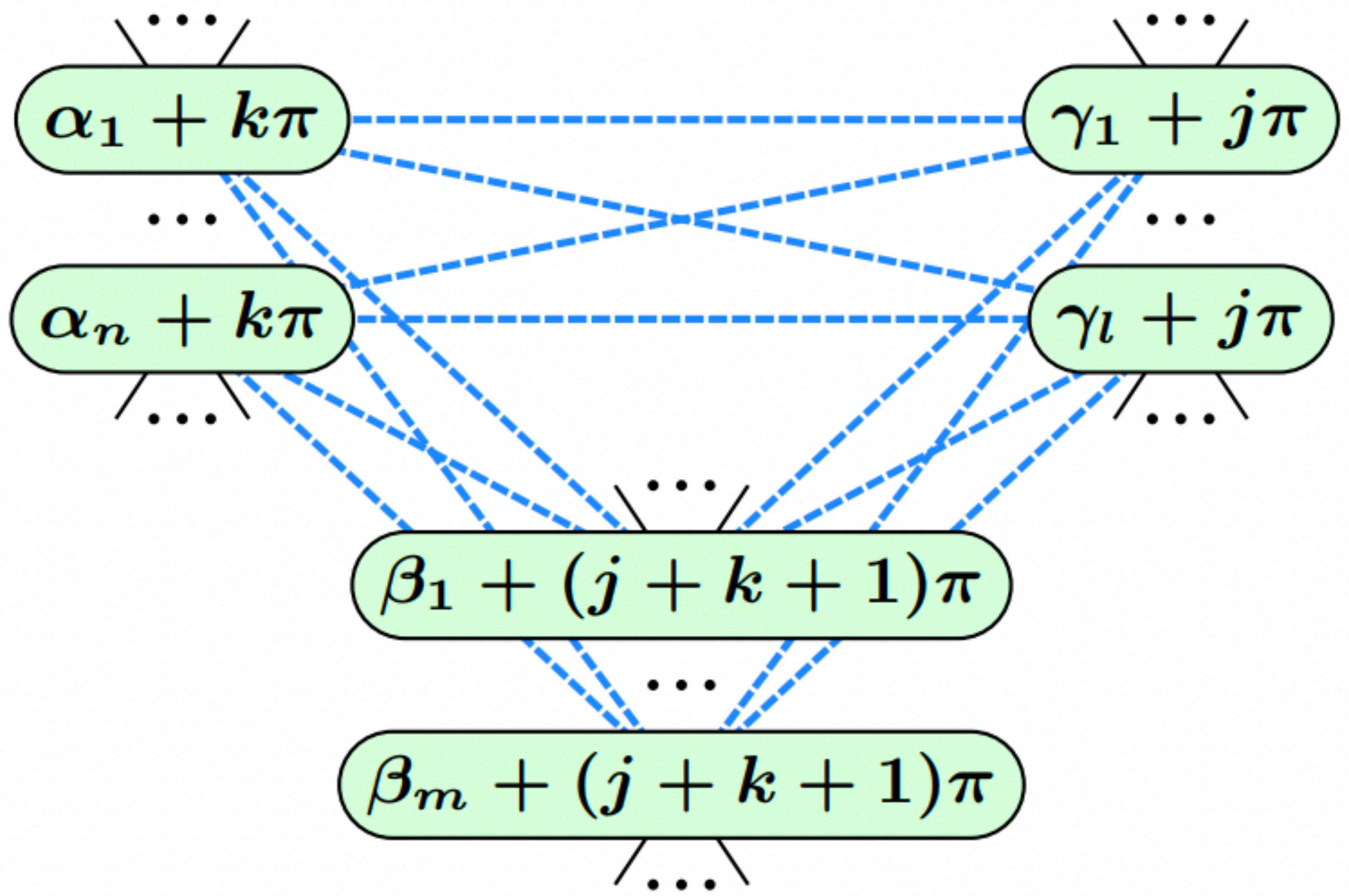


$$= e^{\pm i\pi/4}\sqrt{2}^{\frac{(n-1)(n-2)}{2}}$$



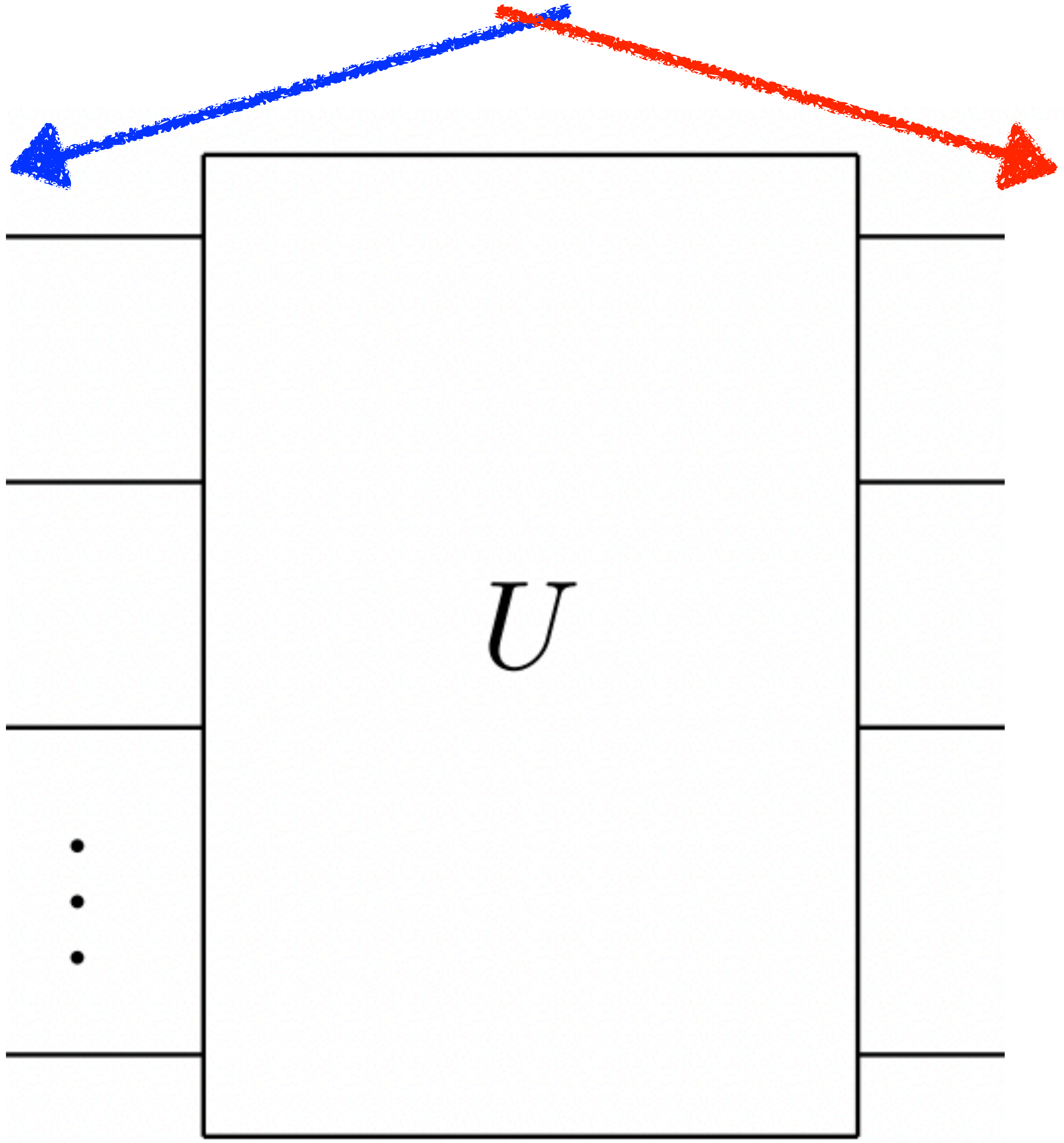


$$= \quad (-1)^{jk}\sqrt{2}^{E}$$



$$E = (n-1)m + (l-1)m + (n-1)(l-1)$$

Calculating an amplitude: $\boxed{\left\langle x_1, \ldots, x_n \right.} \left| U \right| \boxed{\left. 0, \ldots, 0 \right\rangle}$

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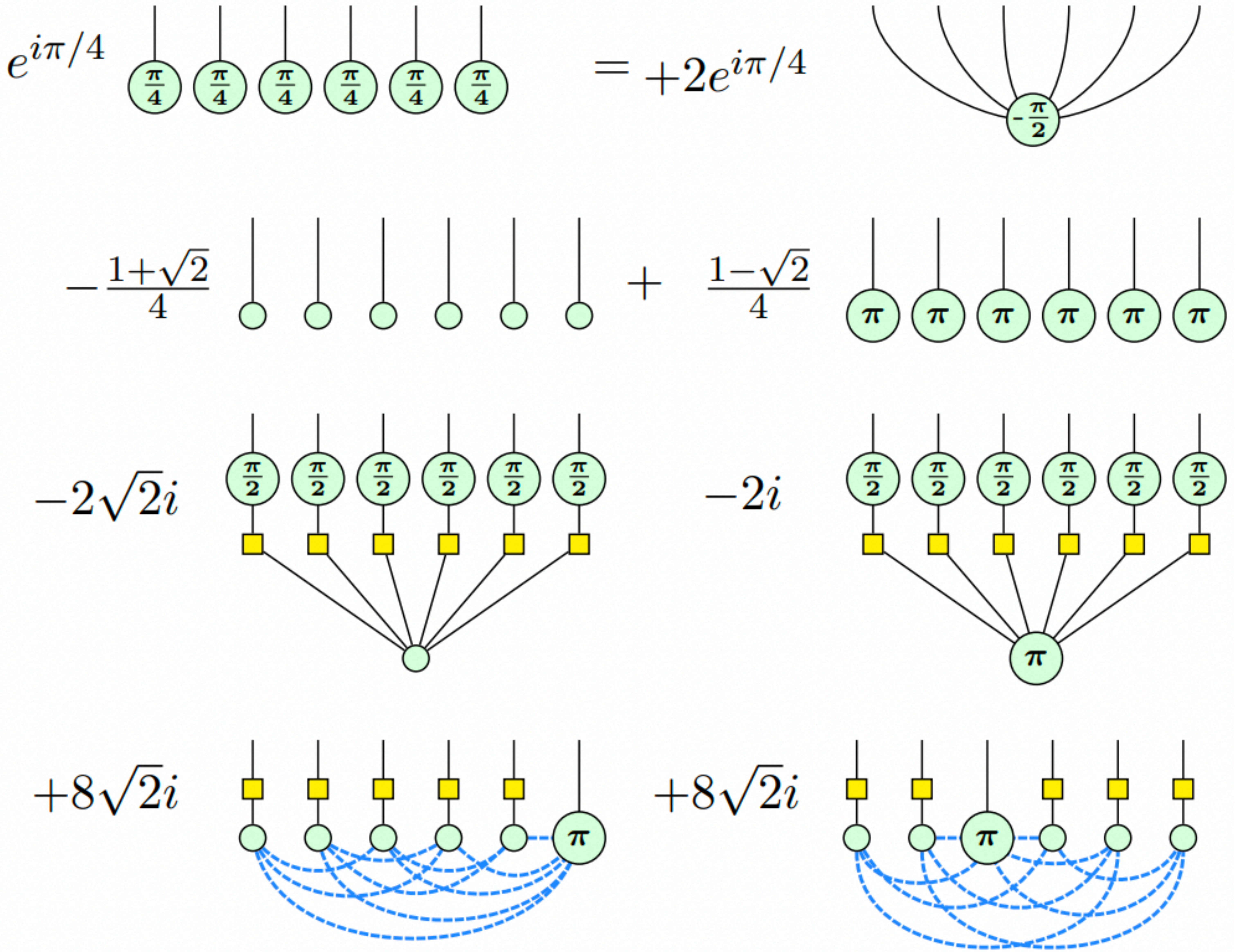
$\frac{3\pi}{4}$ $\frac{\pi}{4}$
$\frac{\pi}{4}$

Then, "*the step where the magic happens*":

Decomposition of $|A\rangle^{\otimes 6}$ into 7 stabilizer terms.

Total number of terms is $2^{\alpha t}$, with $\alpha = \log_2(p)/r$.

In this case:
$\alpha = \log_2(7)/6 \approx 0.468$

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$\frac{7\pi}{4}$
$\frac{\pi}{4}$
$\frac{3\pi}{4}$


$\frac{5\pi}{4}$ $\frac{7\pi}{4}$ $\frac{\pi}{4}$ $\frac{5\pi}{4}$ $\frac{7\pi}{4}$ $\frac{\pi}{4}$ $\frac{5\pi}{4}$ $\frac{\pi}{4}$ $\frac{7\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{7\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{7\pi}{4}$ $\frac{\pi}{4}$ $\frac{5\pi}{4}$ $\frac{\pi}{4}$ $\frac{3\pi}{4}$ $\frac{7\pi}{4}$
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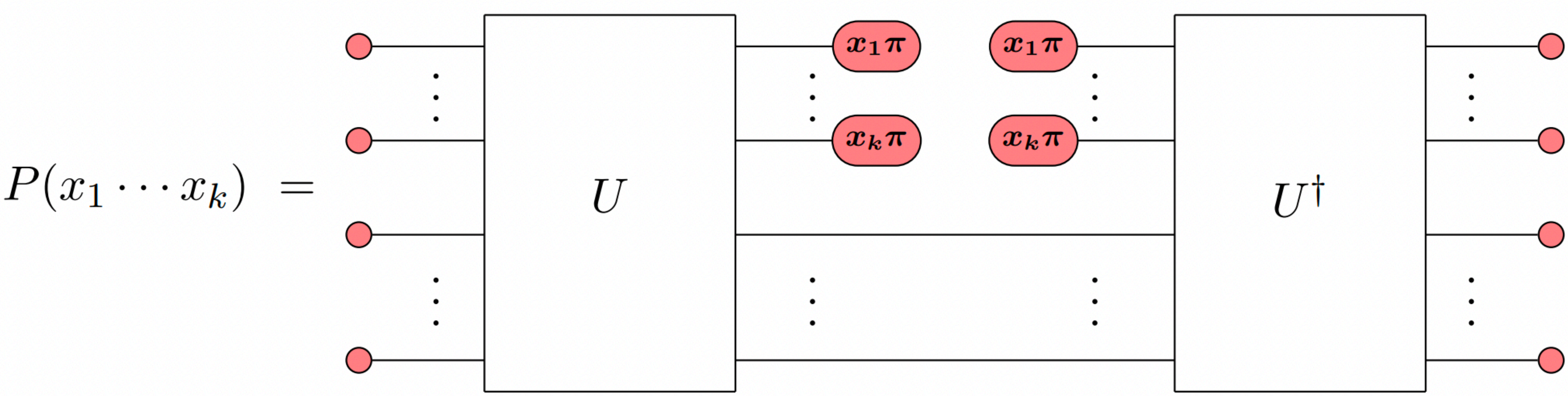
$\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$

Calculating a marginal probability: $P(x_1, \ldots, x_k) = \,?$



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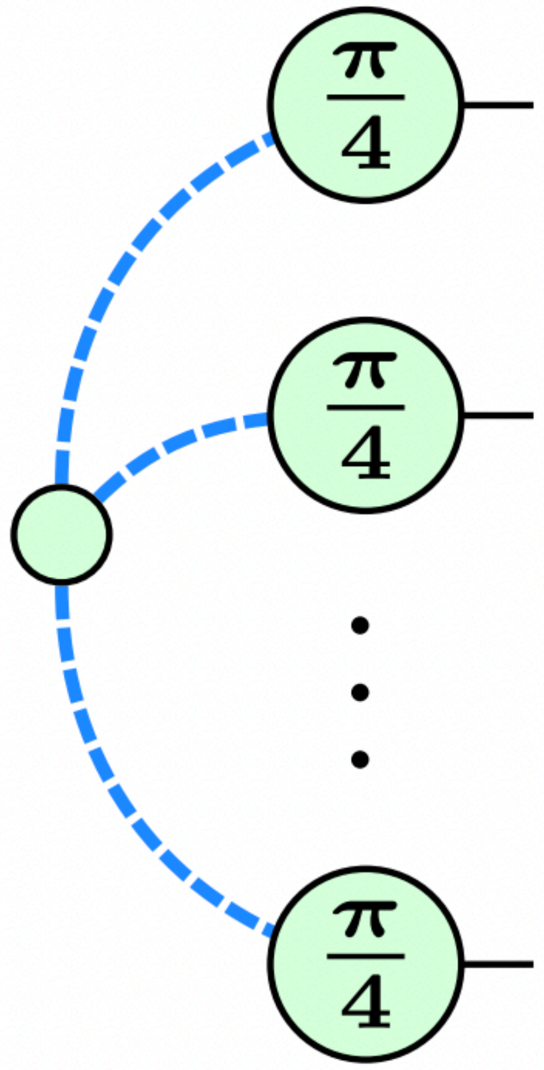
Cat states:

$$\left|\text{cat}_n\right\rangle = \frac{1}{\sqrt{2}}\left(I^{\otimes n} + Z^{\otimes n}\right)\left|A\right\rangle^{\otimes n}$$

$$\left|\text{cat}_n\right\rangle = \frac{1}{\sqrt{2}}$$



n	$\chi(\left\|\text{cat}_n\right\rangle)$	α
3	2	0.333
4	2	0.25
5	3	0.317
6	**3**	0.264

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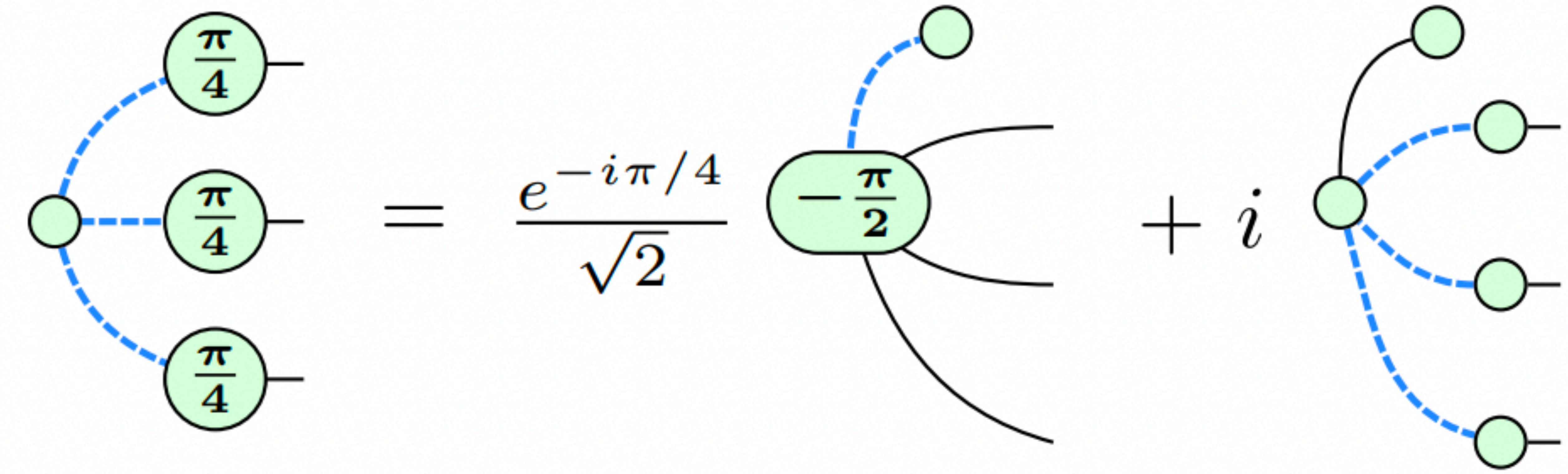

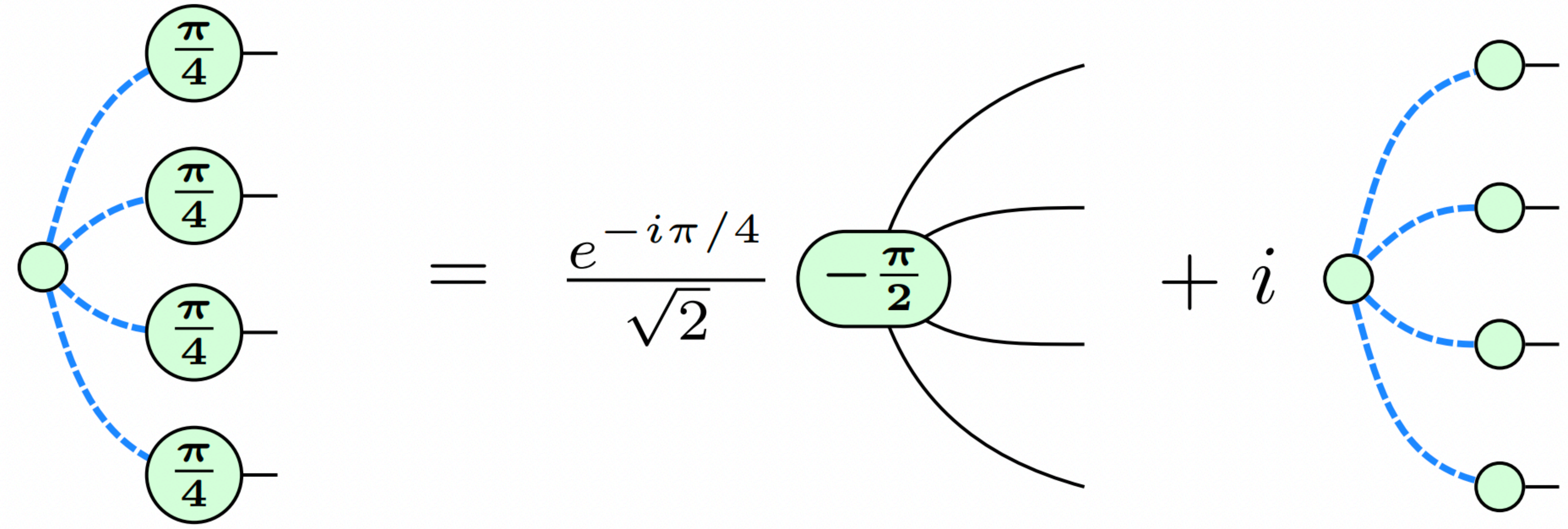

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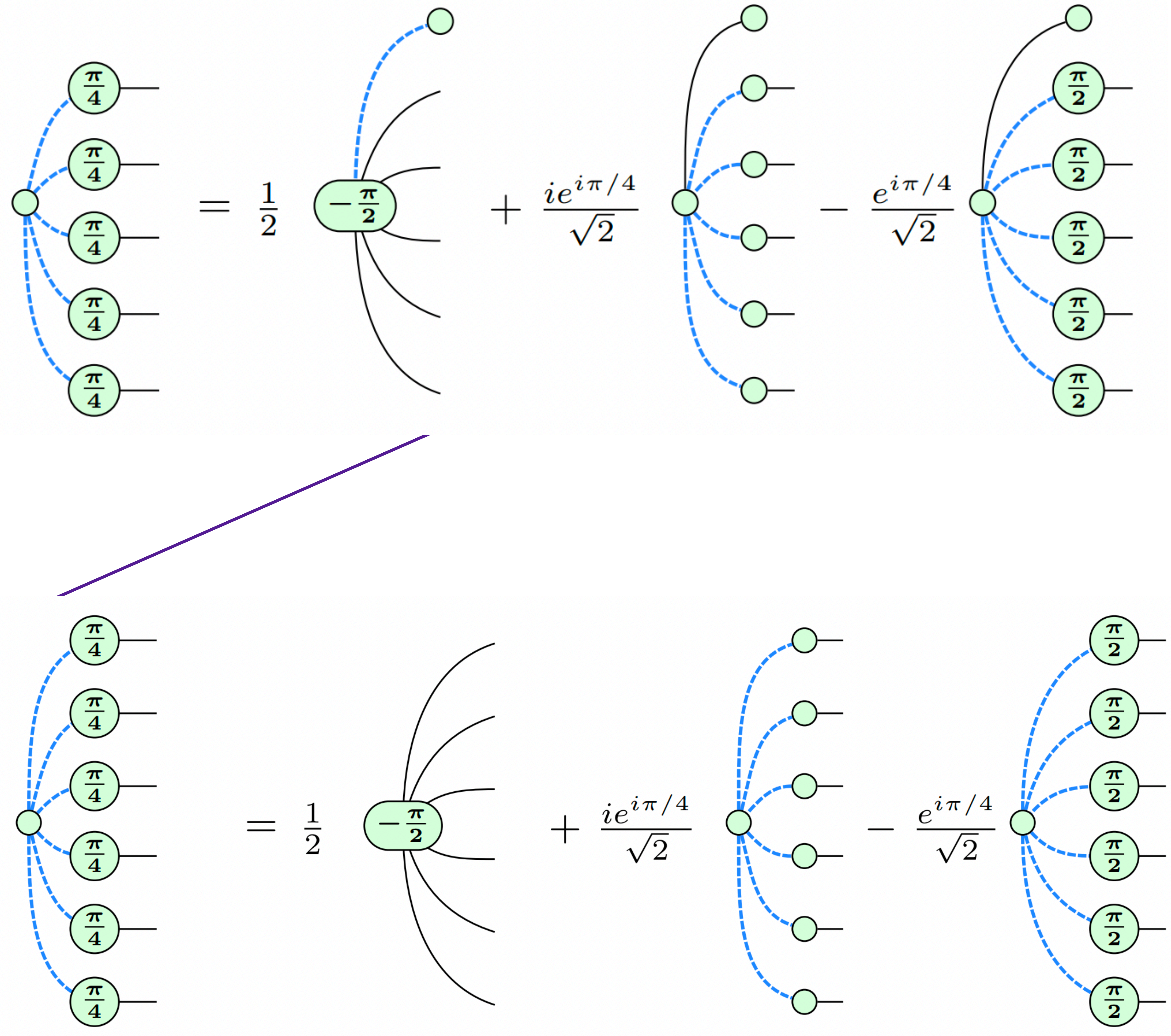
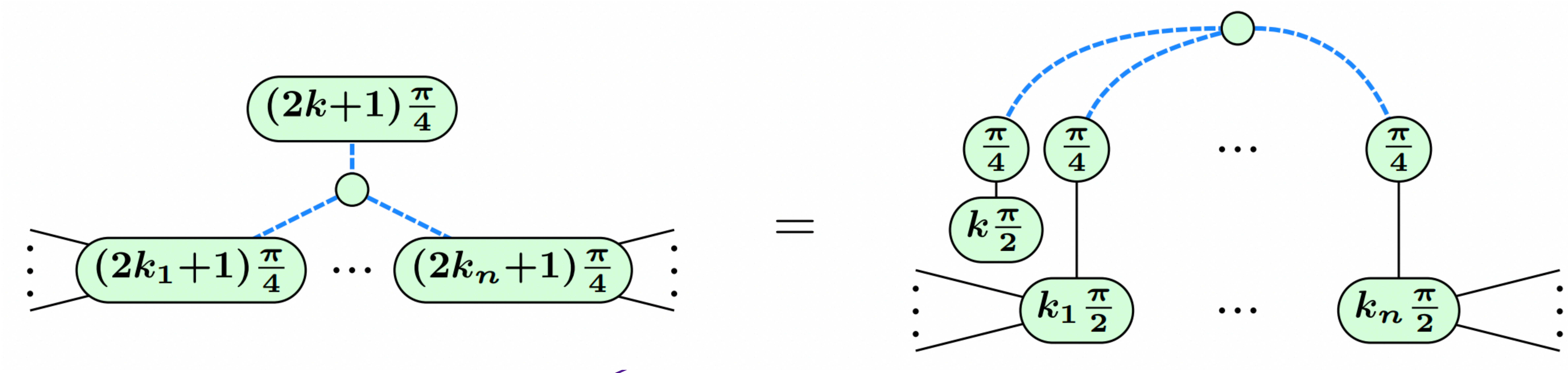
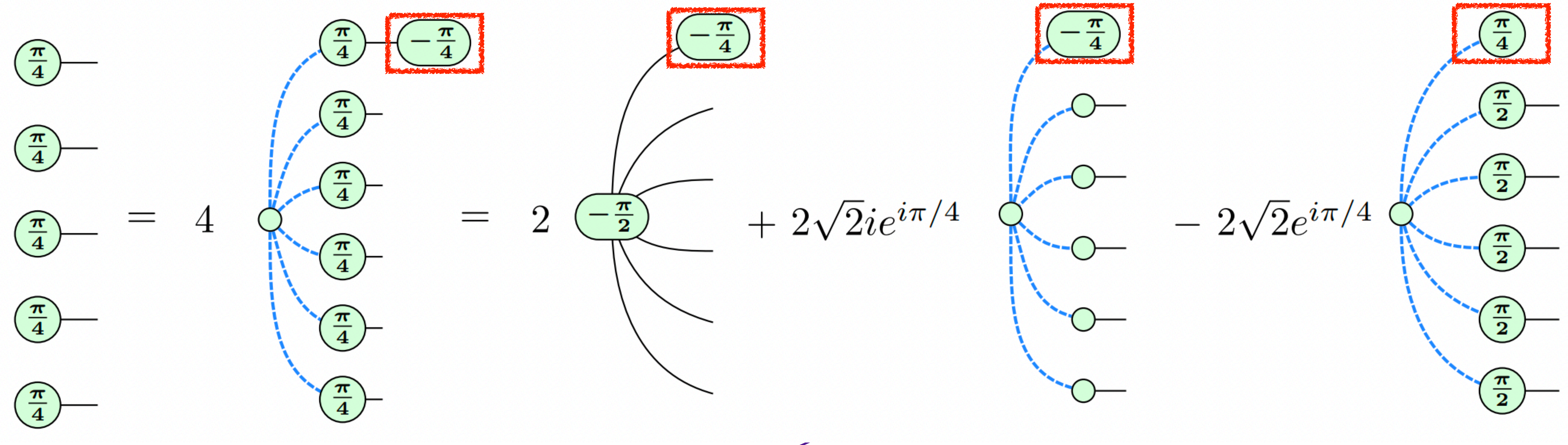


If we can identify these states in the ZX-diagrams, we can use these lower-rank decompositions…



An n-legged phase gadget corresponds to a $|\text{cat}_{n+1}\rangle$ state!



Essentially a 4-to-3 strategy $\Rightarrow \alpha \approx 0.396$!

So, the procedure searches the ZX-diagram according to the following order of importance:

1. a Clifford spider with 4 neighbours ($\alpha = 0.25$),
2. a Clifford spider with 6 neighbours ($\alpha \approx 0.264$),
3. a Clifford spider with 5 neighbours ($\alpha \approx 0.317$),
4. a Clifford spider with 3 neighbours ($\alpha = 1/3 \approx 0.333$),
5. any 5 T-spiders ($\alpha \approx 0.396$).

Thank you for your attention!