Matchgates and classical simulation of quantum circuits

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Presentation outline

- Introduction
- Mathematical formalism
- Classical simulation of quantum circuits
- Extended family of quantum circuits
- Universal quantum computation
- Outlook

Motivation

- Belief that quantum computers are more powerful and efficient than classical computers

Examples $\begin{cases} \text{Shor's algorithm} \\ \text{Universal quantum simulator} \\ \text{Communication complexity} \end{cases}$

- To understand that power an interesting perspective is to study restricted models of computation that can be simulated efficiently by classical devices

Restricted classes of quantum computation $\Rightarrow$ Quantum universality

Clifford gates

Paradigmatic example:

Pauli group = P

Stabilizer circuits $\longrightarrow$ Clifford gates

$$\mathrm{Cliff} = \langle X, Z, H, P, CNOT \rangle$$

$$U \in \mathrm{Cliff} : UPU^\dagger \subseteq P$$

Central objects in:
- Quantum fault-tolerance
- Quantum error correction

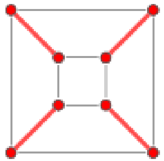
Gottesman-Knill theorem: Stabilizer circuits are classically efficiently simulated

Clifford gates + magic states $\longrightarrow$ Quantum universality

Matchgates

Valiant '02: Matchgates from theory of perfect matchings in graphs

Set of edges s.t. each vertex
is the end point of exactly one edge



(cubical graph)

Matchgates are in general non-unitary → set of 2-qubit unitary gates

nearest-neighbor matchgates

Terhal and DiVincenzo '01: n.n. matchgate circuits $\longleftrightarrow$ non-interacting fermions

Matchgates

Matchgates $\longrightarrow$ 2-qubit unitary gates

$$G(A,B) = \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} & \\ \begin{pmatrix} p & 0 & 0 & q \\ 0 & w & x & 0 \\ 0 & y & z & 0 \\ r & 0 & 0 & s \end{pmatrix} & \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$$

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

Properties: $\begin{cases} A, B \in SU(2) \\ \det A = \det B \end{cases}$

Matchgates

Action of $G(A,B)$

$\nearrow$ A acts on the even parity subspace $(\alpha_{00}|00\rangle + \beta_{11}|11\rangle)$

$\searrow$ B acts on the odd parity subspace $(\alpha_{01}|01\rangle + \beta_{10}|10\rangle)$

$$\boxed{G(A,B)(\alpha_{00}|00\rangle+\alpha_{01}|01\rangle+\beta_{10}|10\rangle+\beta_{11}|11\rangle) = G(A,B)(\alpha_{00},\alpha_{01},\beta_{10},\beta_{11})^T}$$

$$\Updownarrow$$

$$\boxed{A(\alpha_{00},\beta_{11})^T \oplus B(\alpha_{01},\beta_{10})^T}$$

even parity subspace decoupled from odd parity subspace $\longrightarrow$ $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^2 \oplus \mathbb{C}^2$

Main theorem of the paper[1]

Theorem 1.1 Consider any uniform (hence poly-sized) quantum circuit family comprising only $G(A, B)$ gates such that

- the $G(A, B)$ gates act on nearest neighbour (n.n.) lines only,
- the input state is any product state, and
- the output is a final measurement in the computational basis on any single line.

Then the output may be classically efficiently simulated. More precisely, for any k, we can classically efficiently compute the expectation value

$$\langle Z_k \rangle_{\text{out}} = \langle \psi_{\text{out}} | Z_k | \psi_{\text{out}} \rangle = p_0 - p_1 \quad \begin{cases} p_0 = |\langle 0 | \psi_{\text{out}} \rangle|^2 \\ p_1 = |\langle 1 | \psi_{\text{out}} \rangle|^2 \end{cases}$$

[1] Richard Jozsa and Akimasa Miyake. Matchgates and classical simulation of quantum circuits. *Proc. R. Soc. A*, (2008) 464, 3089–3106.

Uniform quantum circuit family

- The notion of *uniform circuit family* shows a connection between the Turing machine model and the circuit model

Circuit family $\longrightarrow$ collection of circuits $\{C_n\}$

input: number x of n bits

output: $C_n(x)$

- A family $\{C_n\}$ is said to be *uniform* if there is some algorithm running on Turing machine which, for input n, generates a description of C_n

$$\boxed{\text{classes of functions computable by uniform circuit family} \quad = \quad \text{classes of functions computable by a Turing machine}}$$

Efficient classical simulation

Uniform circuit family C_n with specified class of :

$$(1)\quad \text{input states} \begin{cases} \text{product states} \\ \text{computational basis states} \end{cases}$$

$$(2)\quad \text{output measurements } Z_k \longrightarrow Z\text{-measurement on any single line}$$

C_n is classically efficiently simulatable if the outcome probabilities can be computed by classical means to m digits of accuracy in $\mathrm{poly}(n, m)$ time

Mathematical formalism

For n-qubit lines we have a set of $2n$ Hermitian operators c_μ satisfying

anticommutation relations

$$\boxed{\{c_\mu, c_\nu\} \equiv c_\mu c_\nu + c_\nu c_\mu = 2\delta_{\mu\nu}\mathbb{I}, \qquad \mu, \nu = 1, \ldots, 2n}$$

Clifford algebra $\mathcal{C}_{2n}$ $\longrightarrow$ elements are arbitrary complex linear combinations of product of generators

$$\boxed{\dim \mathcal{C}_{2n} = 2^{2n} = 2^n \times 2^n}$$

$$\boxed{\sum_{i_1 < \cdots < i_k} A_{i_1 \ldots i_k} c_{i_1} \cdots c_{i_k}}$$

c_μ represented by $2^n \times 2^n$ matrices

Fermionic physics

Formalism of fermionic physics $\longrightarrow$ c_μ Majorana fermions

set of n operators accociated with n fermionic modes $\begin{cases} \text{occupied } (|1\rangle) \\ \text{unoccupied } (|0\rangle) \end{cases}$

creation operator $a_i^\dagger$

annihilation operator a_i

$$\{a_i, a_j\} \equiv a_i a_j + a_j a_i = 0 = \{a_i^\dagger, a_j^\dagger\}, \qquad \{a_i, a_j^\dagger\} = \delta_{ij}\mathbb{I}$$

standard anticommutation relations

$$\boxed{\begin{aligned} c_{2k-1} &= a_k + a_k^\dagger \\ c_{2k} &= -i(a_k - a_k^\dagger) \end{aligned} \qquad k = 1, \ldots, n}$$

$\longrightarrow$ fermionic version of position and momentum operators in bosonic systems

Quadratic Hamiltonians

A quadratic Hamiltonian is an element of $\mathcal{C}_{2n}$

$$H = i \sum_{\mu \neq \nu = 1}^{2n} h_{\mu\nu} c_\mu c_\nu$$

$2n \times 2n$ matrix of coefficients

H being Hermitian $(H = H^\dagger)$

$+$

$c_\mu c_\nu = -c_\nu c_\mu$

w.l.o.g. $h_{\mu\nu}$ be real antisymmetric

$H \longrightarrow$ unitary $U = e^{iH}$

power series

$\Downarrow$ $\Downarrow$

H quadratic $\longrightarrow$ Gaussian operation

Theorem 4.1

Theorem 4.1 Let H be any quadratic Hamiltonian and $U = e^{iH}$ be the corresponding Gaussian operation. Then for all μ,

$$U^\dagger c_\mu U = \sum_{\nu=1}^{2n} R_{\mu\nu} c_\nu$$

where $R \in SO(2n)$. In fact $R = e^{4h}$.

Proof of Theorem 4.1

$$c_\mu = c_\mu(0) \xrightarrow{t} c_\mu(t) = U(t)c_\mu(0)U(t)^\dagger$$

time evolution operator

$$U(t) = e^{iHt}$$

Heisenberg equation of motion $\quad \frac{dc_\mu(t)}{dt} = i[H, c_\mu(t)]$

commutator

$$\boxed{[a,b] = ab - ba}$$

$$i[H, c_\mu(t)] = i\left[i\sum_{k\neq l} h_{kl}c_k c_l, c_\mu(t)\right] = -\sum_{k\neq l} h_{kl}[c_k c_l, c_\mu]$$

$$\nearrow [c_k c_l, c_\mu] \quad k, l \neq \mu$$

$$\searrow [c_\mu c_l, c_\mu]$$

Proof of Theorem 4.1

$$\begin{aligned}[c_k c_l, c_\mu] &= c_k[c_l, c_\mu] + [c_k, c_\mu]c_l \\ &= c_k(2c_l c_\mu) + (2c_k c_\mu)c_l \quad (\{c_\mu, c_\nu\} = 0) \\ &= -2c_k c_\mu c_l + 2c_k c_\mu c_l \\ &= 0\end{aligned}$$

$$\begin{aligned}[c_\mu c_l, c_\mu] &= c_\mu[c_l, c_\mu] + [c_\mu, c_\mu]c_l \\ &= c_\mu(2c_l c_\mu) \qquad (\{c_\mu, c_\nu\} = 0) \\ &= -2c_\mu c_\mu c_l \qquad (c_\mu^2 = \mathbb{I}) \\ &= -2c_l\end{aligned}$$

$$\boxed{\text{Be careful} \quad \longrightarrow \quad [c_k c_\mu, c_\mu]}$$

$$\boxed{\begin{aligned}[c_\mu, c_\nu] &= c_\mu c_\nu - c_\nu c_\mu \\ &= c_\mu c_\nu + c_\mu c_\nu \quad (\{c_\mu, c_\nu\} = 0) \\ &= 2c_\mu c_\nu\end{aligned}}$$

Example

$$\left.\begin{aligned} h_{12}[c_1 c_2, c_1] &= -2h_{12}c_2 \\ h_{21}[c_2 c_1, c_1] &= h_{21}(-[c_1 c_2, c_1]) = h_{12}[c_1 c_2, c_1] = -2h_{12}c_2 \end{aligned}\right\} = -4h_{12}c_2$$

Proof of Theorem 4.1

$$\frac{dc_\mu(t)}{dt} = i[H, c_\mu(t)] = -\sum_{k\neq l} h_{kl}[c_k c_l, c_\mu] = -\sum_l (-4h_{\mu l} c_l)$$

$$\frac{dc_\mu(t)}{dt} = \sum_l 4h_{\mu l} c_l(t) \quad \Rightarrow \quad c_\mu(t) = \sum_l e^{4h_{\mu l}t} c_l(0) \quad \Rightarrow \quad \boxed{c_\mu(t) = \sum_l R_{\mu l}(t) c_l(0)}$$

$$\Downarrow$$

$$c_\mu(t) = U(t) c_\mu(0) U(t)^\dagger$$

antisymmetric matrices are infinitesimal generators of rotations $\longrightarrow$ $t = 1$

$$\boxed{U^\dagger c_\mu U = \sum_{l=1}^{2n} R_{\mu l} c_l}$$

Importance of Theorem 4.1

$U = e^{iH}$ $\longrightarrow$ all products of all generators

power series

$U c_\mu U^\dagger$ $\longrightarrow$ anywhere in 2^{2n} dimensional linear space $\mathcal{C}_{2n}$

Theorem 4.1

$$U^\dagger c_\mu U = \sum_{l=1}^{2n} R_{\mu l} c_l$$ $\longrightarrow$ polynomially small ($2n$-dim) subspace

Representation of Clifford algebra

$2n$ Hermitian operators acting on n qubits

$$c_1 = X_1 \mathbb{I}_2 \ldots \mathbb{I}_n \quad c_3 = Z_1 X_2 \ldots \mathbb{I}_n \quad \ldots \quad c_{2k-1} = Z_1 \ldots Z_{k-1} X_k \mathbb{I}_{k+1} \ldots \mathbb{I}_n = \left(\prod_{j=1}^{k-1} Z_j \right) X_k$$

$$c_2 = Y_1 \mathbb{I}_2 \ldots \mathbb{I}_n \quad c_4 = Z_1 Y_2 \ldots \mathbb{I}_n \quad \ldots \quad c_{2k} = Z_1 \ldots Z_{k-1} Y_k \mathbb{I}_{k+1} \ldots \mathbb{I}_n = \left(\prod_{j=1}^{k-1} Z_j \right) Y_k$$

satisfy $\{c_\mu, c_\nu\} = 2\delta_{\mu\nu}\mathbb{I}$

kth qubit $\left\{ \begin{array}{l} c_{2k-1} \\ c_{2k} \end{array} \right.$

Jordan-Wigner representation

qubits $\longleftrightarrow$ fermions

unique up to unitary equivalence

Classical efficient simulation

$c_\mu \xrightarrow{\text{J-W}}$ n-qubit operators $\Rightarrow$ Gaussian operations (quadratic H) $\longrightarrow$ class of n-qubit unitary gates

$|\psi_{in}\rangle \longrightarrow \boxed{\begin{array}{c}\tilde{U}\\ \text{circuit built of Gaussian gates}\end{array}} \longrightarrow |\psi_{out}\rangle = \tilde{U}|\psi_{in}\rangle$

Theorem 4.1

$$\langle c_\mu \rangle_{out} = \langle \psi_{in}|\tilde{U}^\dagger c_\mu \tilde{U}|\psi_{in}\rangle = \sum_{\nu=1}^{2n} \tilde{R}_{\mu\nu} \langle \psi_{in}|c_\nu|\psi_{in}\rangle$$

product of all $SO(2n)$ matrices

poly-time computable

Classical efficient simulation

input product state $|\psi_{in}\rangle = |x_1\rangle \cdots |x_n\rangle$

$$c_\mu \xrightarrow{\text{J-W}} \text{product operator} \quad P_1 \otimes \cdots \otimes P_n$$

$$\langle \psi_{in} | c_\mu | \psi_{in} \rangle = \prod_{i=1}^{n} \langle x_i | c_\mu | x_i \rangle$$

poly-time computable

$$\uparrow$$

$$\langle c_\mu \rangle_{out}$$

$$\langle Z_k \rangle_{out} = p_0 - p_1$$

J-W

$$Z_k = -i c_{2k-1} c_{2k}$$

Gaussian gates

$$\langle Z_k \rangle_{out} = \langle \psi_{in} | (-i) \tilde{U}^\dagger c_{2k-1} c_{2k} \tilde{U} | \psi_{in} \rangle = \langle \psi_{in} | (-i) (\tilde{U}^\dagger c_{2k-1} \tilde{U}) (\tilde{U}^\dagger c_{2k} \tilde{U}) | \psi_{in} \rangle$$

$$= \sum_{\nu_1 \neq \nu_2 = 1}^{2n} \tilde{R}_{(2k-1)\nu_1} \tilde{R}_{(2k)\nu_2} \langle \psi_{in} | (-i) c_{\nu_1} c_{\nu_2} | \psi_{in} \rangle$$

poly-time computable

Classical efficient simulation

Theorem 1.1 Consider any uniform (hence poly-sized) quantum circuit family comprising only $G(A, B)$ gates such that

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Gaussian gates in J-W representation

c_1, c_2 — qubit line 1

c_3, c_4 — qubit line 2

quadratic Hamiltonians involve

$$\boxed{\begin{aligned} -ic_1c_2 &= Z\mathbb{I} & -ic_2c_3 &= XX \\ ic_1c_3 &= YX & -ic_2c_4 &= XY \\ ic_1c_4 &= YY & -ic_3c_4 &= \mathbb{I}Z \end{aligned}}$$

→ trace free

→ preserve the even and odd parity subspaces

⬇

$SU(2) \oplus SU(2)$ decomposition

Gaussian gates in J-W representation

Idea

(1) construct the X, Y, Z Pauli operators acting in the two parity subspaces

(2) generate the two $SU(2)$'s by exponentiation

Example

$$\frac{1}{2}(XX+YY)(\alpha_{01}|01\rangle+\beta_{10}|10\rangle)=(\beta_{10}|01\rangle+\alpha_{01}|10\rangle)$$
$$\frac{1}{2}(XX+YY)(\alpha_{00}|00\rangle+\beta_{11}|11\rangle)=0$$

$\Rightarrow$

odd parity subspace

$$X(\alpha_{01},\beta_{10})^T=(\beta_{10},\alpha_{01})^T$$
$$X(\alpha_{00},\beta_{11})^T=0$$

even parity subspace

Gaussian gates in J-W representation

Gaussian operations

$$U = e^{iH}$$

$$H = i \sum_{\mu \neq \nu}^{4} h_{\mu\nu} c_\mu c_\nu$$

→

n.n. $G(A, B)$ gates

qubit lines $1 + 2$

For any pair of consecutive lines → all n.n. $G(A, B)$ gates

all n.n. $G(A, B)$ gates are Gaussian for the J-W representation

Theorem 1.1 proved

Gaussian quantum circuits and Clifford gates

Pauli group $\mathcal{P}_n$ $\longrightarrow$ $P_1 \otimes \cdots \otimes P_n$ $\qquad P_j \in \{I, X, Y, Z\}$

Clifford operation U $\longrightarrow$ $U^\dagger \mathcal{P}_n U \subseteq \mathcal{P}_n$

J-W representation comprises Pauli products

$$\boxed{\begin{array}{c} c_\mu \\ \{c_\mu, c_\nu\} = 2\delta_{\mu\nu}\mathbb{I} \end{array}} \longrightarrow \boxed{\begin{array}{c} c'_\mu = V^\dagger c_\mu V \quad \text{for any unitary } V \\ \{c_\mu, c_\nu\} = 2\delta_{\mu\nu}\mathbb{I} \end{array}}$$

Gaussian quantum circuits and Clifford gates

Efficient classical simulation $\begin{cases} \text{quadratic Hamiltonian property (anticomm. relations)} & (1) \\ \text{product structure of J-W representation (product states)} & (2) \end{cases}$

$$\boxed{c'_\mu = V^\dagger c_\mu V \qquad \text{for any Clifford unitary } V}$$

$$H' = i \sum_{\mu \neq \nu}^{2n} h'_{\mu\nu} c'_\mu c'_\nu$$

features (1) + (2) preserved $\Rightarrow$ new class of classically efficiently simulatable quantum circuits

Note: Clifford unitary V $\begin{cases} \text{NOT as Gaussian of original } c_\mu \\ \text{NOT as circuit of n.n. } G(A,B) \end{cases}$

Gaussian quantum circuits and Clifford gates

$$H' = i\sum_{\mu\neq\nu}^{2n} h'_{\mu\nu}c'_{\mu}c'_{\nu}$$

$$c'_{\mu} = V^{\dagger}c_{\mu}V$$

$$\longrightarrow \quad H' = V^{\dagger}\Big(i\sum_{\mu\neq\nu}^{2n} h'_{\mu\nu}c_{\mu}c_{\nu}\Big)V \quad \longrightarrow \quad U_{new} = V^{\dagger}U_{old}V$$

n.n. $G(A, B)$

$$V|\psi_{in,old}\rangle \longrightarrow \boxed{\text{old simulatable circuits}} \text{ — } M \quad VZ_kV^{\dagger}$$

Extension of class of inpute states and output measurements maintining classical efficiency

Universal quantum computation

Matchgates are extremely close to a universal set of gates

Universal quantum computation $\Rightarrow$

XY-interaction

$H \propto X \otimes X + Y \otimes Y$

any pair of qubits

$(\det \mathbb{I} = -\det X)$ $\tilde{G}(I, X) =$ SWAP gate

n.n. $G(A, B)$ + SWAP gate

trivial resource

Crucial condition $\longrightarrow$ nearest-neighbors interaction

Outlook

Matchgates $G(A,B)$ $\longrightarrow$ $A \oplus B$ (even parity$\oplus$odd parity)

Clifford algebra operators c_μ $\longrightarrow$ $\{c_\mu, c_\nu\} = 2\delta_{\mu\nu}\mathbb{I}$

quadratic H $\longrightarrow$ Gaussian gates $U = e^{iH}$ $\Rightarrow$ $\boxed{U^\dagger c_\mu U = \sum_{\nu=1}^{2n} R_{\mu l} c_l}$

$+$

J-W representation

$\Leftarrow$

circuits of n.n. $G(A,B)$
product input states
Z_k measurements

classically efficiently simulatable

Clifford unitaries $\longrightarrow$ new class of classically efficiently simulatable quantum circuits

Universal quantum computation $\Rightarrow$ n.n. $G(A,B)$ + SWAP gate

Thank you for your attention!

References

[1] Richard Jozsa and Akimasa Miyake. Matchgates and classical simulation of quantum circuits. *Proc. R. Soc. A*, (2008) 464, 3089–3106.

[2] Barbara M. Terhal and David P. DiVincenzo. Classical simulation of noninteracting-fermion quantum circuits. *Physical Review A*, (2002), 65(3):032325.

Theorem

Theorem 5.1 Let $H = i\sum_{\mu,\nu} h_{\mu\nu}c_\mu c_\nu$ be any quadratic Hamiltonian with corresponding Gaussian gate $V = e^{iH}$ on n qubits. Then, V as an operator on n qubits is expressible as a circuit of $O(n^3)$ n.n. $G(A,B)$ gates, i.e. $V = V_N \cdots V_1$ where each $U_j = e^{iH_j}$ having $H_j = i\sum_{\mu,\nu} h_{\mu\nu}c_\mu c_\nu$ with the sum involving only four c's associated with two n.n. lines.