Wigner negativity and contextuality are equivalent for continuous-variable quantum measurements

Robert I. Booth

March 9, 2022

Robert I. Booth, Ulysse Chabaud, and Pierre-Emmanuel Emeriau. “Contextuality and Wigner Negativity Are Equivalent for Continuous-Variable Quantum Measurements”. Nov. 2021. arXiv: 2111.13218

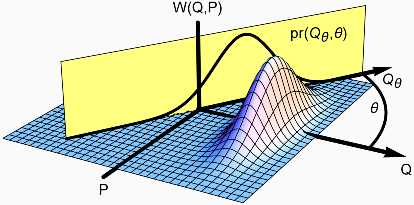
What's the Wigner function?

Wigner's function

The Wigner function associated to a quantum state of N particles is a *real-valued* function on the phase space $\mathbb{R}^{2N}$.

Wigner's function

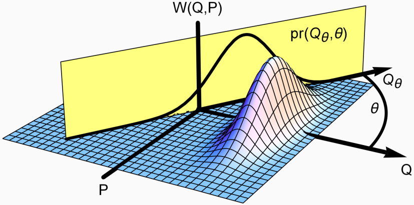
The Wigner function associated to a quantum state of N particles is a *real-valued* function on the phase space $\mathbb{R}^{2N}$.



Wigner's function

The Wigner function associated to a quantum state of N particles is a *real-valued* function on the phase space $\mathbb{R}^{2N}$.

Key property: its marginals give the probability distributions for position and momentum measurements.

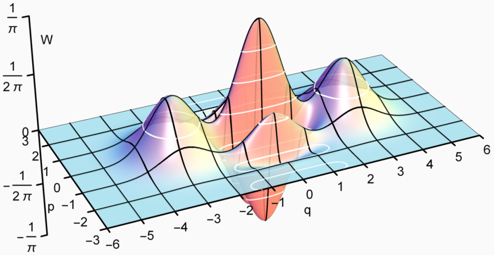


A quasi-probability distribution

When it is non-negative, it can be interpreted as a kind of classical probability distribution.

A quasi-probability distribution

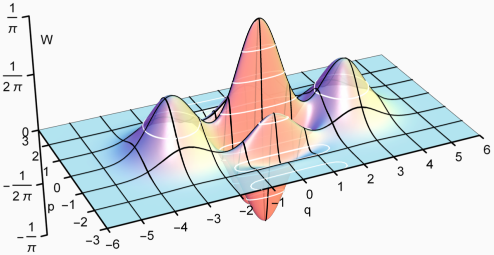
When it is non-negative, it can be interpreted as a kind of classical probability distribution.



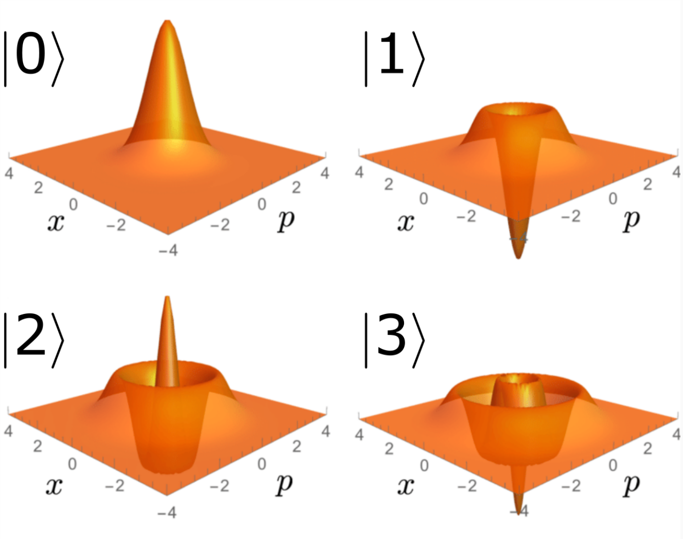
A quasi-probability distribution

When it is non-negative, it can be interpreted as a kind of classical probability distribution.

It often takes negative values. In quantum optics, this is taken to a be an indicator of non-classicality.



Often takes negative values



Negativity relates to incompatibility of measurements

Recall the canonical commutation relations:

$$[Q, P] = i\hbar.$$

Negativity relates to incompatibility of measurements

Recall the canonical commutation relations:

$$[Q, P] = i\hbar.$$

The laws of quantum theory permits one to make simultaneous measurements of commuting observables but not of non-commuting observables.

Negativity relates to incompatibility of measurements

Recall the canonical commutation relations:

$$[Q, P] = i\hbar.$$

The laws of quantum theory permits one to make simultaneous measurements of commuting observables but not of non-commuting observables.

One can show that the negative area of the Wigner function is always proportional to at most a few $\hbar$.

Negativity is necessary for Quantum speedup

Negativity of the Wigner function is a necessary resource for quantum speedup.

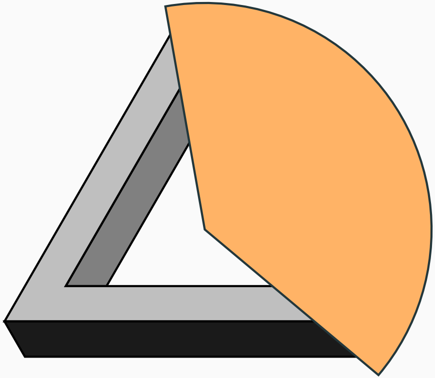
$$\rho_0 \longrightarrow \rho_1 \longrightarrow \cdots \longrightarrow \rho_t \Rightarrow \text{classical outcome}$$

Mari and Eisert's[1] generalisation of Gottesman-Knill theorem: if for all i, $W_i \geq 0$ (including measurement and providing that local probability distributions may be sampled efficiently) then computation can be simulated efficiently.

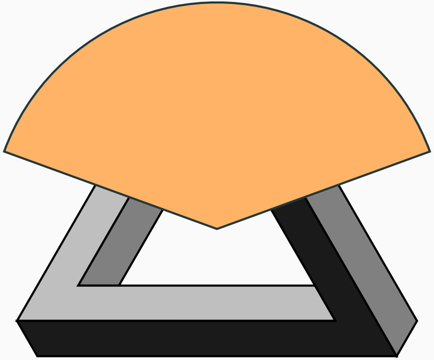
[1] Andrea Mari and Jens Eisert. "Positive Wigner functions render classical simulation of quantum computation efficient". In: *Physical review letters* 109.23 (2012), p. 230503.

Continuous-variable contextuality

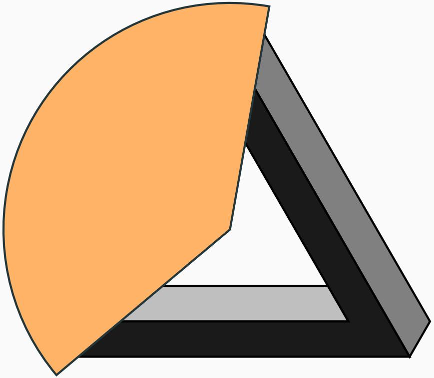
Intuition?



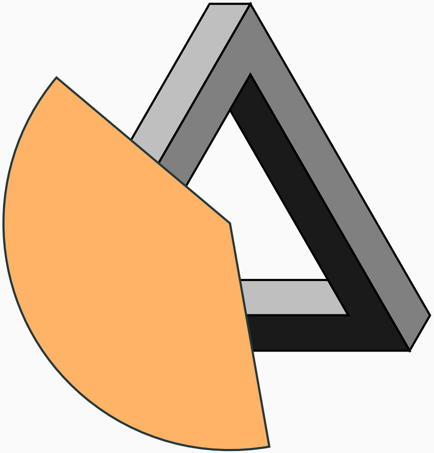
Intuition?



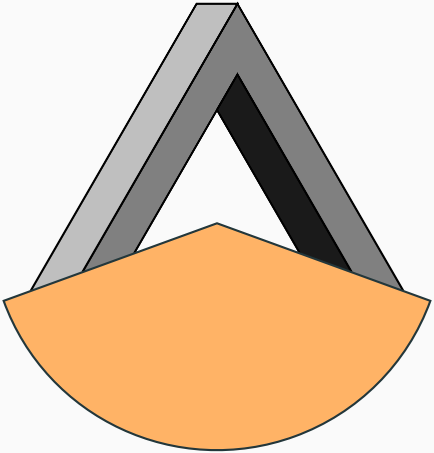
Intuition?



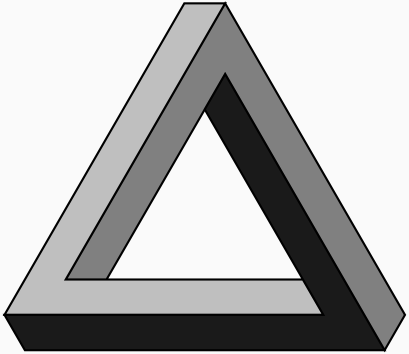
Intuition?



Intuition?



Intuition?

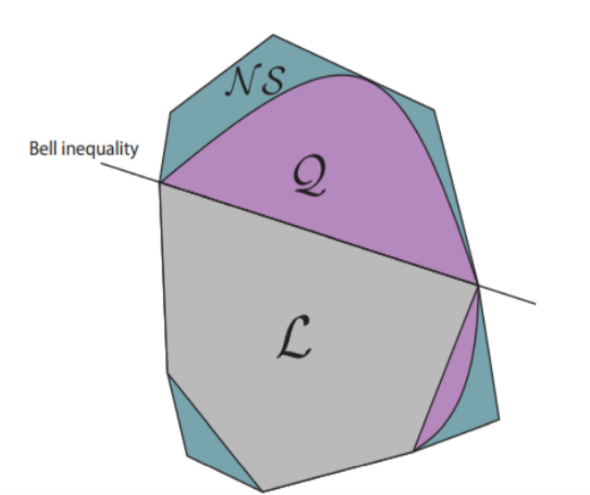


What is contextuality?

Correlations are *locally consistent* but cannot be explained *globally*.

What is contextuality?

Correlations are *locally consistent* but cannot be explained *globally*.



Sheaf-theoretic framework for contextuality

Two main ingredients:

- measurement scenarios;

Sheaf-theoretic framework for contextuality

Two main ingredients:

- measurement scenarios;
- empirical models.

Measurement scenario

A **measurement scenario** is a triple $\langle X, \mathcal{M}, O \rangle$ where:

- X a finite set of measurements

$$\text{e.g.} \quad X = \{a, a', b, b'\}$$

Measurement scenario

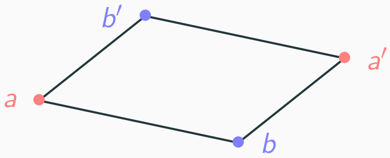
A **measurement scenario** is a triple $\langle X, \mathcal{M}, O \rangle$ where:

- X a finite set of measurements

  e.g. $X = \{a, a', b, b'\}$

- $\mathcal{M}$ the (maximal) contexts

  e.g. $\mathcal{M} = \{\{a, b\}, \{a, b'\}, \{a', b\}, \{a', b'\}\}$



Measurement scenario

A **measurement scenario** is a triple $\langle X, \mathcal{M}, O \rangle$ where:

- X a finite set of measurements

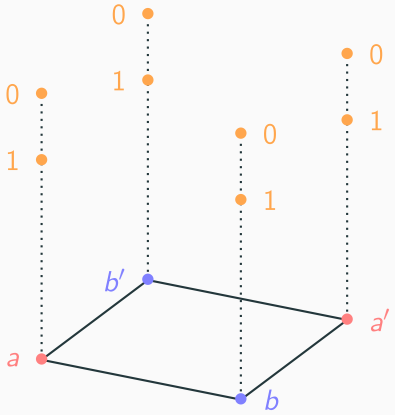
  e.g. $X = \{a, a', b, b'\}$

- $\mathcal{M}$ the (maximal) contexts

  e.g. $\mathcal{M} = \{\{a, b\}, \{a, b'\}, \{a', b\}, \{a', b'\}\}$

- $O = (O_x)_{x \in X}$ a family of outcomes sets

  e.g. $O = \{0, 1\}$



Measurement scenario

A **measurement scenario** is a triple $\langle X, \mathcal{M}, O \rangle$ where:

- X a finite set of measurements

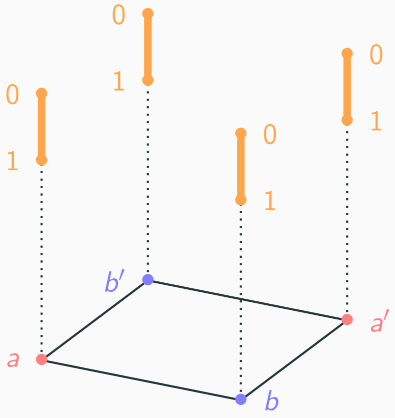
  e.g. $X = \{a, a', b, b'\}$

- $\mathcal{M}$ the (maximal) contexts

  e.g. $\mathcal{M} = \{\{a, b\}, \{a, b'\}, \{a', b\}, \{a', b'\}\}$

- $\boldsymbol{O} = (\boldsymbol{O}_x)_{x \in X}$ a family of measurable spaces

  e.g. $O = \mathbb{R}$ or $O = [0, 1]$



Empirical model

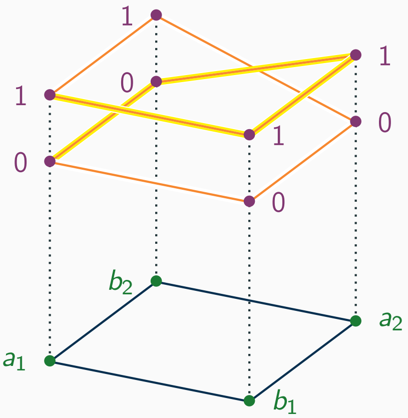
On a **measurement scenario** $\langle X, \mathcal{M}, O \rangle$ an empirical model e specifies the joint probability distribution for each context.

Empirical model

On a **measurement scenario** $\langle X, \mathcal{M}, O \rangle$ an empirical model e specifies the joint probability distribution for each context.

e.g. a PR box.

A	B	00	01	10	11
a_1	b_1	$\frac{1}{2}$	0	0	$\frac{1}{2}$
a_1	b_2	$\frac{1}{2}$	0	0	$\frac{1}{2}$
a_2	b_1	$\frac{1}{2}$	0	0	$\frac{1}{2}$
a_2	b_2	0	$\frac{1}{2}$	$\frac{1}{2}$	0



Empirical model

Definition (Empirical model)

An empirical model on a measurement scenario $\langle X, \mathcal{M}, \boldsymbol{O} \rangle$ is a family $e = (e_C)_{C \in \mathcal{M}}$, where e_C is a probability measure on the space $\boldsymbol{O}_C$ for each maximal context $C \in \mathcal{M}$. It satisfies the compatibility conditions:

$$\forall C, C' \in \mathcal{M}, \quad e_C|_{C \cap C'} = e_{C'}|_{C \cap C'} \tag{1}$$

Noncontextuality

Definition (Noncontextuality or extendability)

An empirical model e on a measurement scenario $\langle X, \mathcal{M}, \boldsymbol{O} \rangle$ is said to be noncontextual (or extendable) if there exists a global probability measure μ on global assignments $\boldsymbol{O}_X$ such that $\forall C \in \mathcal{M}, e_C = \mu|_C$.

Hidden-variable model

Definition (hidden-variable model)

A hidden-variable model on a measurement scenario $\langle X, \mathcal{M}, \boldsymbol{O} \rangle$ consists of the triple $\langle \boldsymbol{\Lambda}, p, (k_C)_{C \in \mathcal{M}} \rangle$ where:

- $\boldsymbol{\Lambda} = \langle \Lambda, \mathcal{F}_\Lambda \rangle$ is the measurable space of hidden variables,
- p is a probability distribution on $\boldsymbol{\Lambda}$,
- for each context $C \in \mathcal{M}$, k_C is a probability kernel between the measurable spaces $\boldsymbol{\Lambda}$ and $\boldsymbol{O}_C$ satisfying the following compatibility condition:

$$\forall C, C' \in \mathcal{M}, \forall \lambda \in \Lambda, \quad k_C(\lambda, -)|_{C \cap C'} = k_{C'}(\lambda, -)|_{C \cap C'} \tag{2}$$

Hidden-variable model

Definition (hidden-variable model)

A hidden-variable model on a measurement scenario $\langle X, \mathcal{M}, \boldsymbol{O} \rangle$ consists of the triple $\langle \boldsymbol{\Lambda}, p, (k_C)_{C \in \mathcal{M}} \rangle$ where:

- $\boldsymbol{\Lambda} = \langle \Lambda, \mathcal{F}_\Lambda \rangle$ is the measurable space of hidden variables,
- p is a probability distribution on $\boldsymbol{\Lambda}$,
- for each context $C \in \mathcal{M}$, k_C is a probability kernel between the measurable spaces $\boldsymbol{\Lambda}$ and $\boldsymbol{O}_C$ satisfying the following compatibility condition:

$$\forall C, C' \in \mathcal{M}, \forall \lambda \in \Lambda, \quad k_C(\lambda, -)|_{C \cap C'} = k_{C'}(\lambda, -)|_{C \cap C'} \tag{2}$$

For every measurable set of joint outcomes $B \in \mathcal{F}_C$,

$$e_C(B) = \int_\Lambda k_C(-, B) \, \mathrm{d}\, p = \int_{\lambda \in \Lambda} k_C(\lambda, B) \, \mathrm{d}\, p(\lambda) \tag{3}$$

FAB theorem

Theorem (Fine–Abramsky–Brandenburger theorem)

Equivalence between:

- *extendability;*
- *deterministic hidden-variable model;*
- *factorisable hidden-variable model.*

Main question

At an intuitive level, these two notions of non-classicality seek to capture similar features of quantum theory. But the question remains:

Main question

At an intuitive level, these two notions of non-classicality seek to capture similar features of quantum theory. But the question remains:

What is the precise relationship between Wigner negativity and contextuality?

State of the art

Wigner positivity sometimes implies non-contextuality

- Robert W Spekkens. "Negativity and contextuality are equivalent notions of nonclassicality". In: *Physical review letters* 101.2 (2008), p. 020401
- Konrad Banaszek and Krzysztof Wódkiewicz. "Nonlocality of the Einstein-Podolsky-Rosen State in the Wigner Representation". In: *Physical Review A* 58.6 (Dec. 1998), pp. 4345–4347
- Zeng-Bing Chen et al. "Maximal Violation of Bell's Inequalities for Continuous Variable Systems". In: *Physical Review Letters* 88.4 (Jan. 2002), p. 040406

Actual equivalence results

- Mark Howard et al. "Contextuality Supplies the Magic for Quantum Computation". In: *Nature* 510.7505 (June 2014), pp. 351–355. arXiv: 1401.4174
- Nicolas Delfosse et al. "Equivalence between Contextuality and Negativity of the Wigner Function for Qudits". en. In: *New Journal of Physics* 19.12 (Dec. 2017), p. 123024
- Robert Raussendorf et al. "Contextuality and Wigner Function Negativity in Qubit Quantum Computation". In: *Physical Review A* 95.5 (May 2017), p. 052334. arXiv: 1511.08506

And in continuous variables?

?

Equivalence between contextuality and Wigner negativity

Equivalence between contextuality and Wigner negativity

The measurement scenario

Definition

We fix the measurement scenario $\langle X, \mathcal{M}, \boldsymbol{O} \rangle$ as follows:

- the set of measurement labels consists of all points in phase-space i.e. all points generated by linear combinations of Q_k and P_j;

Measurement scenario

Definition

We fix the measurement scenario $\langle X, \mathcal{M}, \boldsymbol{O} \rangle$ as follows:

- the set of measurement labels consists of all points in phase-space i.e. all points generated by linear combinations of Q_k and P_j;
- the maximal contexts are maximal subsets of X of commuting quadratures;

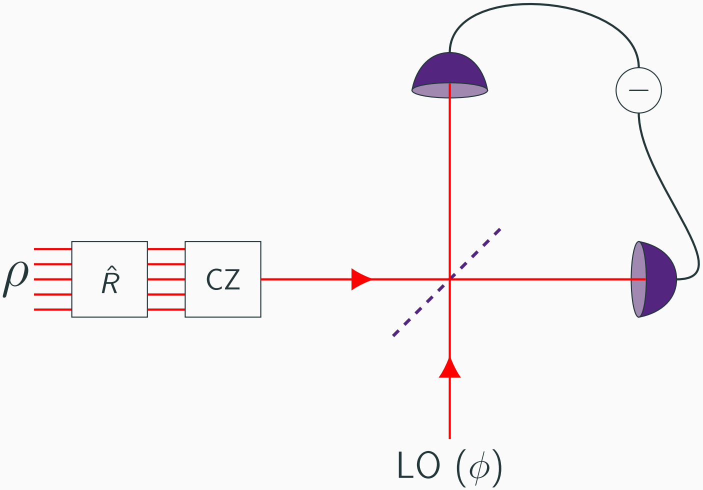
Measurement scenario

Definition

We fix the measurement scenario $\langle X, \mathcal{M}, \boldsymbol{O} \rangle$ as follows:

- the set of measurement labels consists of all points in phase-space i.e. all points generated by linear combinations of Q_k and P_j;
- the maximal contexts are maximal subsets of X of commuting quadratures;
- for each $\boldsymbol{x} \in X$, $\boldsymbol{O_x} := \langle \mathbb{R}, \mathcal{B}_{\mathbb{R}} \rangle$ so that for any set of measurement labels $U \subseteq X$, $O_U \cong \mathbb{R}^U$ can be seen as the set of functions from U to $\mathbb{R}$ with its product σ-algebra $\mathcal{F}_U$.

Measurement scenario



Equivalence between contextuality and Wigner negativity

Admissible empirical models

Which empirical models?

- Empirical models on previously defined measurement scenario. Recall $O_U \cong \mathbb{R}^U$.

Which empirical models?

- Empirical models on previously defined measurement scenario. Recall $O_U \cong \mathbb{R}^U$.
- Quantum! In particular verify the Born rule.

Equivalence between contextuality and Wigner negativity

The equivalence

Main result

Theorem

If ρ is the density operator of a quantum system with Hilbert space $L^2(\mathbb{R}^N)$, and $N \geqslant 2$, there is a deterministic hidden-variable model for the measurements in $\mathcal{M}$ on ρ if and only if the Wigner function of ρ is both integrable and non-negative.

Proof ideas

Three barriers to the proof:

Proof ideas

Three barriers to the proof:

- An arbitrary noncontextual HVM can be pretty weird...

.

Proof ideas

Three barriers to the proof:

- An arbitrary noncontextual HVM can be pretty weird...
  $\rightarrow$ there is always a HVM where the hidden variables are just the global value assignments. In our case, $\mathbb{R}^X$.

Proof ideas

Three barriers to the proof:

- An arbitrary noncontextual HVM can be pretty weird...
  $\to$ there is always a HVM where the hidden variables are just the global value assignments. In our case, $\mathbb{R}^X$.
- But the Wigner function is defined on the phase space X, not $\mathbb{R}^X$...

.

Proof ideas

Three barriers to the proof:

- An arbitrary noncontextual HVM can be pretty weird...
  $\rightarrow$ there is always a HVM where the hidden variables are just the global value assignments. In our case, $\mathbb{R}^X$.
- But the Wigner function is defined on the phase space X, not $\mathbb{R}^X$...
  $\rightarrow$ value assignements can be taken to be linear, so the HVM can be replaced by one on the dual of $X^* \cong X$.
- The probability measure of the HVM and the Wigner function now have the same Fourier transform.

Implications

This equivalence gives a stronger grounding for both notions as indicators of non-classicality:

- Foundationally, negativity of the Wigner function no longer means that just this classical representation is bad, but in fact none exist whatsoever;

Implications

This equivalence gives a stronger grounding for both notions as indicators of non-classicality:

- Foundationally, negativity of the Wigner function no longer means that just this classical representation is bad, but in fact none exist whatsoever;
- Computationally, it implies that contextuality is necessary to obtain any kind of quantum computational advantage in the standard model of CV quantum computation.

Outlook

- Better methods for witnessing contextuality in an experimental setting; continuous-variable Bell inequalities?

Outlook

- Better methods for witnessing contextuality in an experimental setting; continuous-variable Bell inequalities?
- Stronger link between measures of non-classicality?

Outlook

- Better methods for witnessing contextuality in an experimental setting;

  continuous-variable Bell inequalities?

- Stronger link between measures of non-classicality?
- Characterising Wigner positive states?

Outlook

- Better methods for witnessing contextuality in an experimental setting; continuous-variable Bell inequalities?
- Stronger link between measures of non-classicality?
- Characterising Wigner positive states?
- Other s-QPDs?

Outlook

- Better methods for witnessing contextuality in an experimental setting;

  continuous-variable Bell inequalities?
- Stronger link between measures of non-classicality?
- Characterising Wigner positive states?
- Other s-QPDs?
- Extending these methods to treat different measurement scenarios corresponding to different phase spaces.