Free transformations in the resource theory of contextuality

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This talk

- Pre-print available at arXiv:2104.11241 [quant-ph].

Quantum Physics

[Submitted on 22 Apr 2021]

Closing Bell: Boxing black box simulations in the resource theory of contextuality

Rui Soares Barbosa, Martti Karvonen, Shane Mansfield

This chapter contains an exposition of the sheaf-theoretic framework for contextuality emphasising resource-theoretic aspects, as well as some original results on this topic. In particular, we consider functions that transform empirical models on a scenario S to empirical models on another scenario T, and characterise those that are induced by classical procedures between S and T corresponding to 'free' operations in the (non-adaptive) resource theory of contextuality. We proceed by expressing such functions as empirical models themselves, on a new scenario built from S and T. Our characterisation then boils down to the non-contextuality of these models. We also show that this construction on scenarios provides a closed structure in the category of measurement scenarios.

Comments: 36 pages. To appear as part of a volume dedicated to Samson Abramsky in Springer's Outstanding Contributions to Logic series
Subjects: **Quantum Physics (quant-ph)**; Logic in Computer Science (cs.LO); Category Theory (math.CT)
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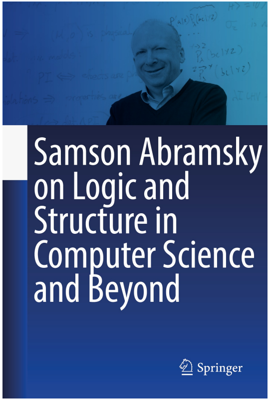

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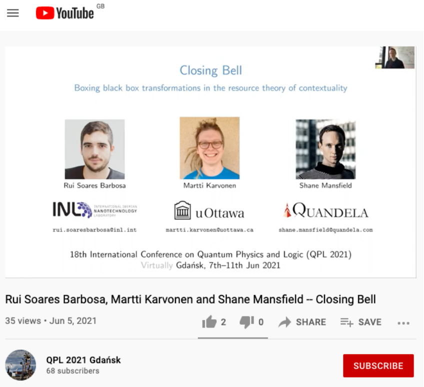
- (Abridged version of) this talk at QPL 2021: `y2u.be/rShNOuaim_U`.



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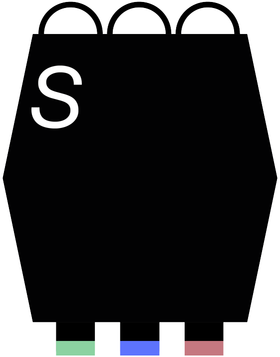
  - $[-, -]$ provides a **closed structure** on the category of measurement scenarios

Contextuality

Type or interface: measurement scenario



- Interaction with system: perform measurements and observe respective outcomes

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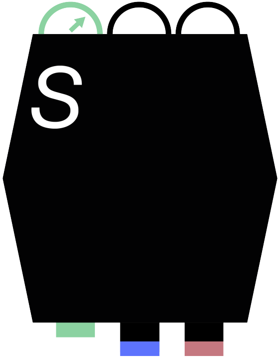

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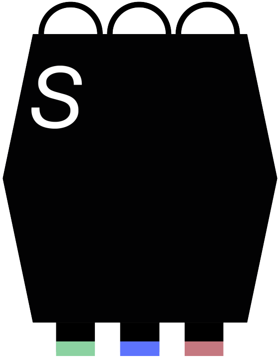

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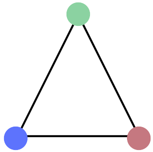

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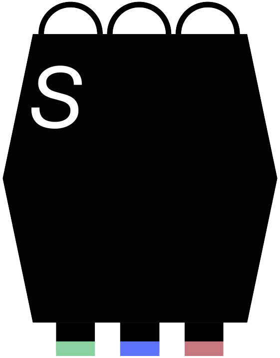

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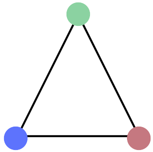

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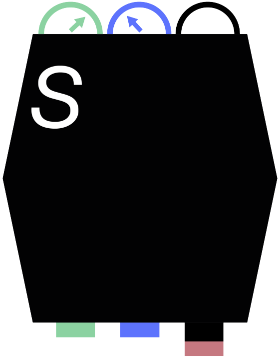

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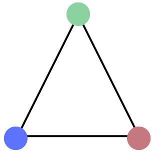

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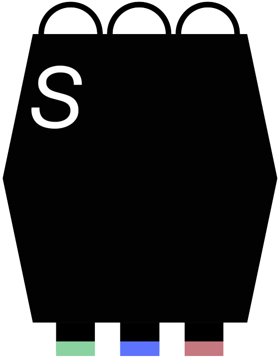

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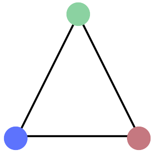

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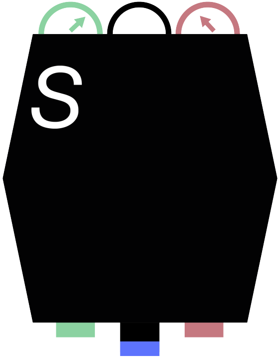

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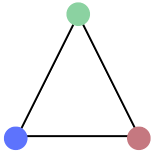

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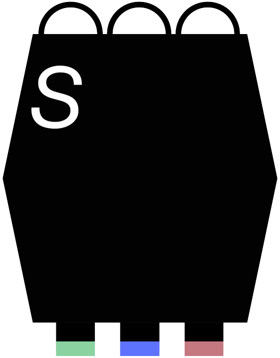

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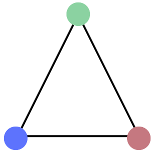

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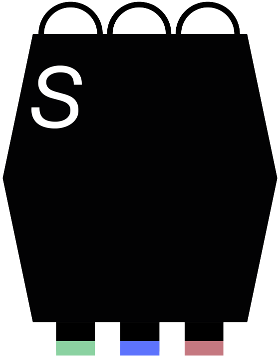

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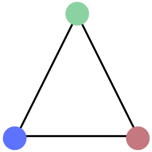

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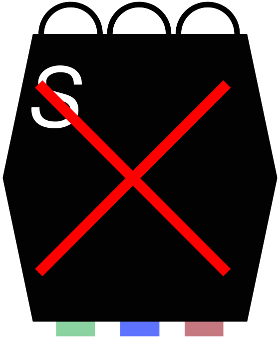

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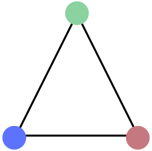

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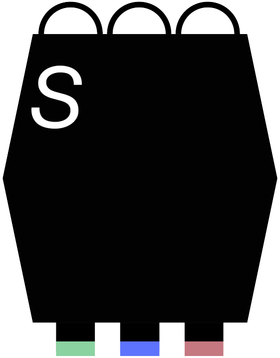

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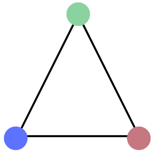

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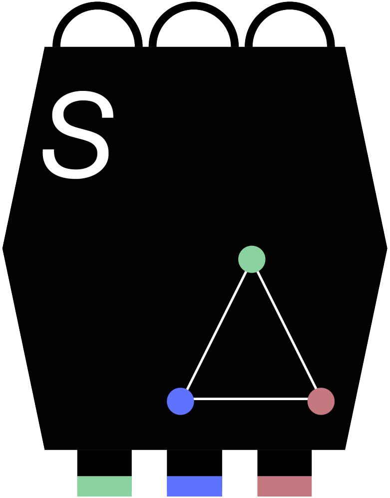

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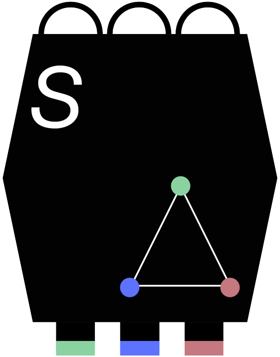

Measurement scenario $S = \langle X_S, \Sigma_S, O_S \rangle$:

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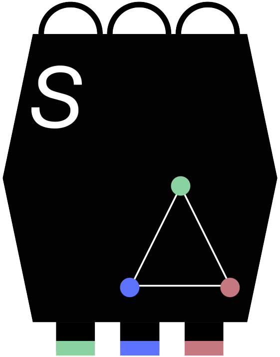

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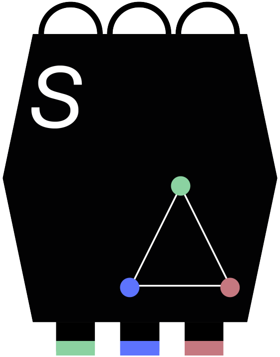

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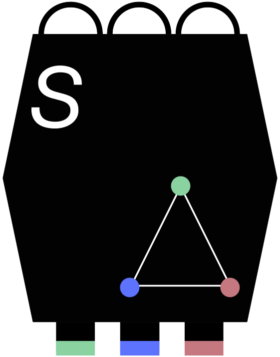

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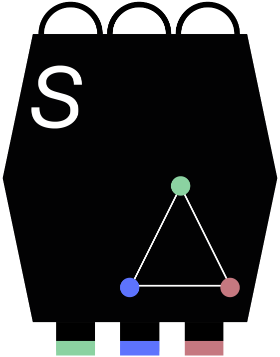

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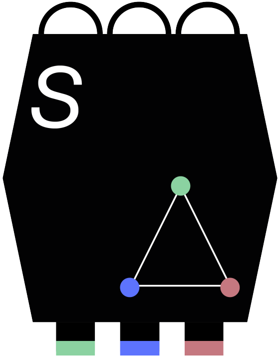

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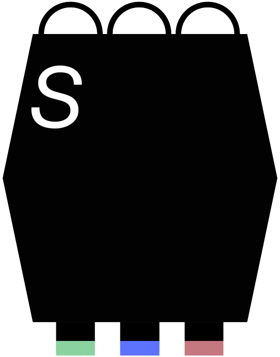
Behaviour: empirical model



- Behaviour of system is described by measurement statistics

		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y				
y	z				
x	z				

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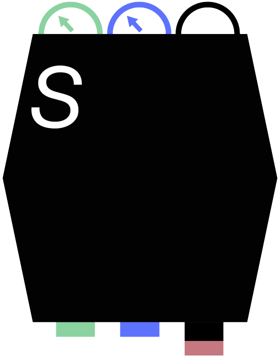

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y	z				
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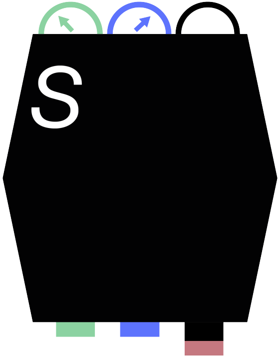

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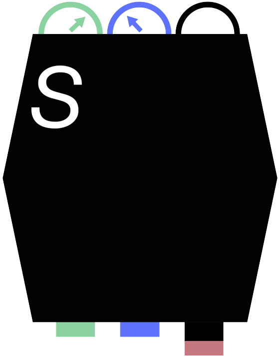

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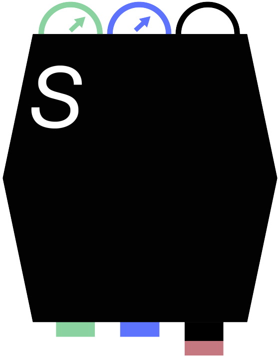

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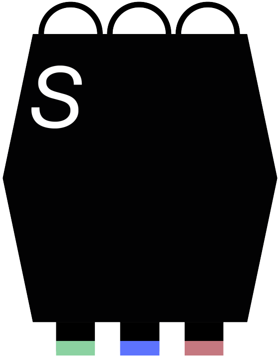
x	z				

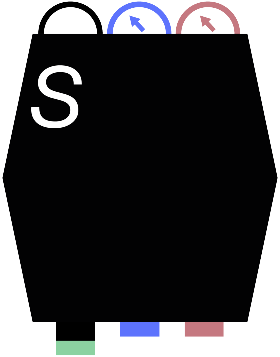
Behaviour: empirical model



- Behaviour of system is described by measurement statistics

		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z				
x	z				

Behaviour: empirical model



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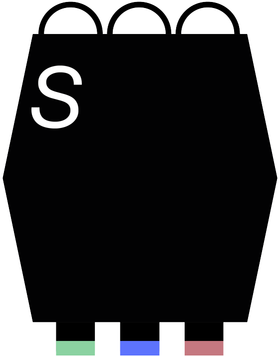
		$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$			
x	z				

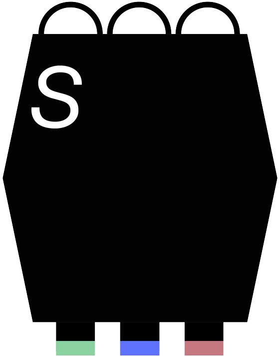
Behaviour: empirical model



- Behaviour of system is described by measurement statistics

		$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

Behaviour: empirical model



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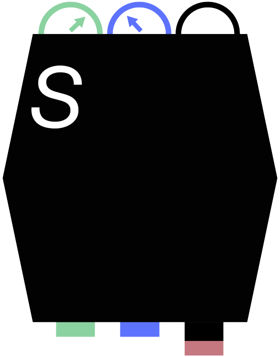
		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

Behaviour: empirical model



- Behaviour of system is described by measurement statistics

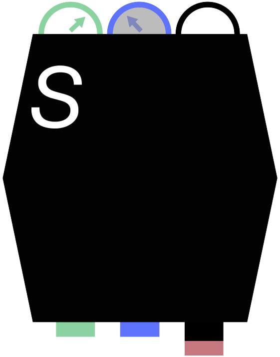
		$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$P(\mathsf{x},\mathsf{y} \mapsto a, b)$$

Behaviour: empirical model



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		$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
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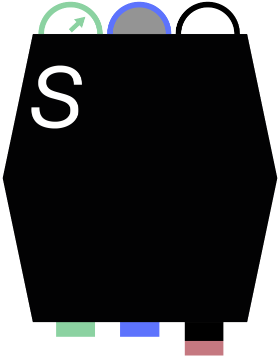


- Behaviour of system is described by measurement statistics

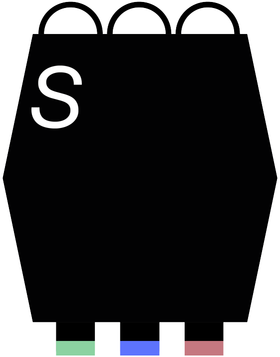
		$(\mathbf{0},\mathbf{0})$	$(\mathbf{0},\mathbf{1})$	$(\mathbf{1},\mathbf{0})$	$(\mathbf{1},\mathbf{1})$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_{b} P(\mathbf{x}, \mathbf{y} \mapsto a, b)$$

Behaviour: empirical model



- Behaviour of system is described by measurement statistics

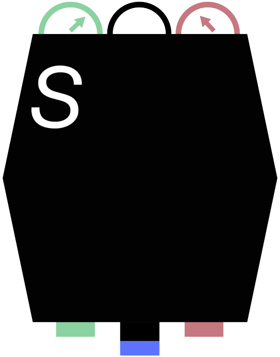
		(0, 0)	(0, 1)	(1, 0)	(1, 1)
x	y	3/8	1/8	1/8	3/8
y	z	3/8	1/8	1/8	3/8
x	z	1/8	3/8	3/8	1/8

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(x, y \mapsto a, b)$$

Behaviour: empirical model



- Behaviour of system is described by measurement statistics

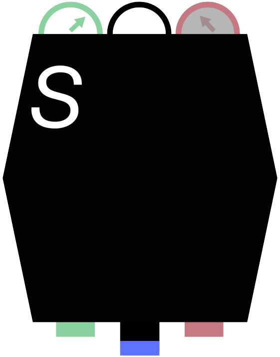
		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathbf{x}, \mathbf{y} \mapsto a, b) \qquad\qquad P(\mathbf{x}, \mathbf{z} \mapsto a, c)$$

Behaviour: empirical model



- Behaviour of system is described by measurement statistics

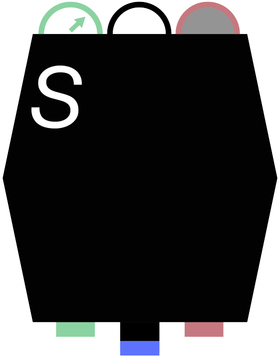
		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathbf{x}, \mathbf{y} \mapsto a, b) \qquad\qquad P(\mathbf{x}, \mathbf{z} \mapsto a, c)$$

Behaviour: empirical model



- Behaviour of system is described by measurement statistics

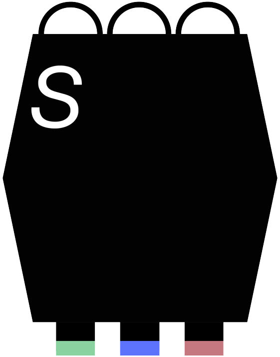
		(0,0)	(0,1)	(1,0)	(1,1)
x	y	3/8	1/8	1/8	3/8
y	z	3/8	1/8	1/8	3/8
x	z	1/8	3/8	3/8	1/8

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_b P(\mathbf{x}, \mathbf{y} \mapsto a, b) \qquad \sum_c P(\mathbf{x}, \mathbf{z} \mapsto a, c)$$

Behaviour: empirical model



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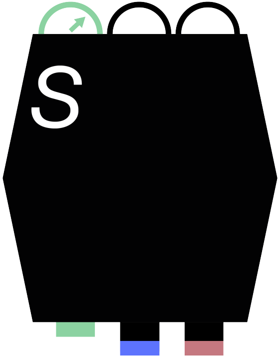
		(0, 0)	(0, 1)	(1, 0)	(1, 1)
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

- Marginal distributions agree

$$\sum_{b} P(\mathsf{x}, \mathsf{y} \mapsto a, b) \quad = \quad \sum_{c} P(\mathsf{x}, \mathsf{z} \mapsto a, c)$$

Behaviour: empirical model



- Behaviour of system is described by measurement statistics

		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
y	z	$3/8$	$1/8$	$1/8$	$3/8$
x	z	$1/8$	$3/8$	$3/8$	$1/8$

No-signalling / no-disturbance

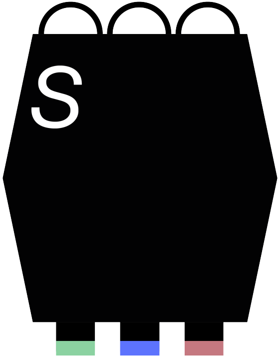
- Marginal distributions agree

$$\sum_b P(\mathsf{x}, \mathsf{y} \mapsto a, b) \qquad \sum_c P(\mathsf{x}, \mathsf{z} \mapsto a, c)$$

$$=$$

$$P(\mathsf{x} \mapsto a)$$

Behaviour: empirical model



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		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
x	y	$3/8$	$1/8$	$1/8$	$3/8$
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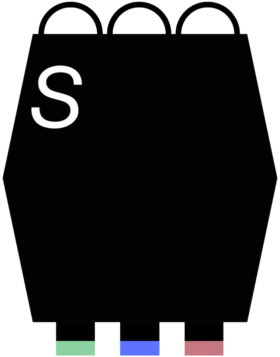
Behaviour: empirical model

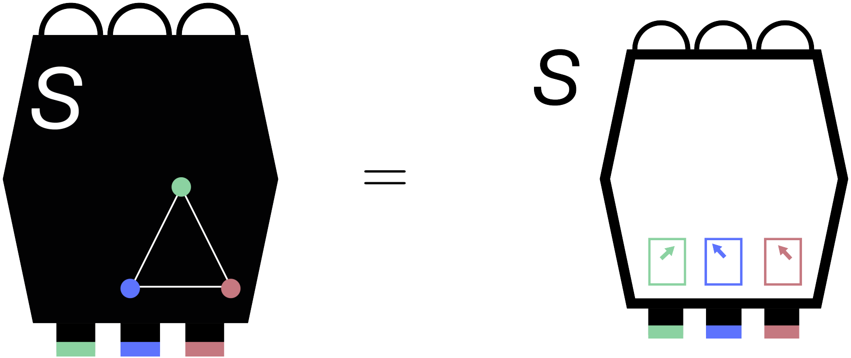


Empirical model $e : S$ is a family $\{e_\sigma\}_{\sigma \in \Sigma_S}$ where:

- e_σ is a probability distribution on the set of joint outcomes $\mathbf{O}_{S,\sigma} := \prod_{x \in \sigma} O_{S,x}$
- These satisfy no-disturbance:
  if $\tau \subset \sigma$, then $e_\sigma|_\tau = e_\tau$.

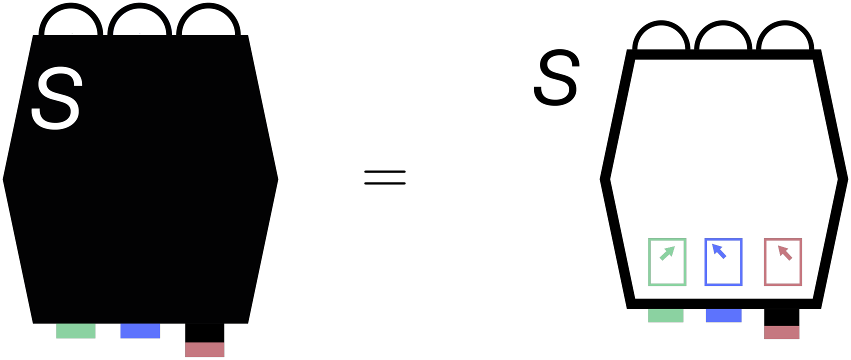
Contextuality

Deterministic model



Contextuality

Deterministic model



Contextuality

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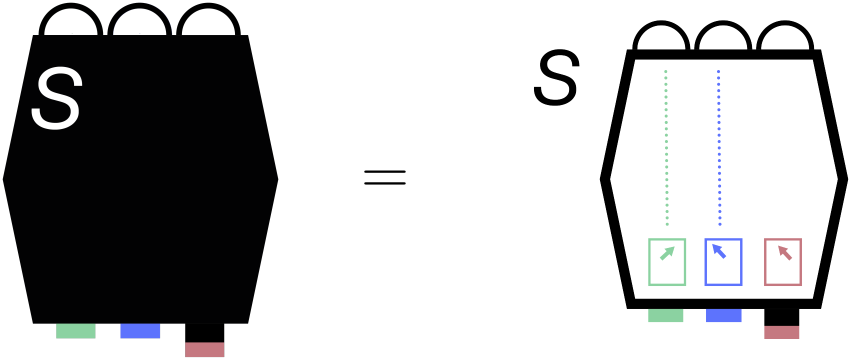

Contextuality

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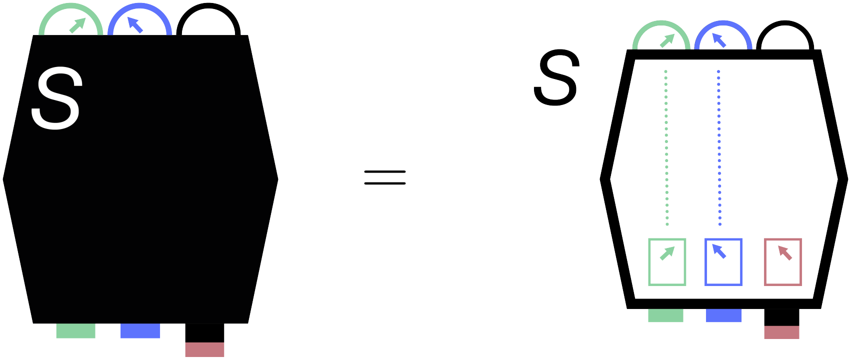
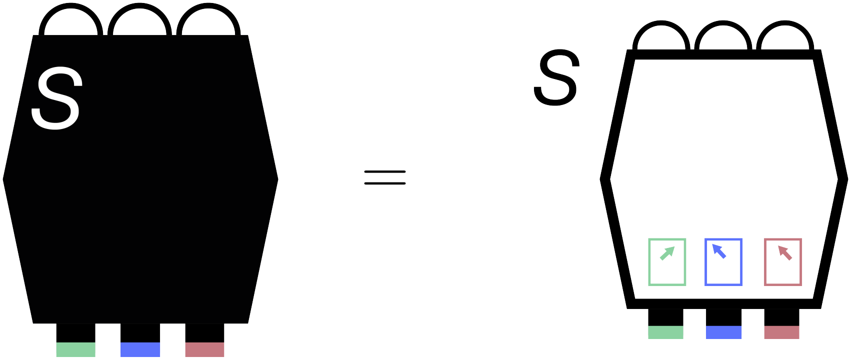


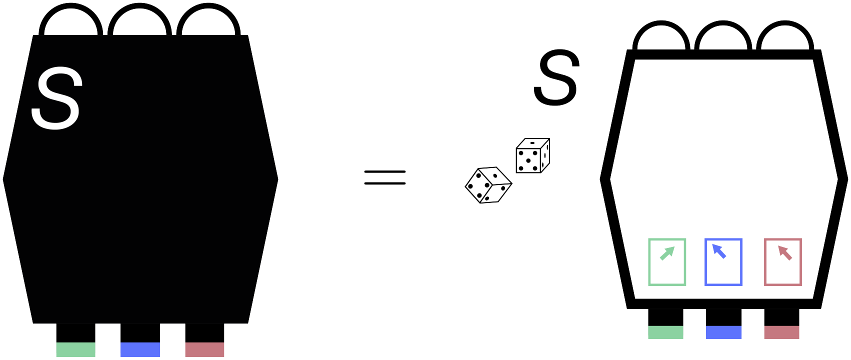
Contextuality

Deterministic model



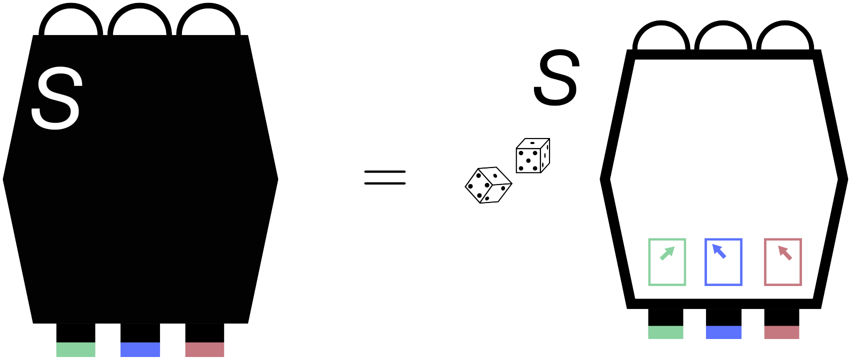
Contextuality

Non-contextual model



Contextuality

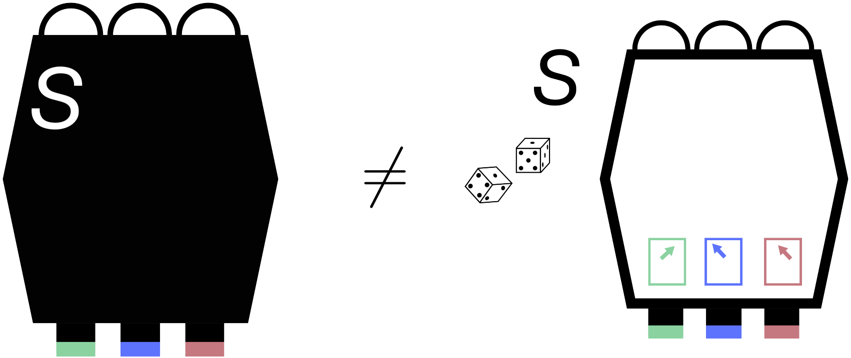
Non-contextual model



$\exists$ probability distribution d on $\mathbf{O}_{S,X_S} = \prod_{x \in X_S} O_{S,x}$ such that $d|_\sigma = e_\sigma$ for all $\sigma \in \Sigma_S$.

Contextuality

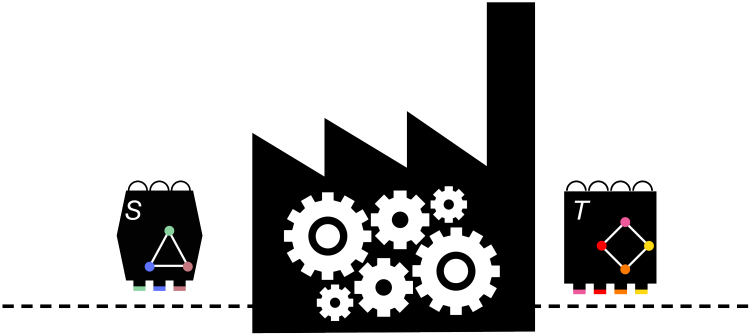
Contextual model



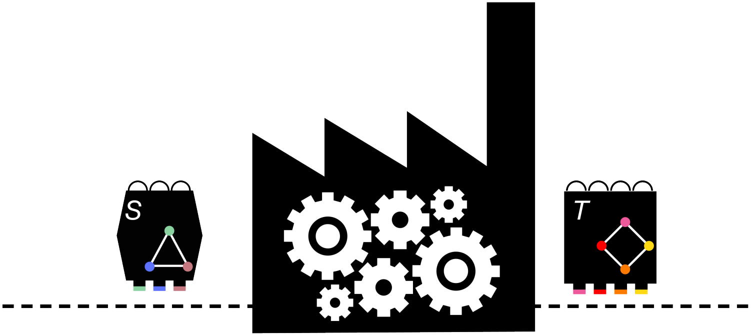
$\nexists$ probability distribution d on $\mathbf{O}_{S,X_S} = \prod_{x \in X_S} O_{S,x}$ such that $d|_\sigma = e_\sigma$ for all $\sigma \in \Sigma_S$.

Resource theory of contextuality

Resource theories

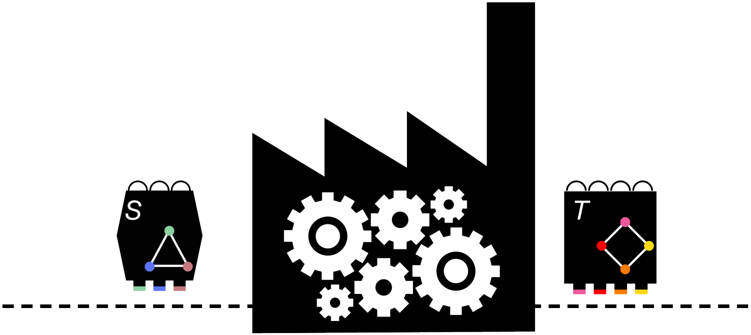


Resource theories



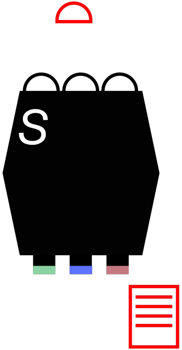
- Consider ‘free’ (i.e. classical) operations:

Resource theories



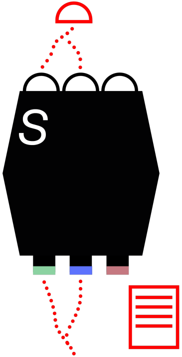
- Consider ‘free’ (i.e. classical) operations:
  (classical) procedures that use a box of type S to simulate a box of type T

Experiments and procedures



- An S-**experiment** is a protocol for an interaction with the box S:
  - which measurements to perform;
  - how to interpret their joint outcome into an outcome of the intended type.

Experiments and procedures



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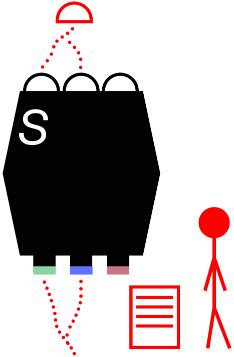

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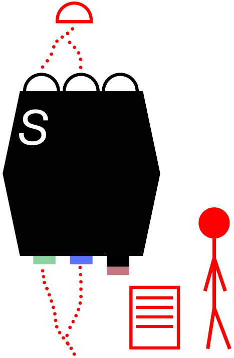

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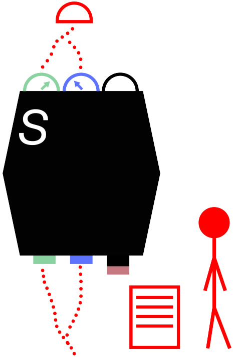
Experiments and procedures



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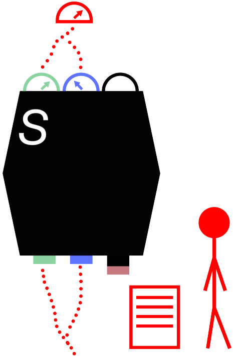
Experiments and procedures



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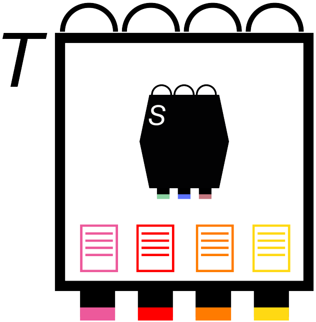
Experiments and procedures



- An S-**experiment** is a protocol for an interaction with the box S:
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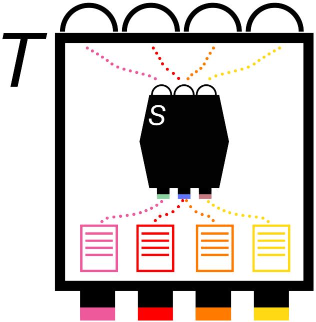
- A **deterministic procedure** $S \longrightarrow T$ specifies an S-experiment for each measurement of T

Experiments and procedures



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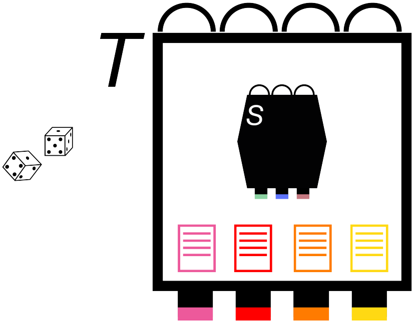
Experiments and procedures



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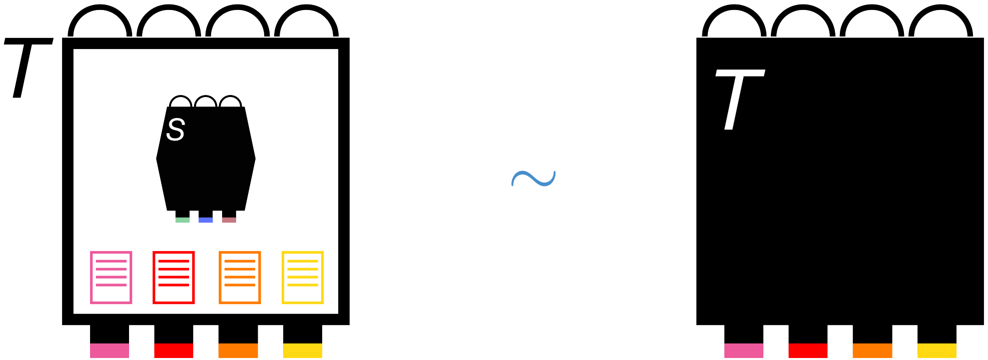
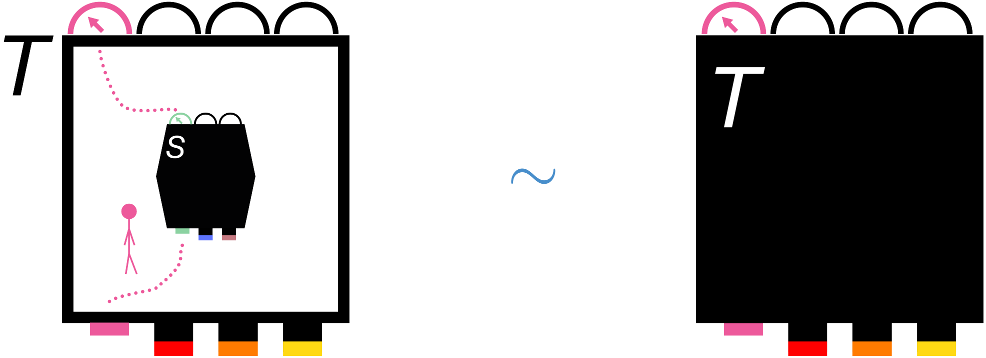
  - which measurements to perform;
  - how to interpret their joint outcome into an outcome of the intended type.
- A **deterministic procedure** $S \longrightarrow T$ specifies an S-experiment for each measurement of T
- A **classical procedure** is a probabilistic mixture of deterministic procedures.

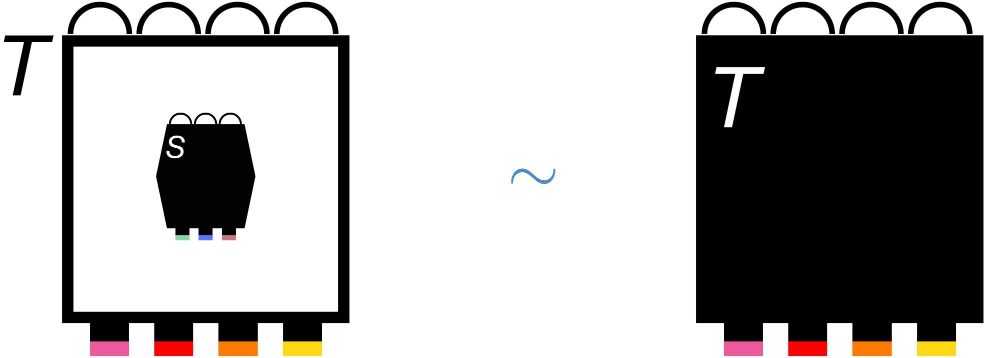
Classical procedures and simulations



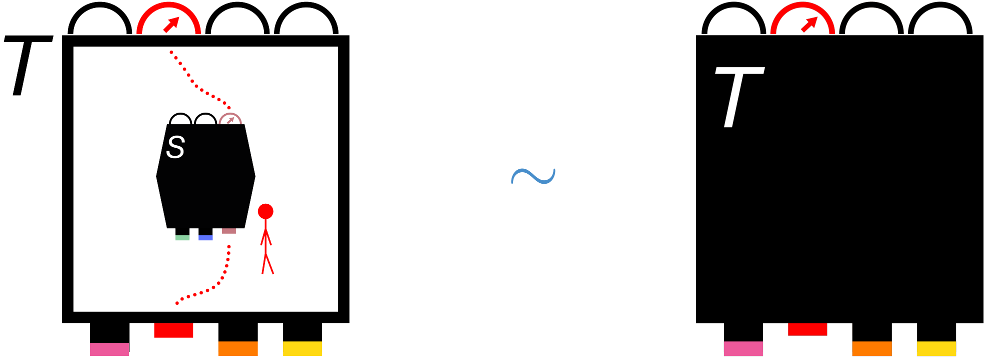
Classical procedures and simulations



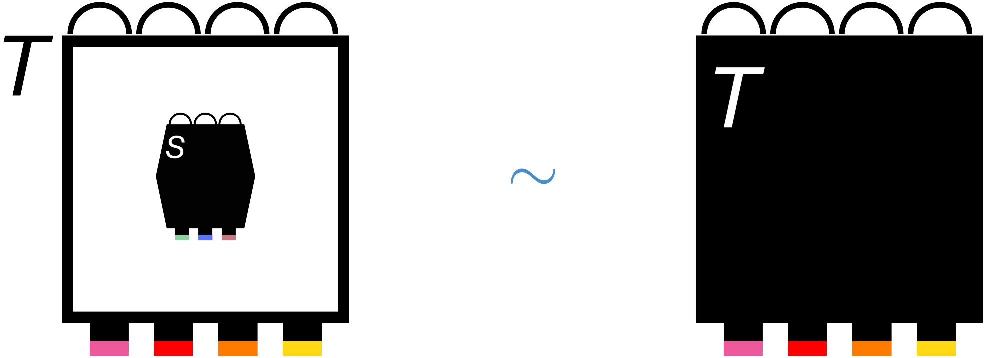
Classical procedures and simulations



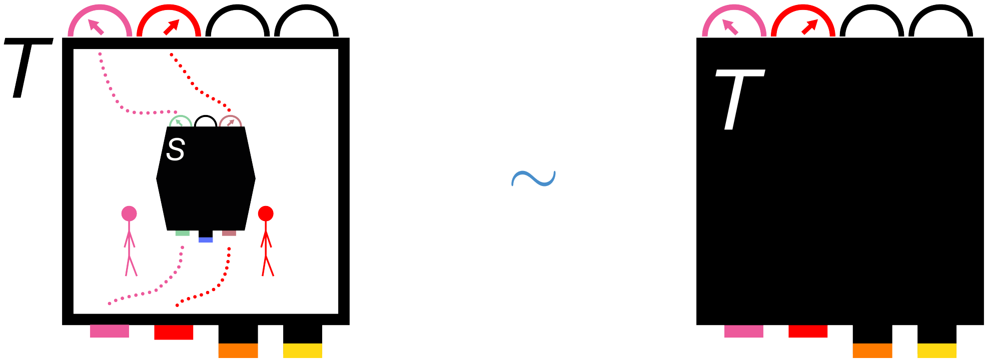
Classical procedures and simulations



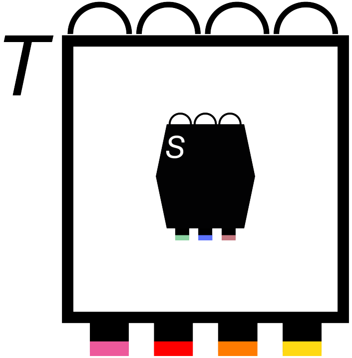
Classical procedures and simulations



Classical procedures and simulations



Classical procedures and simulations



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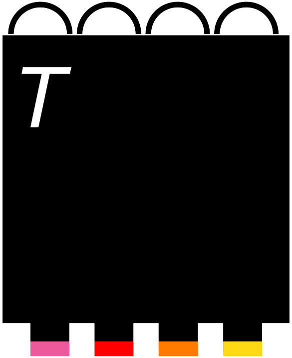
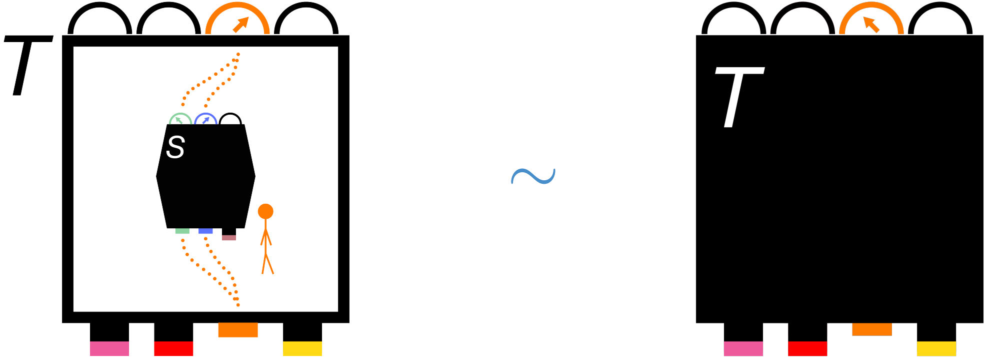
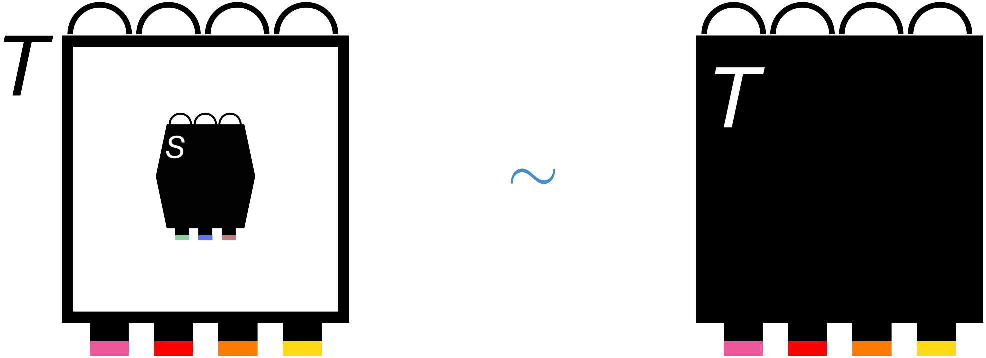


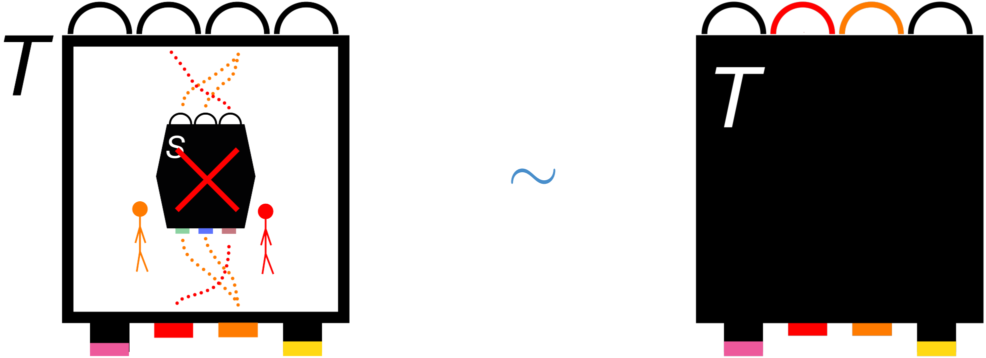
Classical procedures and simulations



Classical procedures and simulations



Classical procedures and simulations



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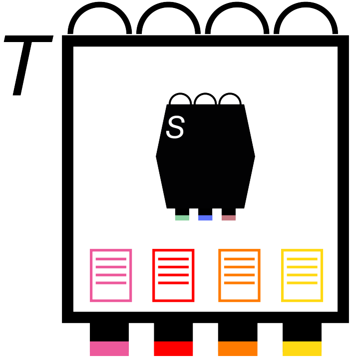

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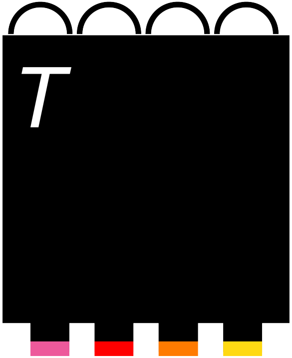

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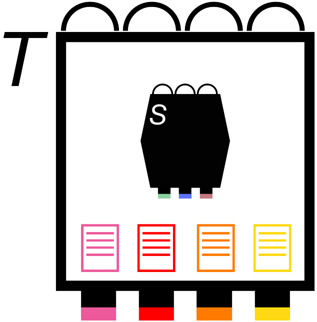

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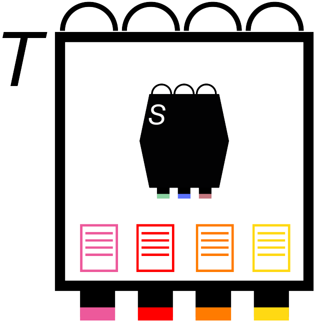

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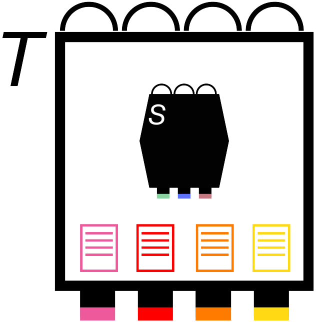

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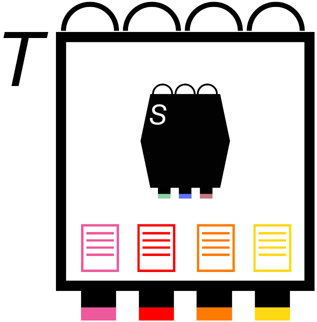

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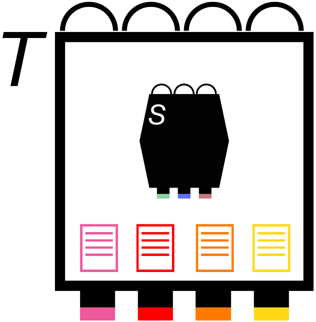

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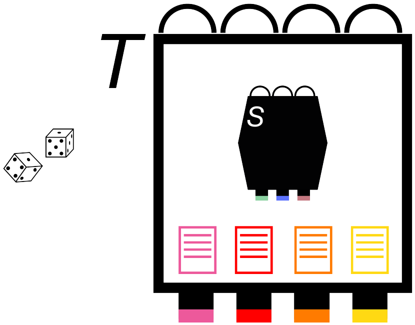

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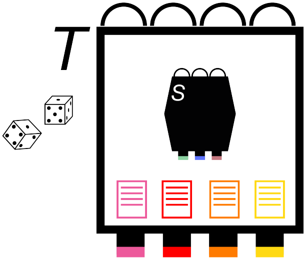
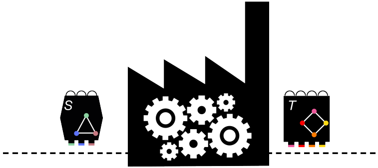
Probabilistic procedure $f : S \longrightarrow T$ is $f = \sum_i r_i f_i$ where $r_i \geq 0$, $\sum_i r_i = 1$, and $f_i : S \longrightarrow T$ deterministic procedures.

Classical simulations

- A classical procedure induces a (convex-preserving) map between empirical models:

$$f : S \longrightarrow T \qquad \rightsquigarrow \qquad \mathbf{Emp}(f) : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$$





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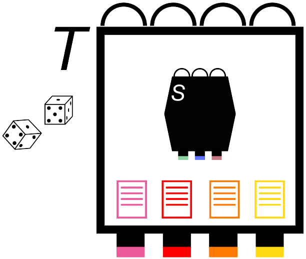

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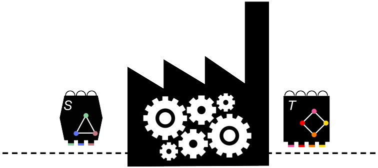
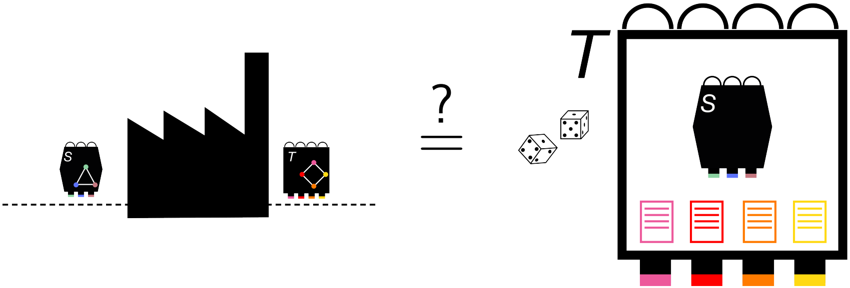


- Which black-box transformations arise in this fashion?

Main question and sketch of the answer

Main question

Given $F : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$, can it be realised by an experimental procedure? I.e. is there a procedure $f : S \longrightarrow T$ s.t. $F = \mathbf{Emp}(f)$?



Relativising contextuality

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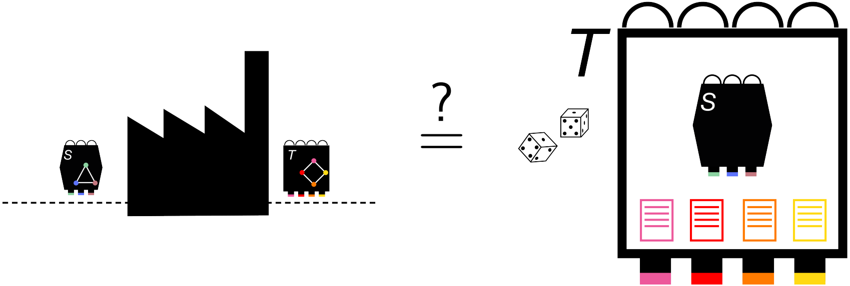

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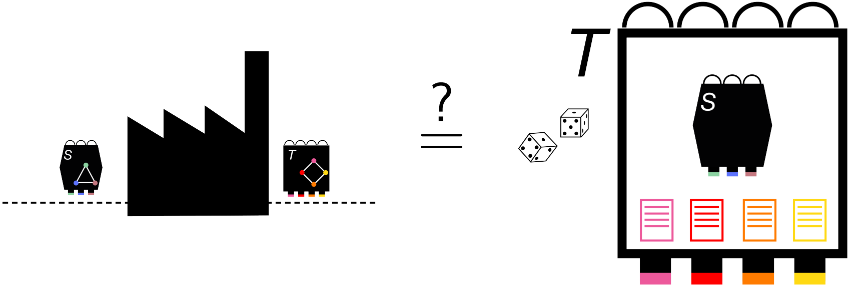

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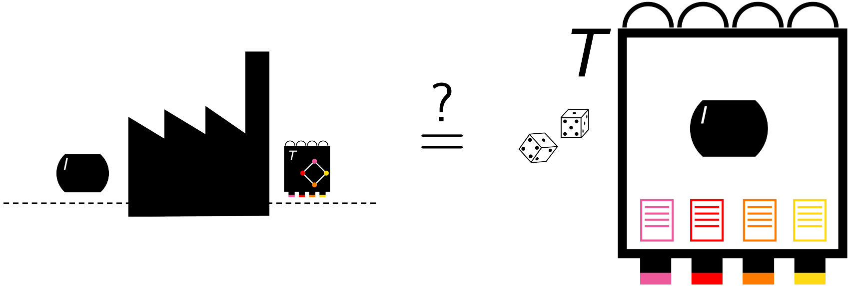

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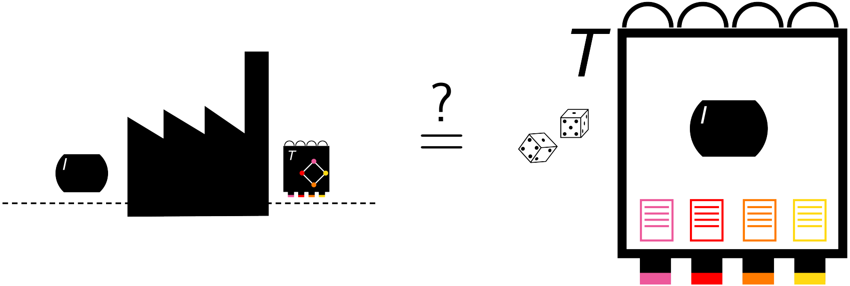

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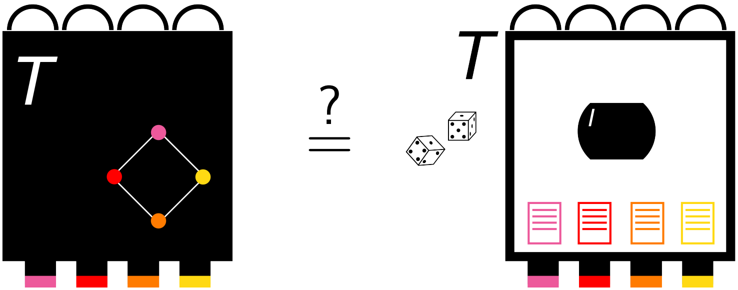

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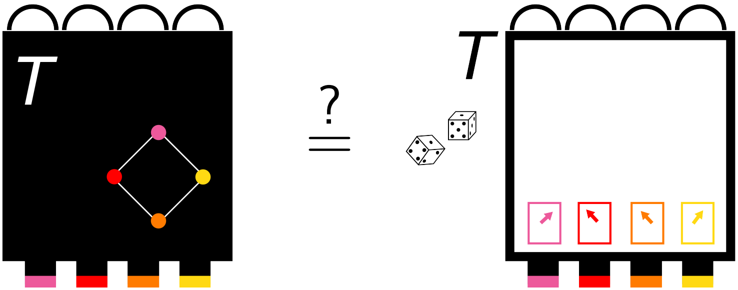

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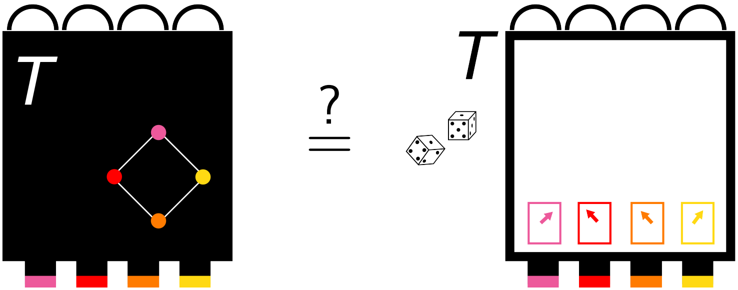
(Non-contextual models are those which can be simulated from nothing.)



From objects to morphisms . . .

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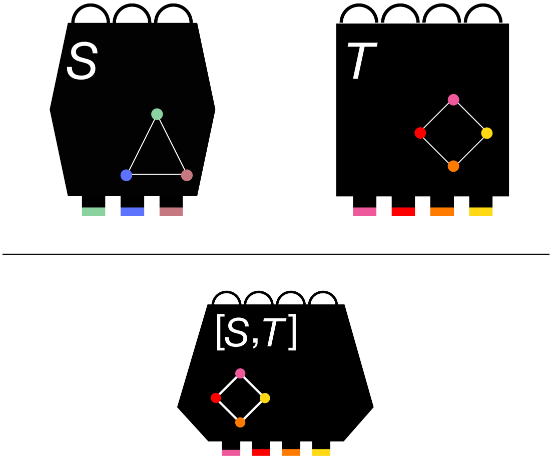
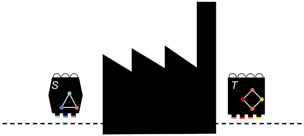
Answering the question by internalisation



From two scenarios S and T, we build a new scenario $[S, T]$.

Answering the question by internalisation



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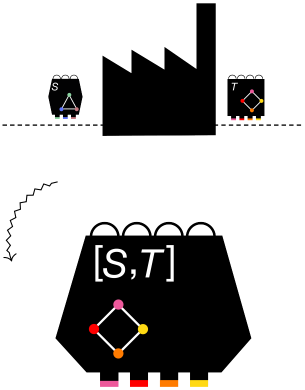

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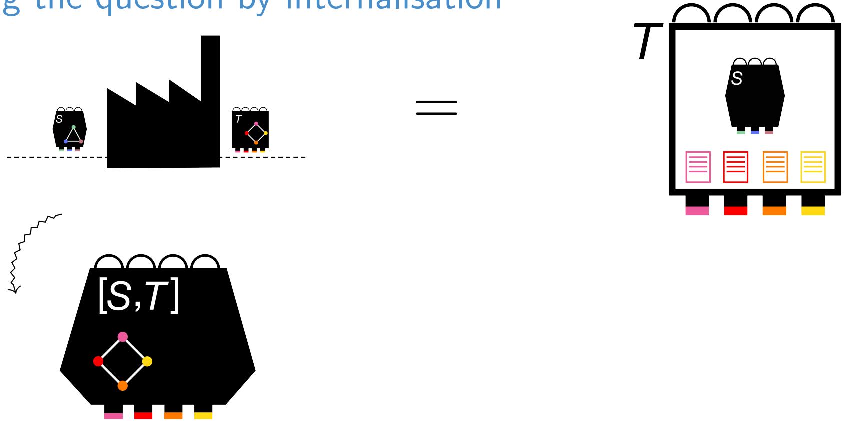

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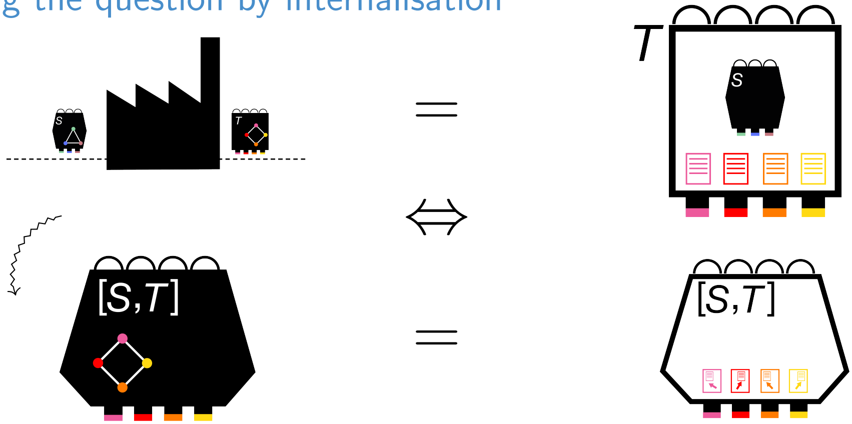

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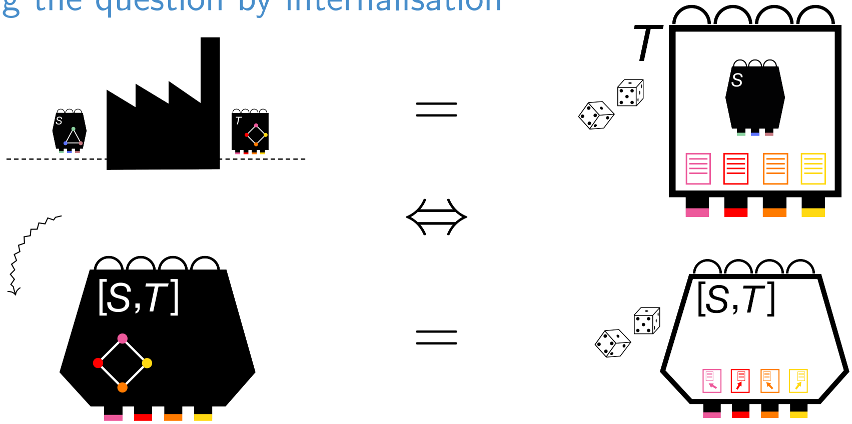

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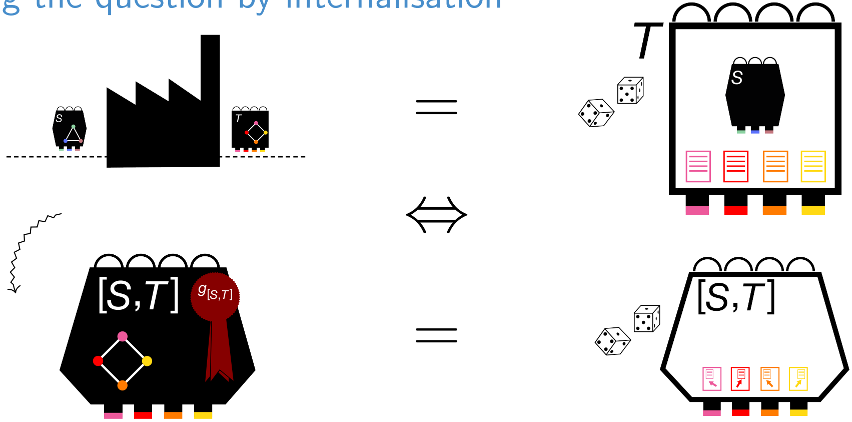

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F is realised by a deterministic procedure iff e_F is deterministic and satisfies $g_{[S,T]}$.

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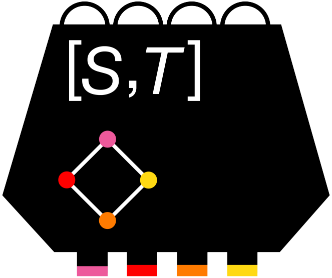
Further details

The hom scenario [S,T]



- **Measurements** are those of T.

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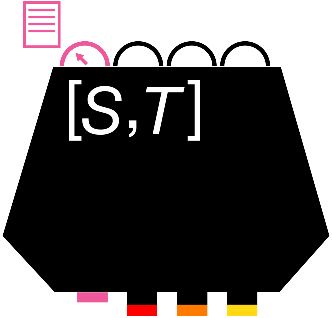

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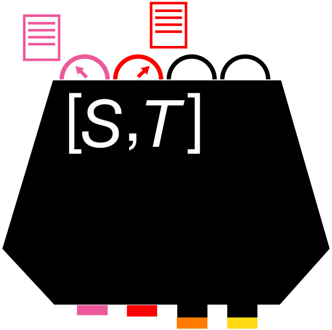

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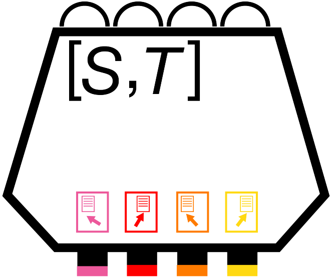

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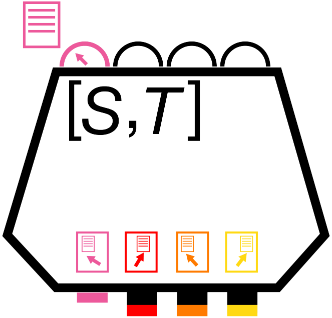

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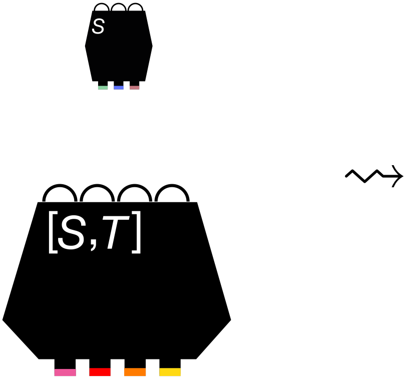

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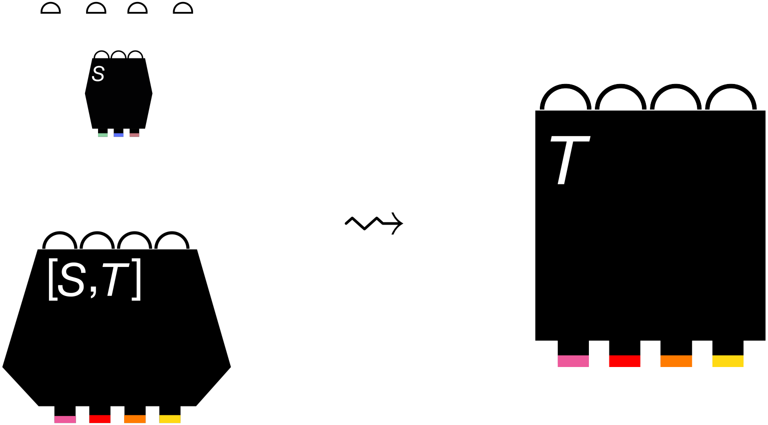

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Facts:

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Similarly, $\sum r_i f_i$ is induced by an experiment if each U_{f_i} is a compatible set of measurements.

Internalisation

As before, a convex-preserving map $F : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$ is determined by its action on $\mathbf{Det}(S)$.

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Lemma

A convex-preserving function $F : \mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$ *induces a canonical no-signalling empirical model* $e_F : [S, T]$.

Main results

Theorem

F is induced by a classical procedure iff e_F is non-contextual and satisfies $g_{[S,T]}$.

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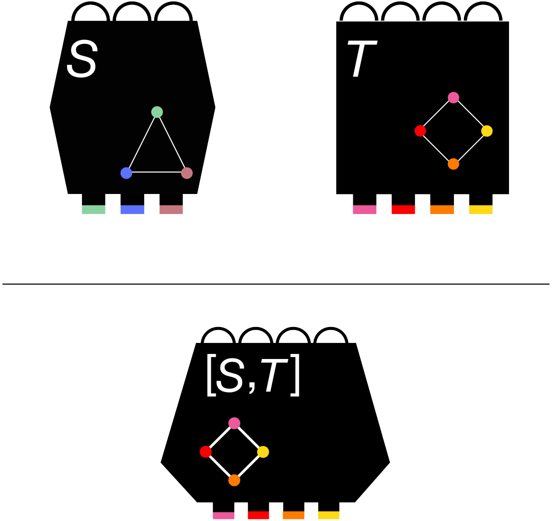
Caveat: adding predicates



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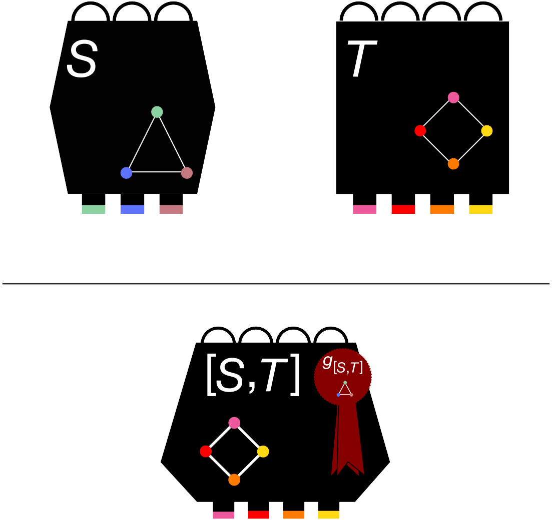

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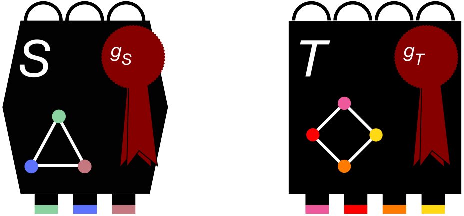

$g_S : S \longrightarrow \mathbf{2}$ $\quad$ $g_T : T \longrightarrow \mathbf{2}$

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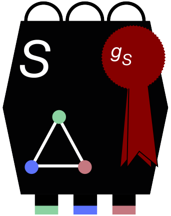

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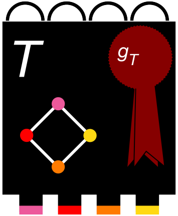
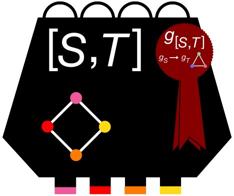


$g_T : T \longrightarrow \mathbf{2}$



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- A morphism $f : \langle S, g \rangle \longrightarrow \langle T, h \rangle$ is given by a procedure $f : S \longrightarrow T$ such that

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Theorem
$[-,-]$ *(appropriately modified) makes this category into a closed category.*

Closed structure

Getting closure

$$[S, T] \ \text{“}\otimes\text{”} \ S \longrightarrow T$$

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$$
\begin{array}{c}
[S, T] \ \text{“}\otimes\text{”} \ S \longrightarrow T \\
\Big\downarrow{\scriptstyle S \cong [I,S]} \\
[S, T] \ \text{“}\otimes\text{”} \ [I, S] \longrightarrow [I, T]
\end{array}
$$

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$$\Big\downarrow \scriptstyle{S \cong [I, S]}$$

$$[S, T] \ \text{“}\otimes\text{”} \ [{\color{red}I}, S] \longrightarrow [{\color{red}I}, T]$$

$$\Big\downarrow \scriptstyle{\text{generalise}}$$

$$[S, T] \ \text{“}\otimes\text{”} \ [{\color{red}R}, S] \longrightarrow [{\color{red}R}, T]$$

Getting closure

$$[S,T] \text{ ``}\otimes\text{'' } S \longrightarrow T$$

$\rightsquigarrow$ $S \cong [I,S]$

$$[S,T] \text{ ``}\otimes\text{'' } [I,S] \longrightarrow [I,T]$$

$\rightsquigarrow$ generalise

$$[S,T] \text{ ``}\otimes\text{'' } [R,S] \longrightarrow [R,T]$$

$\rightsquigarrow$ curry

$$\boxed{\mathsf{L}^R_{S,T} : [S,T] \longrightarrow [[R,S],[R,T]]}$$

Getting closure

Closed category

$$[-,-] : \mathsf{Scen}^{\mathsf{op}} \times \mathsf{Scen} \longrightarrow \mathsf{Scen}$$

- $i_S : S \xrightarrow{\cong} [I, S]$ natural in S
- $j_S : I \longrightarrow [S, S]$ extranatural in S (identity transformations)
- $\mathsf{L}^R_{S,T} : [S, T] \longrightarrow [[R, S], [R, T]]$ natural in S, T, extranatural in R (curried composition)
- $+$ reasonable coherence axioms

Outlook

Further questions

- External characterisation of adaptive procedures?
  Note that $[S, T]$ can be defined in the adaptive case, but there is no obvious way of building a canonical adaptive empirical model out of a convex-preserving function $\mathbf{Emp}(S) \longrightarrow \mathbf{Emp}(T)$.

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- Doing the same possibilistically?

- Does the set of all predicates on S generalise partial Boolean algebras to arbitrary measurement compatibility structures?

- Examining the closed structure?
  Note that it's not monoidal wrt. the usual monoidal structure, but seems closed wrt a 'directed' tensor product.

Questions...

?