Contextualities: from Foundations to Applications

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Contextuality in the early days
- From von Neumann to Kochen-Specker
- Problems in contextuality
- New noncontextuality proposals
- Lesson: contextuality is not a quantum property
- Lesson: “solution to some loopholes” but not to all

Contextuality-by-Default
- Careful with names, definitions, etc.

Generalized Contextuality
- Operational-Probabilistic Theory
- Ontological Models
- Ontological models vs hidden variables
- Noncontextuality hypothesis
- Negativity

Applying contextuality
- Oblivious communication
- Isomorphisms between scenarios: simplest and CHSH
- Isomorphisms between Bell scenarios and PM scenarios
- Quantum set
- Lesson about showing contextual advantages

Contextuality in the early days

As we could learn from Michael's presentation [9] [1]: the early days of quantum theory were confusing.

1. "Spooky action at a distance".
2. Two seemingly different formalisms for quantum theory: Heisenberg's and Schrödinger's.
3. No good ideas for what the quantum states represented.
4. What are the possible interpretations that can be draw from particle-wave duality.
5. Can hidden variables "complete" quantum theory?

[1] On EPR and Bell-local polytopes.

Let's attempt to complete it....

We suppose that we can complete the set of observables $\mathfrak{A}$ with a set of hidden-variables Λ.

$$\mathfrak{A} \to \mathfrak{A} \times \Lambda$$

The hidden-variables $\lambda \in \Lambda$ are such that they "present" the value of the observables. Calling the value of observable $a \in \mathfrak{A}$ by $\lambda(a)$ we have that we can define a valuation function by $v(a, \lambda) := \lambda(a)$.

Let $\mathfrak{A} := \mathrm{Mat}(\mathbb{C}, n)^{\mathbb{R}}$. A noncontextual hidden-variable is a map $v : \mathfrak{A} \times \Lambda \to \mathbb{R}$, such that $v(a, \lambda) \in \sigma(a)$,[2]

1. $v(a^2, \lambda) = v(a, \lambda)^2, \forall a \forall \lambda$.
2. $v(\mathbb{1}_n, \lambda) = 1$, $\forall \lambda$

Recall from [5, 6] that a quantum context is a set of compatible (comutative) quantum operators.

[2] Spectrum of the (self-adjoint) operator a.

Letting (Λ, Σ, μ) be a probability space, we recover the statistics of quantum theory with

$$\mathbb{E}[a] = \int_{\Lambda} v(a, \lambda) \mathrm{d}\mu(\lambda) \tag{1}$$

Theorem

For $n \geq 2$ there exists no linear noncontextual hidden variable model for quantum theory. [3]

[3] Originally in [44], great discussion in [15].

Linearity is too strong

$$a = \alpha a_1 + \beta a_2 \implies v(a, \lambda) = \alpha v(a_1, \lambda) + \beta v(a_2, \lambda)$$

$$\sigma_r := \frac{\sqrt{2}}{2}(\sigma_z + \sigma_y)$$

$$v(\sigma_z, \lambda), v(\sigma_y, \lambda), v(\sigma_r, \lambda) \in \{-1, 1\} \implies \pm 1 = \frac{\sqrt{2}}{2}(\pm 1 \pm 1)$$

Theorem

For $n = 2$ there exists a quasi-linear noncontextual hidden variable model for quantum theory. For $n \geq 3$ there exists no quasi-linear noncontextual hidden variable model for quantum theory.[4]

[4] See [18, 2].

Assumptions of the KS theorem

(a) If every observable $\mathscr{A}$ is associated with a self-adjoint operator a, then for every function f the observable $f(\mathscr{A})$ is associated with the operator $f(a)$.

(b) $v(f(a), \lambda) = f(v(a, \lambda))$.

(c) The conditions above imply quasi-linearity, i.e., linearity for compatible observables. If $[a_1, a_2] = 0$, $a = \alpha a_1 + \beta a_2$ imply $v(a, \lambda) = \alpha v(a_1, \lambda) + \beta v(a_2, \lambda)$. [5]

- Measurements review the properties of states λ.
- The valuations v do not depend in the contexts that they are being considered. In other words, this can be described as outcome noncontextual.

[5] $[a_1, a_2] = 0 \Rightarrow \exists h, g, c : a_1 = g(c), a_2 = h(c)$. [17]

Since we are assuming QT we can supose the existence of observables that are described by projective operators.
For any valuation v, or equivalently any hidden-variable λ we have that the value assignments onto projections must satisfy[6]

$$v(P_1 + \cdots + P_n, \lambda) = v(P_1, \lambda) + \cdots + v(P_n, \lambda) \tag{2}$$

[6] Proof in [15].

Since valuations return values in the spectrum, these can be understood as yes/no questions. Each valuation result can be described by two colors (we will later construct graphs with respect to compatible measurements)

Every columns is a basis for $\mathbb{C}^4$. Each of these vectors define a 1-dimensional projection, so that to each columns we can assign a value one for only one vector: $v(P_1+P_2+P_3+P_4,\lambda) = v(P_1,\lambda)+v(P_2,\lambda)+v(P_3,\lambda)+v(P_4,\lambda) = v(\mathbb{1},\lambda) = 1$.[7]

$(0,0,0,1)$	$(0,0,0,1)$	$(1,-1,1,-1)$	$(1,-1,1,-1)$	$(0,0,1,0)$	$(1,-1,-1,1)$	$(1,1,-1,1)$	$(1,1,-1,1)$	$(1,1,1,-1)$
$(0,0,1,0)$	$(0,1,0,0)$	$(1,-1,-1,1)$	$(1,1,1,1)$	$(0,1,0,0)$	$(1,1,1,1)$	$(1,1,1,-1)$	$(-1,1,1,1)$	$(-1,1,1,1)$
$(1,1,0,0)$	$(1,0,1,0)$	$(1,1,0,0)$	$(1,0,-1,0)$	$(1,0,0,1)$	$(1,0,0,-1)$	$(1,-1,0,0)$	$(1,0,1,0)$	$(1,0,0,1)$
$(1,-1,0,0)$	$(1,0,-1,0)$	$(0,0,1,1)$	$(0,1,0,-1)$	$(1,0,0,-1)$	$(0,1,-1,0)$	$(0,0,1,1)$	$(0,1,0,-1)$	$(0,1,-1,0)$

[7] Original article [8].

Red is one and blue is zero

$(0,0,0,1)$	$(0,0,0,1)$	$(1,-1,1,-1)$	$(1,-1,1,-1)$	$(0,0,1,0)$	$(1,-1,-1,1)$	$(1,1,-1,1)$	$(1,1,-1,1)$	$(1,1,1,-1)$
$(0,0,1,0)$	$(0,1,0,0)$	$(1,-1,-1,1)$	$(1,1,1,1)$	$(0,1,0,0)$	$(1,1,1,1)$	$(1,1,1,-1)$	$(-1,1,1,1)$	$(-1,1,1,1)$
$(1,1,0,0)$	$(1,0,1,0)$	$(1,1,0,0)$	$(1,0,-1,0)$	$(1,0,0,1)$	$(1,0,0,-1)$	$(1,-1,0,0)$	$(1,0,1,0)$	$(1,0,0,1)$
$(1,-1,0,0)$	$(1,0,-1,0)$	$(0,0,1,1)$	$(0,1,0,-1)$	$(1,0,0,-1)$	$(0,1,-1,0)$	$(0,0,1,1)$	$(0,1,0,-1)$	$(0,1,-1,0)$

Red is one and blue is zero

$(0,0,0,1)$	$(0,0,0,1)$	$(1,-1,1,-1)$	$(1,-1,1,-1)$	$(0,0,1,0)$	$(1,-1,-1,1)$	$(1,1,-1,1)$	$(1,1,-1,1)$	$(1,1,1,-1)$
$(0,0,1,0)$	$(0,1,0,0)$	$(1,-1,-1,1)$	$(1,1,1,1)$	$(0,1,0,0)$	$(1,1,1,1)$	$(1,1,1,-1)$	$(-1,1,1,1)$	$(-1,1,1,1)$
$(1,1,0,0)$	$(1,0,1,0)$	$(1,1,0,0)$	$(1,0,-1,0)$	$(1,0,0,1)$	$(1,0,0,-1)$	$(1,-1,0,0)$	$(1,0,1,0)$	$(1,0,0,1)$
$(1,-1,0,0)$	$(1,0,-1,0)$	$(0,0,1,1)$	$(0,1,0,-1)$	$(1,0,0,-1)$	$(0,1,-1,0)$	$(0,0,1,1)$	$(0,1,0,-1)$	$(0,1,-1,0)$

Red is one and blue is zero

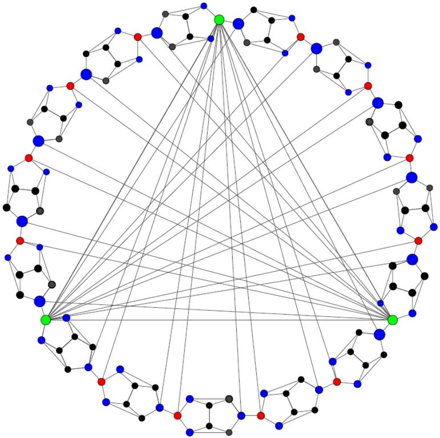
$(0,0,0,1)$	$(0,0,0,1)$	$(1,-1,1,-1)$	$(1,-1,1,-1)$	$(0,0,1,0)$	$(1,-1,-1,1)$	$(1,1,-1,1)$	$(1,1,-1,1)$	$(1,1,1,-1)$
$(0,0,1,0)$	$(0,1,0,0)$	$(1,-1,-1,1)$	$(1,1,1,1)$	$(0,1,0,0)$	$(1,1,1,1)$	$(1,1,1,-1)$	$(-1,1,1,1)$	$(-1,1,1,1)$
$(1,1,0,0)$	$(1,0,1,0)$	$(1,1,0,0)$	$(1,0,-1,0)$	$(1,0,0,1)$	$(1,0,0,-1)$	$(1,-1,0,0)$	$(1,0,1,0)$	$(1,0,0,1)$
$(1,-1,0,0)$	$(1,0,-1,0)$	$(0,0,1,1)$	$(0,1,0,-1)$	$(1,0,0,-1)$	$(0,1,-1,0)$	$(0,0,1,1)$	$(0,1,0,-1)$	$(0,1,-1,0)$

Since every vector appear in pair and we have 18 there must be an even number of 1 assignments, but since we have 9 columns this is a contradiction. [8]

$(0,0,0,1)$	$(0,0,0,1)$	$(1,-1,1,-1)$	$(1,-1,1,-1)$	$(0,0,1,0)$	$(1,-1,-1,1)$	$(1,1,-1,1)$	$(1,1,-1,1)$	$(1,1,1,-1)$
$(0,0,1,0)$	$(0,1,0,0)$	$(1,-1,-1,1)$	$(1,1,1,1)$	$(0,1,0,0)$	$(1,1,1,1)$	$(1,1,1,-1)$	$(-1,1,1,1)$	$(-1,1,1,1)$
$(1,1,0,0)$	$(1,0,1,0)$	$(1,1,0,0)$	$(1,0,-1,0)$	$(1,0,0,1)$	$(1,0,0,-1)$	$(1,-1,0,0)$	$(1,0,1,0)$	$(1,0,0,1)$
$(1,-1,0,0)$	$(1,0,-1,0)$	$(0,0,1,1)$	$(0,1,0,-1)$	$(1,0,0,-1)$	$(0,1,-1,0)$	$(0,0,1,1)$	$(0,1,0,-1)$	$(0,1,-1,0)$

[8] See for instance [2] for other proofs of KS-theorem.

KS proof with 117 vectors



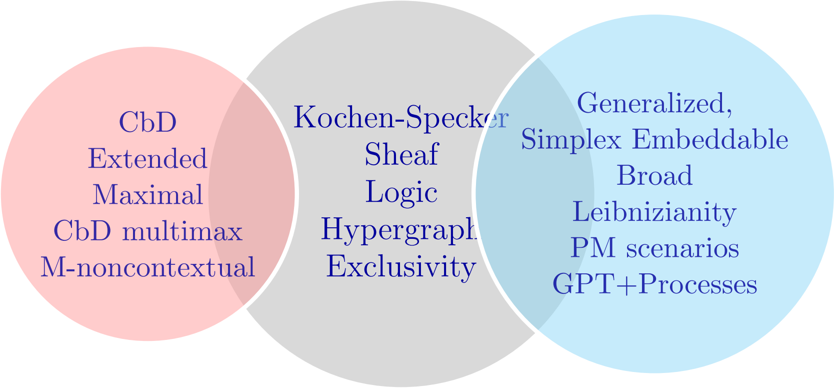
Limitations of the KS theorem

- It assumes Quantum Theory.
- It considers projective measurements and not generalized (noisy) ones.
- It assumes that we can describe and measure a set of perfectly compatible measurements.
- It does not treat different procedures of the quantum formalism (preparations and transformations).

Improving the formalism

- It assumes Quantum Theory. CA, Sheaf, GPT, CbD
- It considers projective measurements and not generalized (noisy) ones. Criticising MKC models, GC
- It assumes that we can describe and measure a set of perfectly compatible measurements. CbD
- It does not treat different procedures of the quantum formalism (preparations and transformations). GC

Noncontextualities:



Local consistency may imply global inconsistency

Important (new) lesson from contextuality

Contextuality is not a quantum property. [9]

[9] PR boxes.

Recall Rui's presentations [5, 6].

$$(X, \mathscr{C}, O) \tag{3}$$

The formalism of the compatibility hypergraph-approach and the sheaf theoretic approach[10] make no reference to quantum theory in their definitions.

[10] And all other recent notions.

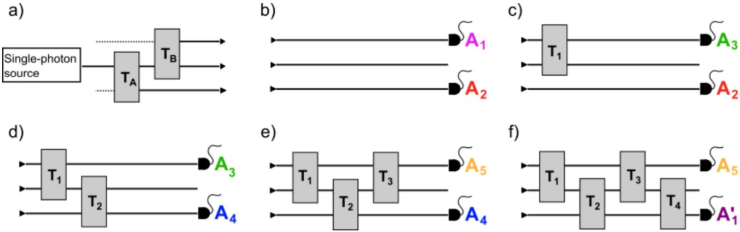
We can build robust noncontextuality inequalities that don't assume quantum theory.

Solving the problem of ruling out experimental loopholes is still an important problem [41]. I will discuss two attempts: Contextuality-by-Default and Generalized contextuality.

Idea: We keep using ideal measurement procedures but relax the use of perfect correlations.

KCBS:

$$\mathbb{E}(A_1A_2) + \mathbb{E}(A_2A_3) + \mathbb{E}(A_3A_4) + \mathbb{E}(A_4A_5) + \mathbb{E}(A_5A_1) \geq -3$$ [11]



[11] Taken from [25].

Motivation for CbD: $A_1 \neq A_1'$

Note that since there is the need for a large statistical analysis to show that $A_1 = A_1'$, a possibility is to consider these errors non-essential. Using CbD it is possible to relax these bounds.

Every measurement in a different context corresponds essentially to a different property: random variables M_q^c.

We then impose noncontextuality by existing a global distribution that maximize the probability that random variables in the same connection are equal. [12]

[12] See [42, 3, 19].

Important lesson: careful with names.

Many approaches differ only by name and objects of study, and do not change essentially. (e.g., CbD and extended noncontextuality). One focus in random variables (CbD) and the other on behaviors (Extended).

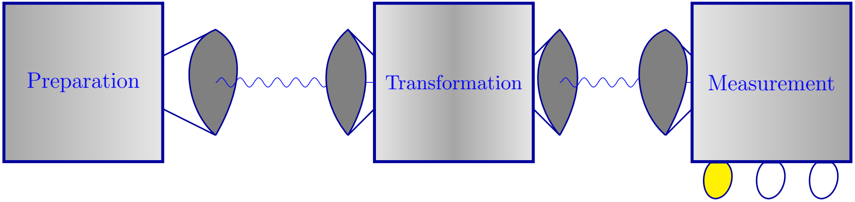
Sometimes different descriptions go by the same name (e.g., transformation contextuality). These 3 papers [30, 27, 38] have different descriptions of the what should be transformation noncontextuality.

Let's now maintain perfect correlations but allow for imperfect procedures.

Generalized Contextuality

An operational theory is described by set of possible laboratory instructions (processes/primitives), and a rule for assigning probabilities. [13]

[13] See [45] and references therein.


Preparation
Transformation
Measurement

$$B := \{p(k|M,T,P)\}$$

$$P \to \rho$$

$$M \to \text{POVM's}$$

$$T \to \text{CPTP}$$

Given $\mathbb{P} := \{P_1, P_2, P_3, P_4\}$, for any $[k|M]$ possible, we have,

$$p(k|M, P_1) + p(k|M, P_2) = p(k|M, P_3) + p(k|M, P_4)$$

$$P \simeq P' \tag{4}$$

$$\forall [k|M], \forall T, p(k|M,T,P) = p(k|M,T,P') \tag{5}$$

$$\mathbb{B} = (\mathbb{P}, \mathbb{M}, \mathbb{O}_M, \mathbb{E}_P, \mathbb{E}_M) \tag{6}$$

$\mathbb{B}_{si}$

$$\mathbb{P} = \{P_1, P_2, P_3, P_4\} \tag{7}$$

$$\mathbb{M} = \{M_1, M_2\} \tag{8}$$

$$\mathbb{O}_M = \{0, 1\} \tag{9}$$

$$\mathbb{E}_M = \emptyset \tag{10}$$

$$\mathbb{E}_P :\Longleftrightarrow \ \frac{1}{2}P_1 + \frac{1}{2}P_2 \simeq \frac{1}{2}P_3 + \frac{1}{2}P_4 \tag{11}$$

$$B \to \begin{pmatrix} p(1|M_1, P_1) \\ \vdots \\ p(k|M_I, P_J) \end{pmatrix} \in \mathbb{R}^{|K|\times|I|\times|J|}$$

Prepare-and-measure polytope

$$p(k|M_i, P_j) \geq 0$$

$$\sum_k p(k|M_i, P_j) = 1$$

$$p(k|M_i, P_1) + p(k|M_i, P_2) = p(k|M_i, P_3) + p(k|M_I, P_4)$$

The operational theory does not explain "why" the probabilities arising are the way they are.

$$\lambda \in \Lambda$$

Each preparation P_j is actually preparing an ontic state λ.
For each preparation P_j we associate a probability distribution μ_{P_j} over Λ. [14]

[14] We can generalize this to measures.

Each measurement M is associated to a probability distribution $\xi_{[\cdot|M]}(\lambda)$[15] over K, the set of outcomes, given a prepared $\lambda \in \Lambda$.

[15] We can generalize this to Markov kernels.

Any behavior $B := \{p(k|M_i, P_j)\}$ is explained at the ontological model as,

$$p(k|M_i, P_j) = \sum_{\lambda \in \Lambda} \xi_{[k|M_i]}(\lambda)\mu_{P_j}(\lambda)$$

The name is different because of the philosophical purposes. But many times, they refer to the same thing (and other times they don't!).

Noncontextuality hypothesis

$$\sum_j \alpha_j P_j \simeq \sum_j \beta_j P_j \implies \sum_j \alpha_j \mu_{P_j} = \sum_j \beta_j \mu_{P_j}$$

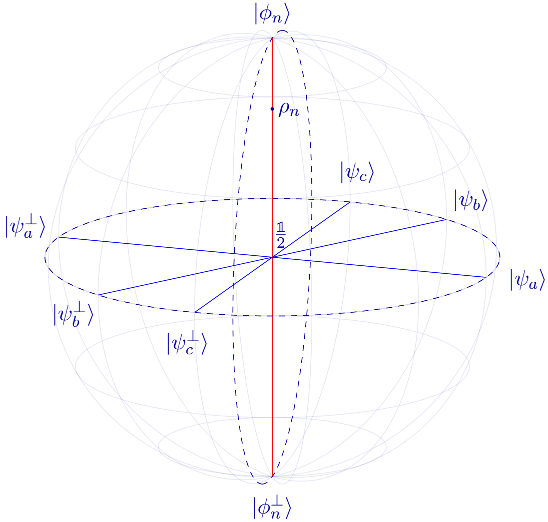
$$\sum_{k,i} \alpha_{[k|M_i]}[k|M_i] \simeq \sum_{k,i} \beta_{[k|M_i]}[k|M_i] \implies$$
$$\sum_{k,i} \alpha_{[k|M_i]}\xi_{[k|M_i]} = \sum_{k,i} \beta_{[k|M_i]}\xi_{[k|M_i]}$$

Theorem

Quantum theory is contextual for preparations, transformations and unsharp measurements, for any dimension of Hilbert space larger then, or equal to 2.

1. Outcome determinism for sharp measurements. This means that any time we have a measurement procedure that is sharp the functions $\xi_{[\cdot|M]}(\lambda)$ answer yes/no questions.
2. Two preparation procedures that are distinguishable with a one-shot measurement must have a non-overlapping epistemic description in the ontological model.
3. Convex combinations of procedures are again valid procedures in the operational theory and they are mapped to the same ontological quantities.

Proof


$|\phi_n\rangle$
ρ_n
$|\psi_c\rangle$
$|\psi_b\rangle$
$|\psi_a^\perp\rangle$
$\frac{\mathbb{1}}{2}$
$|\psi_a\rangle$
$|\psi_b^\perp\rangle$
$|\psi_c^\perp\rangle$
$|\phi_n^\perp\rangle$

$$W(p,q)$$

Recall that our descriptions captures phase space representations in terms of the space Λ. In quantum terms, [16]

$$\mathrm{Tr}(\rho E_k) = p(k|M,P) = \int_\Lambda \xi_{[k|M]}(\lambda)\mu_P(\lambda)\mathrm{d}\lambda \tag{12}$$

A generalized noncontextual model exists iff any attempt of describing quantum physics in phase space imply in a quasi-probabilistic necessity.

[16] See [39]

Theorem

A quasi-probability representation of quantum theory must have negativity in either its representation of states or measurements (or both). [17]

[17] See [10].

Applying contextuality

Vast number of advantages/literature.

(a) Weak values. (Introduced in [1], Critics [11], Noncontextuality [32, 22])

(b) Quantum Thermodynamic Machines. [28]

(c) Two state discrimination. [37]

(d) Quantum cloning. [29]

(e) MBQC. [12, 34, 7]

(f) Quantum Metrology. [28, 4]

(g) Magic States[18]

Theorem

Quantum computation with magic states can have a quantum speedup only if the Wigner function of the initial magic state is negative. [43].

[18] The important Nature [16].

(h) Communication protocols [31, 35, 36, 48, 14]...
(i) Quantum thermodynamic measurements of Heat/Work substituting TPM (negativity again). [26]
(j) HVM for computation???, [49][19]...

[19] This is a very recent work, interesting!

For reasons of being in the group of Galvão, who first proposed the use of quantum contextuality for communication advantages [13, 40] I will discuss the Parity-Oblivious Multiplexing (POM) game:

Alice generates a random string of bits,

$$x = (1, 0, 0, 1, 1, 1, 1, 0)$$

POM task: Bob

Bob generates a random number associated with the bit position of Alice's string.

$$y \in \{0, 1, 2, \ldots, n-1\}$$

The goal of Bob is to guess correctly the y-th bit in the string x generated by Alice.

$$x_y$$

Alice does not send any information about *any* parity of the string x.

$$\mathrm{Par} := \left\{ r \middle| r \in \{0,1\}^n, \sum_i r_i \geq 2 \right\} \tag{13}$$

We call the procedures that Alice sends to Bob as,

$$P_x$$

The generalized restriction of the POM game is,

$$\forall r \in \mathrm{Par}, \forall [k|M] : \sum_{x|x\cdot r=0} p(k|M, P_x) = \sum_{x|x\cdot r=1} p(k|M, P_x)$$

where $x \cdot r$ means sum module 2 of the bit scalar products. $x \cdot r = \bigoplus_i x_i r_i$.

$$x = (x_0, x_1) \in \{0, 1\}^2$$

$$\mathrm{Par} = \{(1,1)\}$$

$$
\begin{aligned}
(1,1)\cdot(0,0) &= 0\oplus 0 = 0\\
(1,1)\cdot(1,0) &= 1\oplus 0 = 1\\
(1,1)\cdot(0,1) &= 0\oplus 1 = 1\\
(1,1)\cdot(1,1) &= 1\oplus 1 = 0
\end{aligned}
$$

$$p(k|M, P_{(0,0)}) + p(k|M, P_{(1,1)}) = p(k|M, P_{(0,1)}) + p(k|M, P_{(1,0)}).$$

$$p(g = x_y) = \frac{1}{2 \cdot 2^2} \sum_{y \in \{0,1\}} \sum_{x \in \{0,1\}^2} p(g = x_y | M_y, P_x) \tag{14}$$

$1/2 = p(y), 1/2^2 = 1/4 = p(x)$

$$\text{2-bit POM} \equiv \mathbb{B}_{si}$$

$$n\text{-bit POM} \equiv \mathbb{B}$$

Theorem

The optimal bound for the probability of Bob finding the correct value of x_y, in any noncontextual ontological model, is

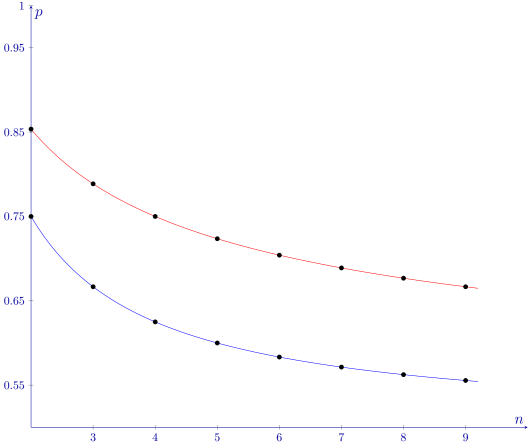
$$p(g = x_y) \leq \frac{1}{2}\left(1 + \frac{1}{n}\right) \tag{15}$$

Probability of winning: classical vs quantum theory[20]

$$p^{Q}(g = x_y) = 0.85$$
$$p^{NC}(g = x_y) = 0.75$$

[20] See [46] and references for simplest scenario.

n-POM: classical vs quantum theory



The simplest scenario we have presented is isomorphic to a Bell scenario with two parties, each with two binary outcome measurements. [37]

The simplest scenario again [33]

$$\mathbb{P} = \{P_1, P_2, P_3, P_4\} \tag{16}$$

$$\mathbb{M} = \{M_1, M_2\} \tag{17}$$

$$\mathbb{O}_M = \{0, 1\} \tag{18}$$

$$\mathbb{E}_M = \emptyset \tag{19}$$

$$\mathbb{E}_P :\Longleftrightarrow \ \frac{1}{2}P_1 + \frac{1}{2}P_2 \simeq \frac{1}{2}P_3 + \frac{1}{2}P_4 \tag{20}$$

Instead of preparations P_1, P_2, P_3, P_4 we let two sources S_0, S_1 be such that, each source S_i chooses between two settings $s_i \in \{0, 1\}$ with equal probability. [21]

We then associate these events of a source choosing a setting $[0|S_0]$ as a preparation procedure $P_1 \equiv P_{[0|S_0]}$ etc, $\forall s : p(s|S) = 1/2$.

$$\frac{1}{2}[0|S_0] + \frac{1}{2}[1|S_0] \simeq \frac{1}{2}[0|S_1] + \frac{1}{2}[1|S_1]$$

[21] For sources framework see [23, 24, 20, 21].

Joint probability of events

$$p(k, s|M, S) = p(k|M, P_{[s|S]})p(s|S)$$

Normalized

$$\sum_{k,s} p(k,s|M,S) = \sum_{k,s} p(k|M,P_{[s|S]})p(s|S) = \sum_{s} p(s|S) = 1$$

$$\mathscr{C} := \{\{M_0, S_0\}, \{M_0, S_1\}, \{M_1, S_0\}, \{M_1, S_1\}\}$$

$$\begin{aligned}
\sum_s p(k, s|M_0, S_0) &= \frac{1}{2}\sum_s p(k|M_0, P_{[s|S_0]}) \\
&= \frac{1}{2}\left(p(k|M_0, P_{[0|S_0]}) + p(k|M_0, P_{[1|S_0]})\right) \\
&= \frac{1}{2}\left(p(k|M_0, P_{[0|S_1]}) + p(k|M_0, P_{[1|S_1]})\right) \\
&= \frac{1}{2}\sum_s p(k|M_0, P_{[s|S_1]}) \\
&= \sum_s p(k, s|M_0, S_1)
\end{aligned}$$

Preparation noncontextuality will imply the existence of an ontological model such that

$$p(k, s|M, S) = \int_\lambda \xi(k|\lambda, M) p(s|S) \mu(\lambda|s, S) \mathrm{d}\lambda \tag{21}$$

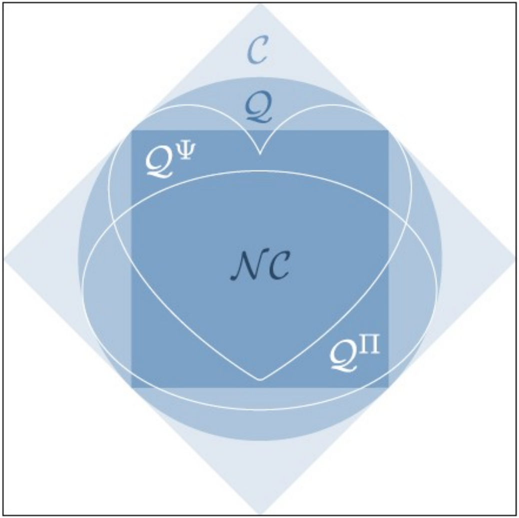
But noticing that $\mu(\lambda|s, S)p(s|S) = \mu(\lambda, s|S) = \mu(s|\lambda, S)\mu(\lambda)$ we get a locally causal model for the behaviour, by letting "Alice" and "Bob" correspond to the devices M and S. [22]

$$p(k, s|M, S) = \int_\lambda \xi_A(k|\lambda, M) p(s|S) \mu_B(s|\lambda, S) \mu(\lambda) \mathrm{d}\lambda \tag{22}$$

[22] See [33] for the other direction of the proof.

Every scenario that has a no measurement equivalences, and that has preparation equivalences that generate one procedure is isomorphic to a Bell scenario.

There are behaviours that are maximally contextual without being described by pure states and sharp measurements.


$\mathcal{C}$
$\mathcal{Q}$
$\mathcal{Q}^{\Psi}$
$\mathcal{NC}$
$\mathcal{Q}^{\Pi}$

- Describe you experimental scenario, that represents an important problem.
- Try to obtain a bound, or a no-go result that represents a constraint into noncontextual models.
- Show that quantum theory overcomes these constraints.

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