Quantum physics needs complex numbers

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Presentation outline

- Complex numbers
- Real versus complex numbers
- Quantum experiments using real numbers
- SWAP scenario
- Outlook

Complex numbers I

Complex numbers were invented to solve equations like $x^2 = -1$

Early days

- Greek engineer Heron of Alexandria
- Arab polymath Al-Khwarizmi → solutions to quadratic equations
- Many Italians mathematicians of Renaissance → solutions to cubic equations

Complex numbers II

Leonhard Euler

Introduced i with the rule $i^2 = -1$ ➡ $x^2 = -1$ → $x_1 = +i$, $x_2 = -i$

I will call it imaginary!

What a ridiculous name!
It should be called lateral.

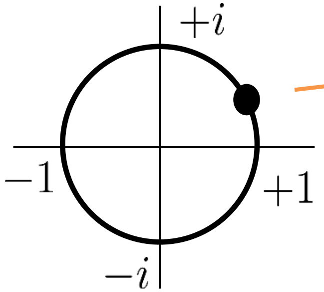
Real part

Complex numbers $z = a + i \cdot b$

Imaginary part

Complex numbers and Physics

Complex numbers are very useful in physics



$$e^{i\cdot\theta}$$

→ Rotations in the unit circle

→ Euler's formumla → $e^{i\cdot\theta} = \cos\theta + i\sin\theta$

Electromagnetism → Electromagnetic waves $\mathbf{E} = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

Special relativity → Lorentz transformations → generalized rotations in 4D space-time

Convenient mathematical tool but not necessary

Complex numbers and Quantum mechanics

Complex numbers connected to the heart of quantum mechanics

1^{st} postulate $\begin{cases} \text{isolated physical system associated with complex Hilbert space } \mathcal{H} \\ \text{state represented by vector } |\psi\rangle \in \mathcal{H} \end{cases}$

Schrödinger equation ⟶ $i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$

imaginary unit

split into real part + imaginary part

Measurement outcome ⟶ real number

Complex numbers and Quantum mechanics

Complex numbers connected to the heart of quantum mechanics

1^{st} postulate { isolated physical system associated with complex Hilbert space $\mathcal{H}$; state represented by vector $|\psi\rangle \in \mathcal{H}$ }

Do we actually need complex numbers?

Schrödinger equation ⟶ $i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$

imaginary unit

split into real part + imaginary part

Measurement outcome ⟶ real number

Real versus complex numbers

Amplitudes in QM are complex numbers → Why should it be so?

QM over real numbers ⟹
- constructive and destructive interference
- unitary tranformations
- absolute squares of amplitudes that are probabilites

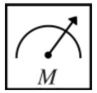
Quantum Information

Teleportation	Grover's algorithm
QKD	
Superdense coding	Bell inequality

← Real numbers

Example I

Phenomenon observed by Bill Wooters

Uniformly random pure state $\sum_{i=1}^{d} a_i |i\rangle$ → 

→ $(p_1, p_2, \ldots, p_d)$ probability distribution

$(p_1, p_2, \ldots, p_d)$ → Uniformly random probability distribution

↘ uniformly random point on the simplex $p_1 + p_2 + \cdots + p_d = 1$

Fineprint: Only if the amplitudes are complex numbers

Example II

A ⟵ unknown mixed state ρ ⟶ B

(many identical copies of ρ)

Can ρ be fully determined from joint statistics of product measurements?

Example: Bell experiment ⟶ certify that A and B share an entangled state

Classical probability distributions ⟶ YES!

Quantum mixed states ⟶ YES but only with complex amplitudes

QM over reals $\quad \rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

→ NOT maximally mixed state

→ indistiguinshable from the maximally mixed state by any local measurement

Why?

of independent real parameters to specify a n-dim mixed state

n-dim mixed state $\longrightarrow$ $n \times n$ Hermitian matrix

Complex $\longrightarrow$ $n + 2\frac{n(n-1)}{2} = n^2$ $\left\{ \begin{array}{l} n \text{ real parameters from diagonal} \\ n(n-1)/2 \text{ complex parameters below the diagonal} \end{array} \right.$

Real $\longrightarrow$ $\frac{n(n+1)}{2}$ real parameters ("Hermitian matrix" = real symmetric matrix)

composite system AB $\xrightarrow{\otimes}$ $d_{AB} = d_A d_B$

- Complex $(n_A n_B)^2 = n_A^2 n_B^2$
- Real $\frac{n_A n_B (n_A n_B + 1)}{2} > \frac{n_A(n_A+1)}{2} \cdot \frac{n_B(n_B+1)}{2}$

Main paper

Quantum physics needs complex numbers

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Complex numbers, i.e., numbers with a real and an imaginary part, are essential for mathematical analysis, while their role in other subjects, such as electromagnetism or special relativity, is far less fundamental. Quantum physics is the only physical theory where these numbers seem to play an indispensible role, as the theory is explicitly formulated in terms of operators acting on complex Hilbert spaces. The occurrence of complex numbers within the quantum formalism has nonetheless puzzled countless physicists, including the fathers of the theory, for whom a real version of quantum physics, where states and observables are represented by real operators, seemed much more natural. In fact, previous works showed that such "real quantum physics" can reproduce the outcomes of any multipartite experiment, as long as the parts share arbitrary real quantum states. Thus, are complex numbers really needed for a quantum description of nature? Here, we show this to be case by proving that real and complex quantum physics make different predictions in network scenarios comprising independent quantum state sources. This allows us to devise a Bell-type quantum experiment whose input-output correlations cannot be approximated by any real quantum model. The successful realization of such an experiment would disprove real quantum physics, in the same way as standard Bell experiments disproved local physics.

Main idea

Question: If we use the standard quantum formalism and restrict the Hilbert spaces to be real, possibly of *larger* dimensions, could we still explain the same phenomena?

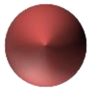
GAME

"Complex" quantum physicist	"Real" quantum physicist
↓	↓
propose experiments using standard QM based on complex Hilbert spaces	simulate the *same* experiment using QM but restricted to real measurerent operators and density matrices

Main goal: Rule out the possibility that the universe is secretely based on real version of QM in order to *simulate* complex QM

Simulation using real Hilbert spaces

ρ



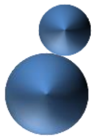
→ "Real" quantum physics can simulate any experiment

Observation

$$P(r) = \mathrm{tr}(\rho\Pi_r) \xrightarrow{P(r)\ \text{real}} P(r) = P(r)^* = \mathrm{tr}(\rho^*\Pi_r^*)$$

→ the *same* experiment explained by complex conjugates as well

enlarge Hilbert space by adding an extra qubit (doubling the dimension) →



$$|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$$

separable mixed state

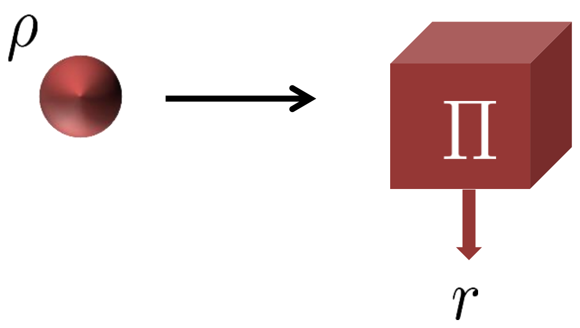
simulation density matrix → $\tilde{\rho} = \frac{1}{2}(\rho\otimes|+i\rangle\langle+i|+\rho^*\otimes|-i\rangle\langle-i|)$

real meaurement operator → $\tilde{\Pi}_r = (\Pi_r\otimes|+i\rangle\langle+i|+\Pi_r^*\otimes|-i\rangle\langle-i|)$

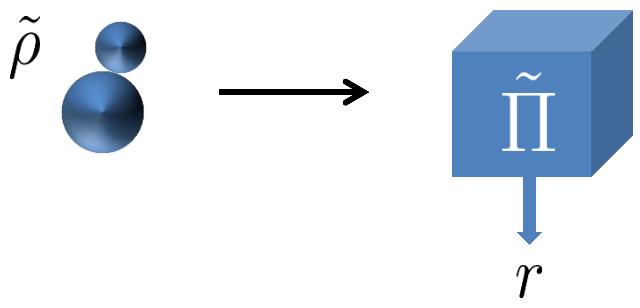
Simulation using real Hilbert spaces

Single quantum system

"Complex" quantum physicist



"Real" quantum physicist

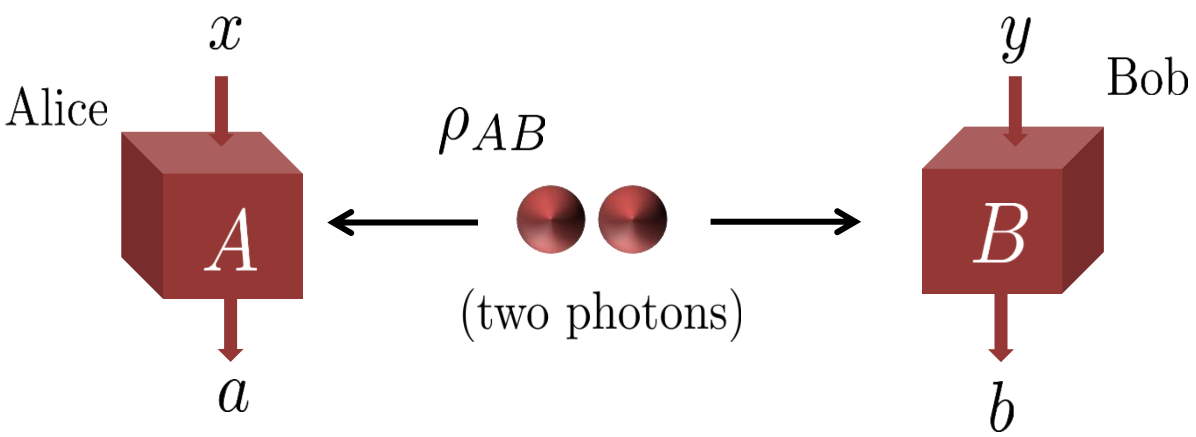


In single lab we can describe quantum experiments without complex numbers

Simulation involving several parties

Experiments involving several distant labs → quantum entanglement, Bell non-locality

→ Bell experiment



- state ρ_{AB} acting on joint space
- $A_{a|x}, B_{b|y}$ are local measurements
- correlations given by $P(a,b|x,y) = \mathrm{tr}(\rho_{AB} A_{a|x} \otimes B_{b|y})$

violation the Bell inequality ⟶ disproves local classical physics

Question: Can "real" quantum physics be falsified by (complex) quantum Bell experiment?

Some important remarks

When there is NO violation of any Bell inequality

↘ measured correlations *can* be reproduced by local deterministic models ➡ real numbers suffice

Bell violation ⟶ *necessary* condition for complex-real gap

↙ NOT *sufficient* ➡

CHSH inequality: $\text{CHSH}(1,2;1,2) = \langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle \leq 2$

$\beta_{\text{CHSH}} = 2\sqrt{2}$
maximal violation ➡ real measurements + real two-qubit state

Looking for complex maximal violation

Next step $\longrightarrow$ Bell inequalities whose maximum quantum violation requires complex numbers

Combination of 3 CHSH inequalities: $\text{CHSH}_3 = \text{CHSH}(1,2;1,2) + \text{CHSH}(1,3;3,4) + \text{CHSH}(2,3;5,6) \leq 6$

$3\beta_{\text{CHSH}} = 6\sqrt{2}$

maximal violation

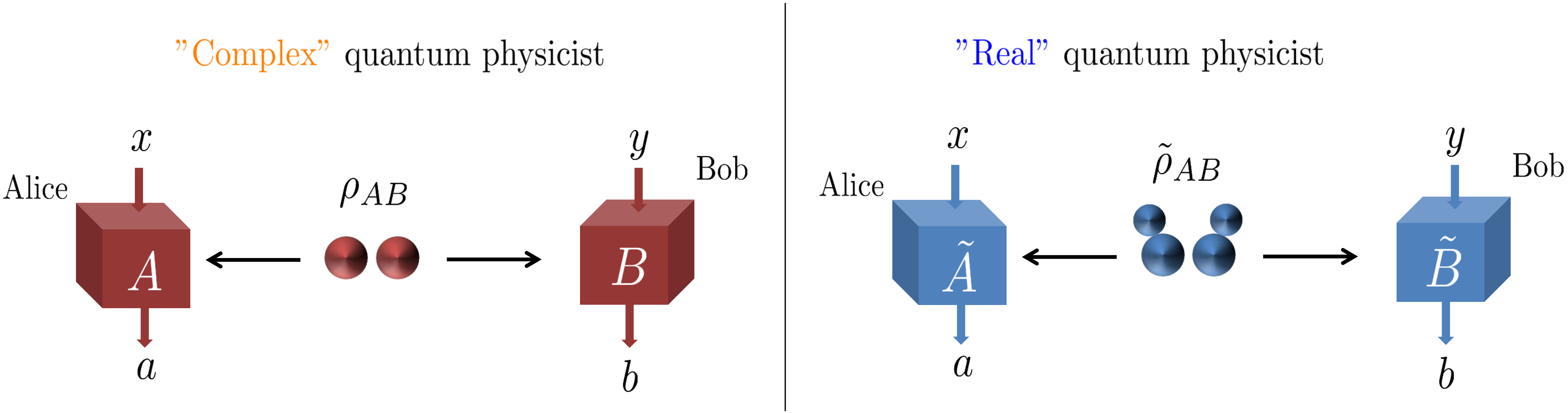
Alice: 3 measurements
Bob: 6 measurements

NOPE! $\longrightarrow$ real quantum Bell experiments reproduce the statistics of *any* Bell experiment[2]

[2] M. McKague, M. Mosca, and N. Gisin, Simulating quantum systems using real hilbert spaces, Phys. Rev. Lett. 102, 020505(2009).

Extra qubit again

Adapt the construction of extra qubit in the multipartite case

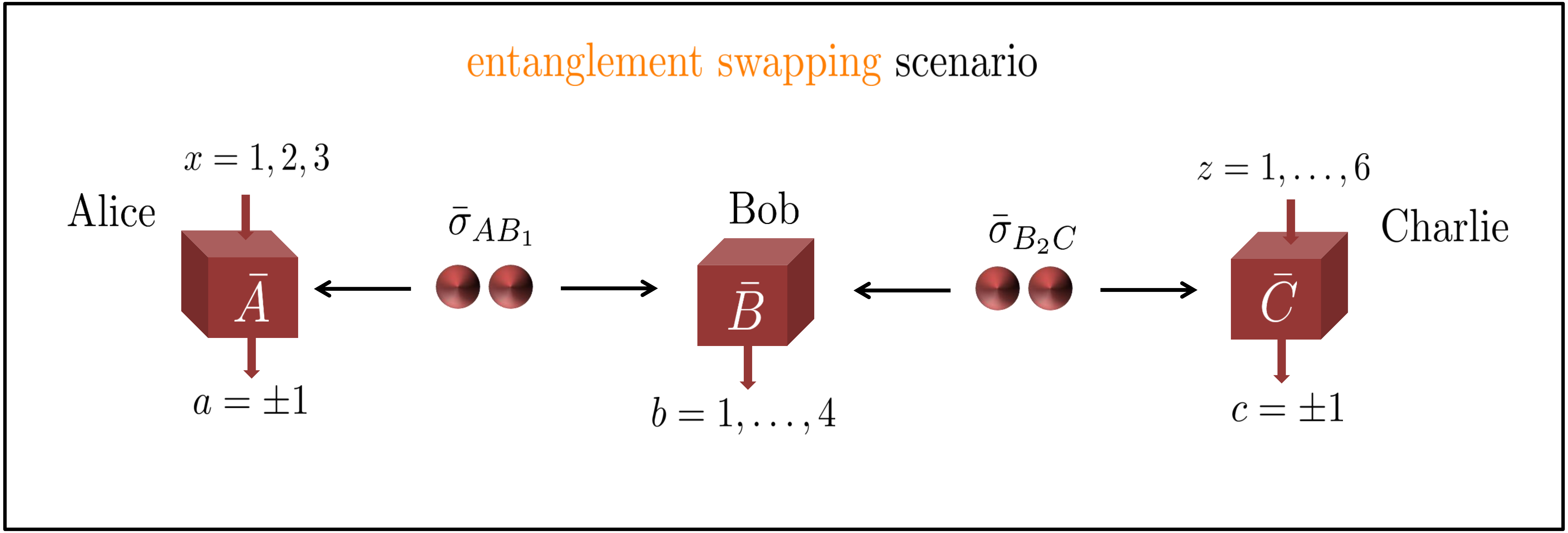


$$\tilde{\rho}_{AA'BB'} = \frac{1}{2}(\rho_{AB} \otimes |+i,+i\rangle\langle +i,+i|_{A'B'} + \rho^*_{AB} \otimes |-i,-i\rangle\langle -i,-i|_{A'B'})$$

$$P(ab|xy) = \mathrm{tr}(\tilde{\rho}_{AB} \tilde{A}_{a|x} \otimes \tilde{B}_{b|y})$$

More than one source

Scenarios with more than one source of entangled states ⟶ Quantum internet



Entanglement swapping scenario

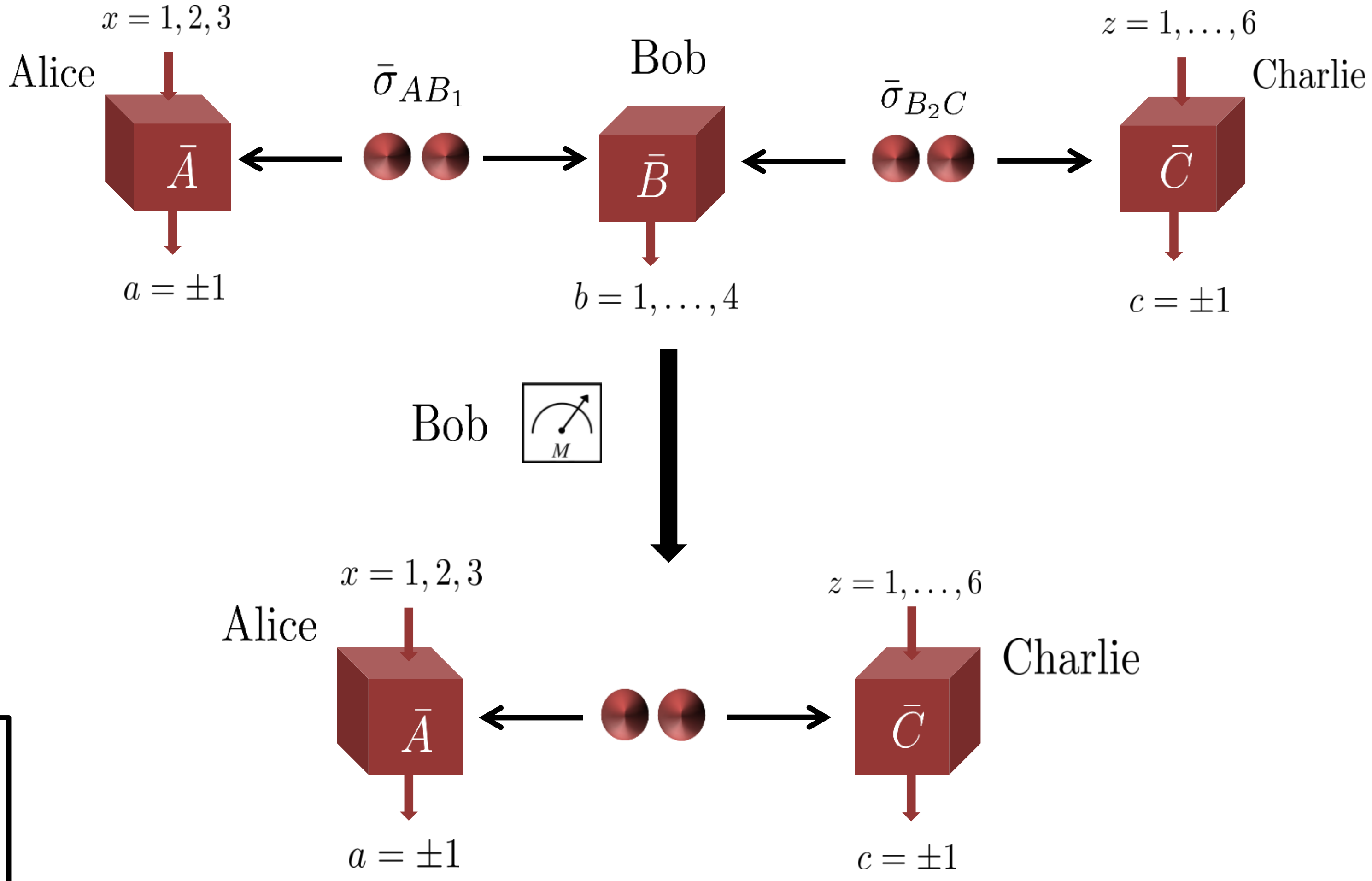
$b = 1, \ldots, 4$

$\rho_{AC}(b)$ (2-qubit entangled state)

Alice + Charlie

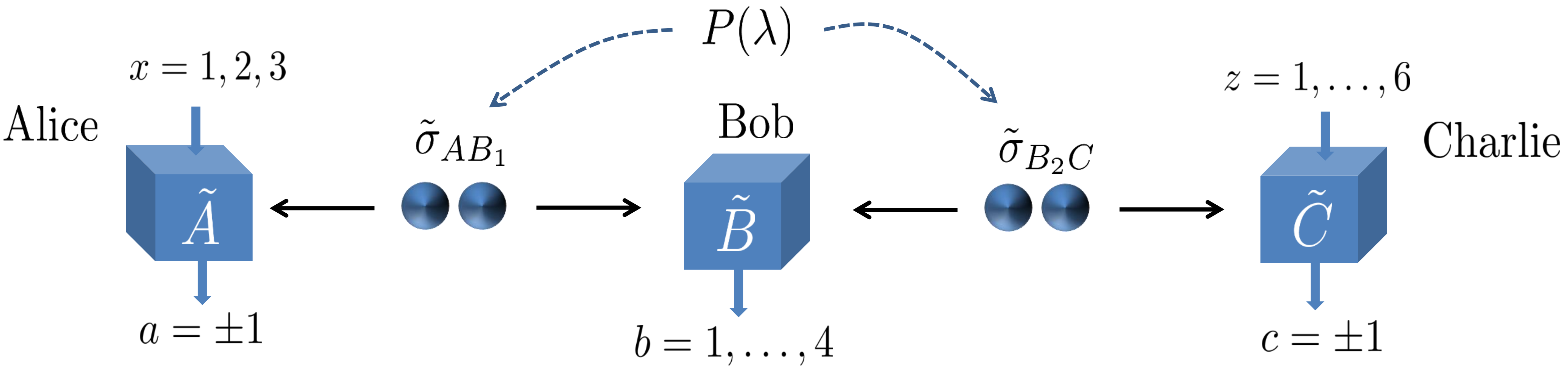
maximal violation of CHSH_3 inequality

NO real simulation is possible!



SWAP scenario

"Real" quantum physicist



Observed statistics must admit: $P(a, b, c|x, z) = \sum_{\lambda} P(\lambda)\, \mathrm{tr}\,\{(\tilde{\sigma}_{AB_1} \otimes \tilde{\sigma}_{B_2C})\underbrace{(\tilde{A}_{a|x} \otimes \tilde{B}_b \otimes \tilde{C}_{c|z})}\}$

projective measurements

Particular SWAP scenario

"Complex" quantum physicist

$$\bar{\sigma}_{AB_1} = \bar{\sigma}_{B_2C} = \Phi^+ = |\phi^+\rangle\langle\phi^+| \longrightarrow |\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Alice's observables: $(\bar{A}_{1|x} - \bar{A}_{-1|x} : x = 1,2,3) \longrightarrow \sigma_Z, \sigma_X, \sigma_Y$

Charlie's observables: $(\bar{C}_{1|z} - \bar{C}_{-1|z} : x = 1,\ldots,6) \begin{cases} D_{ij} = \frac{\sigma_i + \sigma_j}{\sqrt{2}} \\ E_{ij} = \frac{\sigma_i - \sigma_j}{\sqrt{2}} \end{cases} \quad (ij = zx, zy, xy)$

Bob's measurements: $\begin{cases} |\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \\ |\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \end{cases}$

Observed statistics are $\bar{P}(a,b,c|x,z) = \mathrm{tr}\left\{(\bar{\sigma}_{AB_1} \otimes \bar{\sigma}_{B_2C})(\bar{A}_{a|x} \otimes \bar{B}_b \otimes \bar{C}_{c|z})\right\}$

Particular SWAP scenario

three CHSH inequality $\longrightarrow$ Bell-type parameter

$$E_{zx}^{b} = \sum_{a,c=\pm 1} P(a,b,c|x,z)ac$$ $\Rightarrow$ linear functional $\mathcal{T}_b[(E_{zx}^{b}(P)]$

it can be verified that $\boxed{\mathcal{T}_b(\bar{P}) = 6\sqrt{2}\bar{P}(b)}$ for all b with $\bar{P}(b) = \frac{1}{4}$

Alice and Charlie's state, conditioned on Bob's outcome b, *maximally* violate the three CHSH inequality

$$\mathrm{CHSH}_3 = \mathrm{CHSH}(1,2;1,2) + \mathrm{CHSH}(1,3;3,4) + \mathrm{CHSH}(2,3;5,6) \leq 6$$

Main results

Proposition 1. $\bar{P}$ does not admit the following decomposition if we demand the states and the measurements to be real.

$$P(a,b,c|x,z) = \sum_{\lambda} P(\lambda)\,\mathrm{tr}\left\{(\tilde{\sigma}_{AB_1} \otimes \tilde{\sigma}_{B_2C})(\tilde{A}_{a|x} \otimes \tilde{B}_b \otimes \tilde{C}_{c|z})\right\}$$

Theorem 3. For any distribution P admitting the decomposition above with real states and measurements,

$$\boxed{\mathcal{T}(P) \leq 7.6605}$$

Outlook

Complex numbers $\longrightarrow$ convenient mathematical tool

Question: If we use the standard quantum formalism and restrict the Hilbert spaces to be real, possibly of *larger* dimensions, could we still explain the same phenomena?

Single system
+
entangle state shared by 2 parties

} *simulate* with real

SWAP scenario $\longrightarrow$ "real" and "complex" quantum physics give different predictions

Results rely on the assumption of the formalism of tensor product

↘

Real frameworks with the same predictive power

Experimental realization

[Submitted on 15 Mar 2021]

Ruling out real-number description of quantum mechanics

Ming-Cheng Chen, Can Wang, Feng-Ming Liu, Jian-Wen Wang, Chong Ying, Zhong-Xia Shang, Yulin Wu, Ming Gong, Hui Deng, Futian Liang, Qiang Zhang, Cheng-Zhi Peng, Xiaobo Zhu, Adan Cabello, Chao-Yang Lu, Jian-Wei Pan

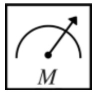
Standard quantum mechanics has been formulated with complex-valued Schrodinger equations, wave functions, operators, and Hilbert spaces. However, previous work has shown possible to simulate quantum systems using only real numbers by adding extra qubits and exploiting an enlarged Hilbert space. A fundamental question arises: are the complex numbers really necessary for the quantum mechanical description of nature? To answer this question, a non-local game has been developed to reveal a contradiction between a multiqubit quantum experiment and a player using only real numbers. Here, based on deterministic and high-fidelity entanglement swapping with superconducting qubits, we experimentally implement the Bell-like game and observe a quantum score of 8.09(1), which beats the real number bound of 7.66 by 43 standard deviations. Our results disprove the real-number description of nature and establish the indispensable role of complex numbers in quantum mechanics.

Comments:	13 pages, 4 figures, submitted
Subjects:	**Quantum Physics (quant-ph)**; Mesoscale and Nanoscale Physics (cond-mat.mes-hall); Mathematical Physics (math-ph)
Cite as:	arXiv:2103.08123 **[quant-ph]**
	(or arXiv:2103.08123v1 **[quant-ph]** for this version)

Proposition 1

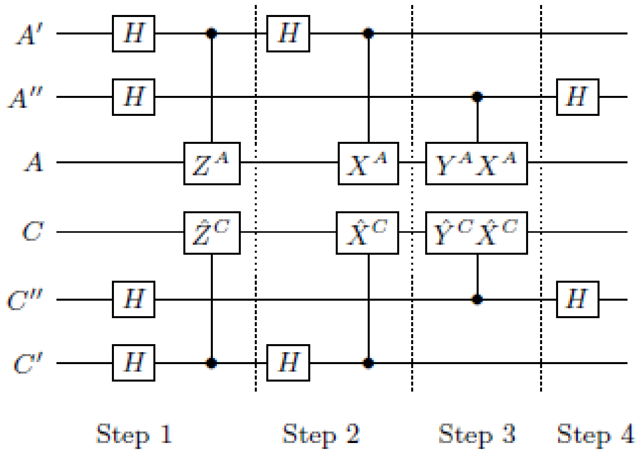
Proof by contradiction

Claim: Suppose that $P(\lambda)$, A, B_1, B_2, C, real states $\tilde{\sigma}^{\lambda}_{AB_1}$, $\tilde{\sigma}^{\lambda}_{B_2C}$, real measurements $\tilde{A}_{a|x}, \tilde{B}_b, \tilde{C}_{c|z}$ such that $\sum_{\lambda} P(\lambda)\,\mathrm{tr}\left\{(\tilde{\sigma}_{AB_1} \otimes \tilde{\sigma}_{B_2C})(\tilde{A}_{a|x} \otimes \tilde{B}_b \otimes \tilde{C}_{c|z})\right\} = \bar{P}(a,b,c|z,x)$

Bob 

$$\psi = \sum_{\lambda} P(\lambda)(\tilde{\sigma}_{AB_1} \otimes \tilde{\sigma}_{B_2C}) \longrightarrow \text{AC system} \longrightarrow \mathrm{tr}_B\left(\tilde{B}_b \psi \tilde{B}_b\right)/P(b) \overset{\text{purification}}{=} |\psi^b\rangle\langle\psi^b|$$

local isometry $U \otimes V$



self-testing protocols

Claim

if the real $|\psi^b\rangle$ saturates $\langle\psi^b|\mathcal{T}|\psi^b\rangle \leq 6\sqrt{2}$

$\Downarrow$ $U \otimes V$

$$\rho^b := \mathrm{tr}_{AC}\left(U \otimes V|\psi^b\rangle\langle\psi^b|\right) = |b\rangle\langle b|_{A'C'} \otimes \left[\frac{\Phi^+ + \Phi^-}{2}\right]_{A''C''}$$

Proposition 1

$$\rho^b := \mathrm{tr}_{AC}\left(U \otimes V |\psi^b\rangle\langle\psi^b|\right) = |b\rangle\langle b|_{A'C'} \otimes \left[\frac{\Phi^+ + \Phi^-}{2}\right]_{A''C''}$$

$$\frac{\Phi^+ + \Phi^-}{2} = \frac{|i\rangle\langle i|^{\otimes 2} + |-i\rangle\langle -i|^{\otimes 2}}{2}$$

$$\rho = \sum_b \bar{P}(b)\rho^b = \underbrace{\frac{\mathbb{I}_{A'B'}}{4} \otimes \left[\frac{|i\rangle\langle i|^{\otimes 2} + |-i\rangle\langle -i|^{\otimes 2}}{2}\right]_{A''C''}}$$

NOT real *separable* state

$$\rho = \underbrace{\sum_\lambda P(\lambda)\, \mathrm{tr}_A(U\tilde{\sigma}_{AB_1}U^\dagger) \otimes \mathrm{tr}_B(V\tilde{\sigma}_{B_2C}V^\dagger)}$$

real *separable* state

CONTRADICTION

Thank you for your attention!

References

[1] Renou et al., Quantum physics needs complex numbers, 2101.10873(2021).

[2] M. McKague, M. Mosca, and N. Gisin, Simulating quantum systems using real hilbert spaces, Phys. Rev. Lett. 102, 020505(2009).

[3] J. Bowles, I. Supić, D. Cavalcanti, and A. Acín, Self-testing of Pauli observables for device-independent entanglement certication, Phys. Rev. A 98, 042336 (2018).