Matchgate hierarchy: A Clifford-like hierarchy for matchgate circuits

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Joint work with Rui and Nadish

Matchgate hierarchy: A Clifford-like hierarchy for deterministic gate teleportation in matchgate circuits

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The Clifford hierarchy, introduced by Gottesman and Chuang in 1999, is an increasing sequence of sets of quantum gates crucial to the gate teleportation model for fault-tolerant quantum computation. Gates in the hierarchy can be deterministically implemented, with increasing complexity, via gate teleportation using (adaptive) Clifford circuits with access to magic states.

We propose an analogous gate teleportation protocol and a related hierarchy in the context of matchgate circuits, another restricted class of quantum circuits that can be efficiently classically simulated but are promoted to quantum universality via access to `matchgate-magic' states. The protocol deterministically implements any n-qubit gate in the hierarchy using adaptive matchgate circuits with magic states, with the level in the hierarchy indicating the required depth of adaptivity and thus number of magic states consumed. It also provides a whole family of novel deterministic matchgate-magic states.

We completely characterise the gates in the matchgate hierarchy for two qubits, with the consequence that, in this case, the required number of resource states grows linearly with the target gate's level in the hierarchy. For an arbitrary number of qubits, we propose a characterisation of the matchgate hierarchy by leveraging the fermionic Stone–von Neumann theorem. It places a polynomial upper bound on the space requirements for representing gates at each level.

Presentation outline

- Motivation
- Matchgate circuits and FLO
- Clifford hierarchy
- Matchgate hierarchy
- Hierarchy gate teleportation with magic states
- 2-qubit characterization
- Fermionic Stone-von Neumann theorem

Motivation

A way to understand the power of quantum computers is to study **restricted classes** of quantum circuits that can be **classically simulated** but become **universal** by the addition of extra **resources**.

Stabiliser sub-theory:

- Clifford circuits
- Gottesman-Knill theorem
- Cliffords + *T* gate
- Non-stabiliser magic states

Matchgate sub-theory:

- Matchgate circuits
- Valiant´s theorem
- Matchgates + *SWAP* gate
- Connected to fermionic linear optics (FLO)
- Fermionic non-Gaussian magic states

Promotion of restricted classes of circuits to quantum universality can be done through **quantum gate teleportation protocol** introduced by Shor and in more generality by Gottesman and Chuang.

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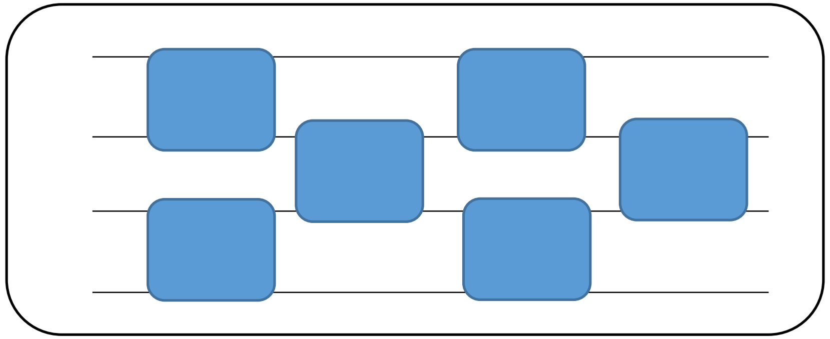
Clifford hierarchy gates can be ***deterministically*** implemented using gate teleportation protocol on stabiliser circuits. They can be performed **fault-tolerantly**.

Matchgate quantum computation

Valiant ´02: Matchgates arose from the theory of *perfect matchings* in the context of counting constraint satisfaction problems.

Matchgate is a 2-qubit unitary gate of the form: $G(A,B) := \begin{pmatrix} A_{11} & 0 & 0 & A_{12} \\ 0 & B_{11} & B_{12} & 0 \\ 0 & B_{21} & B_{22} & 0 \\ A_{21} & 0 & 0 & A_{22} \end{pmatrix} \in \mathcal{U}(4)$

$\boxed{|A| = |B|}$

Matchgate circuits $\longrightarrow$  $\longrightarrow$ $\mathcal{M}_2^{(n)} \subset \mathcal{U}(2^n)$

Action of $G(A,B)$

- *A* acts on the *even-parity* subspace spanned by $\{|00\rangle, |11\rangle\}$
- *B* acts on the *odd-parity* subspace spanned by $\{|01\rangle, |10\rangle\}$

parity-preserving operator

Matchgate quantum computation

$(\mathbb{C}^2)^{\otimes n} =$ *even-parity* subspace $\oplus$ *odd-parity* subspace

eigenspace of $Z^{\otimes n}$ with *+1* eigenvalue   eigenspace of $Z^{\otimes n}$ with *-1* eigenvalue

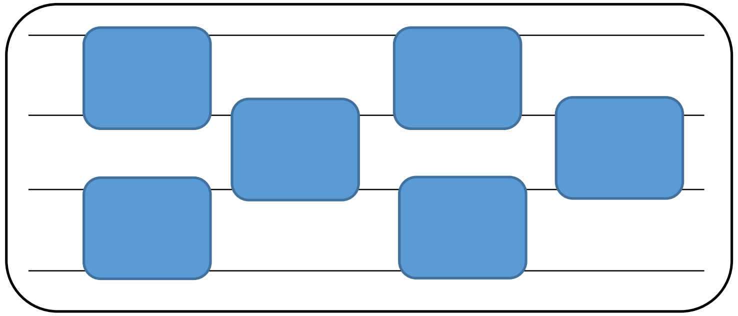
n-qubit operator *M* → *parity-preserving* or *even* if $[M, Z^{\otimes n}] = 0$

n-qubit operator *M* → *parity-reversing* or *odd* if $\{M, Z^{\otimes n}\} = 0$

Matchgate circuits ⟶ parity-preserving (even)

$\boxed{\mathcal{M}_2^{(n)} + X_k} \longrightarrow \mathcal{G}_2^{(n)}$ *generalized matchgate circuits* (***even* + *odd*** operators)

Matchgate quantum computation

computational basis states +  + Z_k *measurement* ⟶ *classically efficiently simulable*

n.n. matchgate circuits

n.n. matchgate circuits + $\left\{ \begin{array}{l} \text{SWAP}^{*} = G(\mathbb{1}, X)\,(|\mathbb{1}| = -|X|) \\ \textbf{non}\text{-matchgate} \end{array} \right\}$ ⟶ *quantum universality*

**fermionic SWAP:*

$$\text{fSWAP} = G(Z, X) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Clifford setting and quantum universality:

- Clifford circuits + **non**-Clifford gate
- Clifford gates + **non**-stabilizer states (´magic´ states)

Hebenstreit et al. ´19:

Quantum universality with **´magic´ states** using **gate teleportation protocol**

pure fermionic non-Gaussian states

probabilistic, may require multiple rounds

Majorana fermions and fermionic linear optics

Physical significance of matchgate circuits $\xrightarrow{\text{Jordan-Wigner transform}}$ quantum evolution of *non-interacting* fermions

Fermionic physics:

n fermionic modes with creation and annihilation operators $a_k^\dagger$ and a_k

2n Hermitian unitary operators $\{c_\mu\}_{\mu=1}^{2n}$ known as *Majorana operators:*

$c_{2k-1} := a_k + a_k^\dagger$ and $c_{2k} := -i(a_k - a_k^\dagger)$ satisfy CAR: $\{c_\mu, c_\nu\} = 2\delta_{\mu\nu}\mathbb{1}$

$$\{c_\mu, c_\nu\} := c_\mu c_\nu + c_\nu c_\mu$$

CAR ➡ presentation of the ***Majorana group*** ➡ $\pm c_{\mu_1} \cdots c_{\mu_m} \quad (\mu_1 < \cdots < \mu_m)$

CAR ➡ presentation of an algebra over $\mathbb{C}$ ➡ elements: $\sum_{m=0}^{2n} \sum_{\mu_1 < \cdots < \mu_m} \alpha_{\mu_1,\ldots,\mu_m} c_{\mu_1} \cdots c_{\mu_m}$

Clifford algebra

Jordan-Wigner transformation

Jordan-Wigner transformation:

- *n* fermionic mode systems $\longleftrightarrow$ *n* qubits (one dimensional chains of spin-1/2 particles)
- *n* fermionic operators $\longleftrightarrow$ *n*-qubit Pauli operators

***n*-mode fermionic state** being represented by ***n*-qubit state**:

computational basis state: $|\mathbf{z}\rangle = |z_1, \ldots, z_n\rangle$ ($\mathbf{z} \in \mathbb{Z}_2^n$) $\longrightarrow$ Fock state: $|\Psi_{\mathbf{z}}\rangle = (a_1^\dagger)^{z_1} \cdots (a_n^\dagger)^{z_n} |\mathbf{0}\rangle$

Majorana operators $\{c_\mu\}_{\mu=1}^{2n}$ represented by *n*-qubit Hermitian unitaries (i.e. in $\mathcal{U}(2^n)$) as:

$$c_{2k-1} = \left(\prod_{i=1}^{k-1} Z_i\right) X_k, \qquad c_{2k} = \left(\prod_{i=1}^{k-1} Z_i\right) Y_k \quad (k = 1, \ldots, n)$$

The monomials $c_{\mu_1} \cdots c_{\mu_m}$ with $\mu_1 < \cdots < \mu_m$ form a basis of $M_{2^n}(\mathbb{C})$ of *n*-qubits

Even and odd operators in fermionic language

n-qubit operator *M* is *even* if $[M, Z^{\otimes n}] = 0$ or *odd* if $\{M, Z^{\otimes n}\} = 0$

Fermionic language:

- An operator is even (resp. odd) if it is a l.c. of even (resp. odd) degree monomials of Majorana operators
- The parity subspace $\mathcal{E}^{(n)}$ (resp. $\mathcal{O}^{(n)}$) is spanned by the monomials $c_{\mu_1} \cdots c_{\mu_{\mathfrak{m}}}$ with $\mathfrak{m}$ even (resp. odd)

Physical states are constrained by the *parity superselection rule* $\longrightarrow$ ~~even + odd~~ number parity states

Example: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$

$$\boxed{\textbf{Physical states are constrained to be eigenstates of } Z^{\otimes n} \textbf{ : fermionic states}}$$

Parity is *conserved* quantity: $|\psi\rangle \in \mathcal{E}^{(n)}$ or $\mathcal{O}^{(n)} \xrightarrow{U \text{ even}} |\psi'\rangle \in \mathcal{E}^{(n)}$ or $\mathcal{O}^{(n)}$

Adjoining ancillary modes enlarges the physically implementable evolutions to include odd unitaries

$$\boxed{\text{Even or odd unitaries} \longrightarrow \textbf{\textit{fermionic}}}$$

Non-interacting (´free´) fermions

Non-interacting (‘free’) fermionic systems are governed by **quadratic Hamiltonians**:

$H = i\sum_{\mu,\nu=1}^{2n} h_{\mu\nu} c_\mu c_\nu$ where $h = (h_{\mu\nu})$ is a $2n$ x $2n$ real antisymmetric matrix

$\downarrow$

$U = e^{iH}$ $\longrightarrow$ *Gaussian* unitary (or fermionic linear optical) $\longrightarrow$ implemented by matchgate circuits

Action of Gaussians on Majorana operators:

$$U c_\mu U^\dagger = \sum_{\nu=1}^{2n} R_{\mu\nu} c_\nu \quad \text{for} \quad R \in \mathrm{SO}(2n)$$

$\downarrow$ conversely

there exists anti-symmetric matrix *h* s.t. $R = e^h$

$\boxed{\text{Gaussian operations} \longleftrightarrow R \in \mathrm{SO}(2n)}$

$\longrightarrow$ ***Classical simulability* of Gaussians (or matchgate circuits)**

Generalized matchgate circuits $\longleftrightarrow$ $R \in \mathrm{O}(2n)$ (reflections)
(adding c_μ to the gate set)

Non-interacting (´free´) fermions

Gaussian states:

An n-qubit state is called *Gaussian* if it arises as the action of a Gaussian operator on the Fock state or equivalently of a matchgate circuit on a computational basis state

Alternative definitions of Gaussianity is given in terms of the operator $\Lambda_n = \sum_{k=1}^{2n} c_k \otimes c_k$:

- A fermionic operator U is Gaussian iff $[\Lambda_n, U^{\otimes 2}] = 0$
- A fermionic state $|\psi\rangle$ is Gaussian iff $\Lambda_n |\psi\rangle^{\otimes 2} = 0$

The above definitions suggest a **general definition** of Gaussianity:

A $2^n \times 2^m$ matrix M is called Gaussian iff $\Lambda_n M^{\otimes 2} = M^{\otimes 2} \Lambda_m$

Clifford hierarchy

Gates in the Clifford hierarchy were introduced by Gottesman and Chuang to analyse resources in the gate teleportation protocol. They can be implemented fault-tolerantly to achieve universality.

Increasing sequence of sets of *n*-qubit gates $\{\mathcal{C}_k^{(n)}\}_{k\in\mathbb{N}}$

The first level of the hierarchy is the *n*-qubit Pauli group $\mathcal{C}_1^{(n)}$ which is generated by the X_i and Z_i Pauli operators.

The second level is the Clifford group: $\mathcal{C}_2^{(n)} := \left\{ U \in \mathcal{U}(2^n) \mid U\mathcal{C}_1^{(n)}U^\dagger \in \mathcal{C}_1^{(n)} \right\}$

The Clifford hierarchy is defined recursively: $\mathcal{C}_{k+1}^{(n)} := \left\{ U \in \mathcal{U}(2^n) \mid U\mathcal{C}_1^{(n)}U^\dagger \in \mathcal{C}_k^{(n)} \right\} \quad (k \geq 1)$

Examples of 3-level gates: Toffoli gate, the *T* gate and the controlled-phase gate *CP.*

(Generalized) matchgate hierarchy

Definition. Fix any $n > 1$, the number of qubits (or fermionic modes). The n-qubit *generalized matchgate hierarchy* is the sequence of n-qubit gates $\{\mathcal{G}_k^{(n)}\}_{k\in\mathbb{N}}$ defined as follows:

- The first level of the hierarchy is the set of unit-norm, l.c. of Majorana operators,

$$\mathcal{G}_1^{(n)} := \left\{\sum_{\mu=1}^{2n} a_\mu c_\mu \mid \mathbf{a} \in \mathbb{R}^{2n}, \|\mathbf{a}\| = 1\right\};$$

ensures unitarity

- Higher levels are defined recursively, for any $k \geq 1$:

$$\mathcal{G}_{k+1}^{(n)} := \left\{U \in \mathcal{U}(2^n) \mid \forall\mu \in \{1,\ldots,2n\}. \boxed{U c_\mu U^\dagger \in \mathcal{G}_k^{(n)} \cap \mathcal{O}^{(n)}}\right\}$$

restricts the gates to be *fermionic*

$$\mathcal{G}_{k+1}^{(n)} := \left\{U \in \mathcal{E}^{(n)} \cup \mathcal{O}^{(n)} \mid \forall\mu.\ U c_\mu U^\dagger \in \mathcal{G}_k^{(n)}\right\}$$

Motivation:

- *Foundational reason*: Physical operations are parity constrained $\longrightarrow$ *fermionic* operators

- *Practical reason*: In the gate teleportation protocol magic states must be ´ freely´ swapped (without the use of SWAP gate), which was shown (Hebenstreit ´19) to be done iff the state is *fermionic.*

Fermionic unitaries

The following lemma characterises **fermionic gates** as the ones that *preserve* matrix parity under conjugation:

Lemma. *Let $U \in \mathcal{U}(2^n)$ be an n-qubit unitary. If conjugation by U preserves the parity of operators, in that it maps even operators to operators and odd operators to odd operators, i.e.*

$$M \in \mathcal{E}^{(n)} \implies UMU^\dagger \in \mathcal{E}^{(n)} \quad \text{and} \quad M \in \mathcal{O}^{(n)} \implies UMU^\dagger \in \mathcal{O}^{(n)}$$

then U is fermionic (i.e. either even or odd).

Corollary. *For all $n, k \geq 1, \mathcal{G}_k^{(n)} \subset \mathcal{E}^{(n)} \cup \mathcal{O}^{(n)}$.* ⟶ Gates in the hierarchy are **fermionic!**

We denote k-level *even* gates as $\mathcal{M}_k^{(n)} := \mathcal{G}_k^{(n)} \cap \mathcal{E}^{(n)}$

$\mathcal{M}_2^{(n)}$ ⟶ unitaries realised by **matchgate circuits**

$\mathcal{G}_2^{(n)}$ ⟶ unitaries realised by **generalised matchgate circuits**

Examples

The SWAP $= G(\mathbb{I}, X)$ is an example of an **even** two-qubit **third-level** gate since:

$$\text{SWAP}\, c_1\, \text{SWAP} = -ic_1c_2c_3 \qquad \text{SWAP}\, c_3\, \text{SWAP} = -ic_1c_3c_4$$
$$\text{SWAP}\, c_2\, \text{SWAP} = -ic_1c_2c_4 \qquad \text{SWAP}\, c_4\, \text{SWAP} = -ic_2c_3c_4$$

Another example is the *CZ* $= G(Z, \mathbb{I}) = \mathrm{diag}(1,1,1,-1)$ and more generally the **controlled-phase** gate $C_{2\pi/2^{k-2}} \coloneqq \mathrm{diag}(1,1,1,e^{\frac{2\pi i}{2^{k-2}}})$ which is a ***k*-level** gate.

Examples of **three-qubit** gates in the **third level**:

$$|x,y,z\rangle \mapsto (-1)^{xz}\,|x,y,z\rangle \quad \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&-1 \end{pmatrix} \quad \begin{matrix}\text{fSWAP}_{[1,3]}\\ |x,y,z\rangle \mapsto (-1)^{xz}\,|z,y,x\rangle\end{matrix}$$

An interesting family of examples is the *n*-qubit $C^{n-1}Z$ gate which is in the $(\boldsymbol{n}+\mathbf{1})$**-th level** of the matchgate hierarchy.

Basic properties of the hierarchy

Closed under scaling by a phase:

Proposition. *For* $n \geq 1$ *and* $k \geq 2$, $U \in \mathcal{G}_k^{(n)}$ *implies* $e^{i\phi}U \in \mathcal{G}_k^{(n)}$ *for all* $\phi \in [0, 2\pi)$.

Closed under right multiplication by Majorana operators:

Proposition. *For* $n \geq 1$ *and* $k \geq 2$, $U \in \mathcal{G}_k^{(n)}$ *implies* $\boxed{Uc_\mu \in \mathcal{G}_k^{(n)}}$ *for all μ.*

bijection between $\mathcal{M}_k^{(n)} := \mathcal{G}_k^{(n)} \cap \mathcal{E}^{(n)}$ and $\mathcal{G}_k^{(n)} \cap \mathcal{O}^{(n)}$

Closed under right multiplication by Majorana operators:

Proposition. *For* $n, k \geq 1$, $U \in \mathcal{G}_k^{(n)}$ *implies* $c_\mu U c_\mu \in \mathcal{G}_k^{(n)}$ *for all μ.*

Nested levels:

Proposition. *For* $n, k \geq 1$, $\mathcal{G}_k^{(n)} \subset \mathcal{G}_{k+1}^{(n)}$.

Closed under tensor product:

Proposition. *For* $n, m \geq 1$ *and* $k \geq 2$, $U \in \mathcal{G}_k^{(n)}$ and $V \in \mathcal{G}_k^{(m)}$ implies $U \otimes V \in \mathcal{G}_k^{(n+m)}$

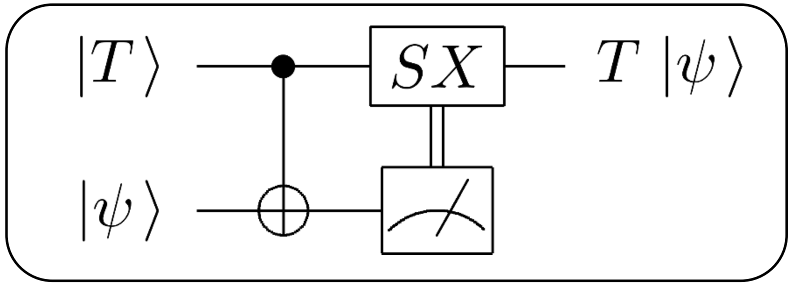
Magic states and quantum universality

Quantum universality:

1. **Clifford gate set** *(free operations)* + **non-Clifford gate** *(resourceful gate)*
2. **Clifford gate set** + ***´magic´ states*** (and adaptive measurements)

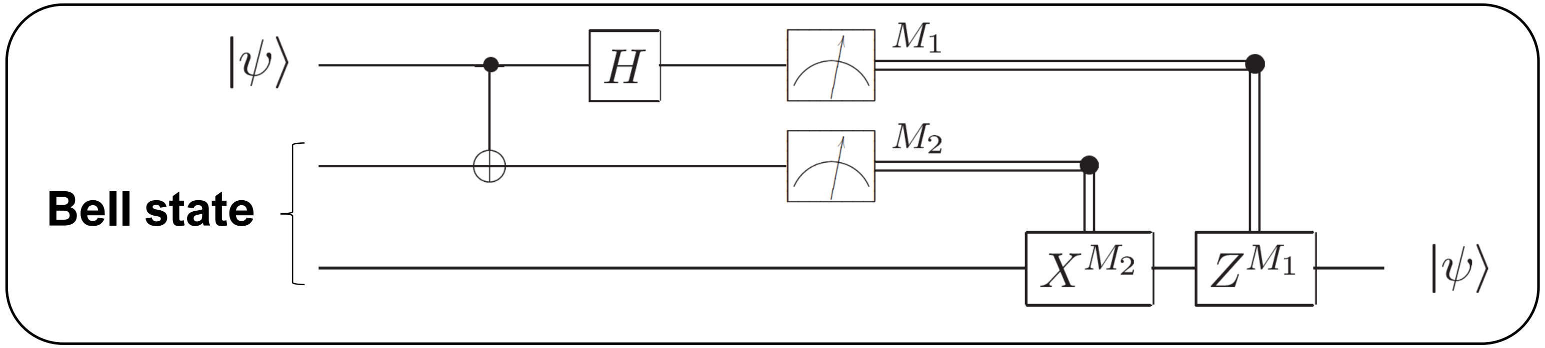
Paradigmatic example:

(resourceful) ***T* gate** which can be implemented using the magic states $|\mathrm{T}\rangle := \dfrac{|0\rangle + \mathrm{e}^{\mathrm{i}\,\pi/4}\,|1\rangle}{\sqrt{2}}$ using the so called *T-gadget*



Gadget that consumes single copy of $|\mathrm{T}\rangle$ state to *deterministically* implement a *T* gate

Quantum teleportation protocol



Quantum universality of matchgates

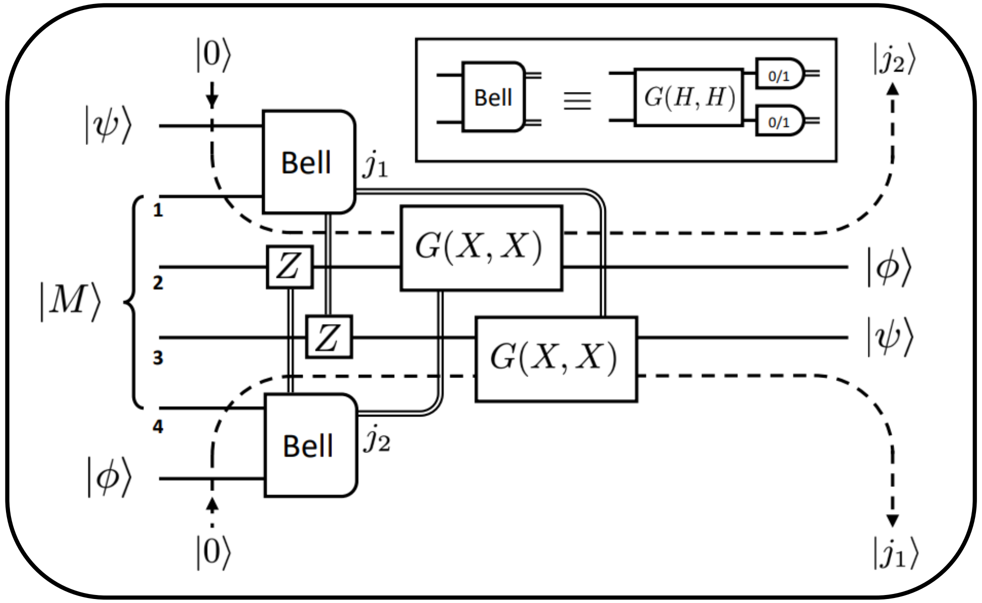
n.n. matchgate circuits $+ \left\{ \begin{array}{l} \text{SWAP} = G(\mathbb{1}, X)(|\mathbb{1}| = -|X|) \\ \textbf{non}\text{-matchgate} \end{array} \right\}$ $\longrightarrow$ *quantum universality*

Hebenstreit et al. ´19:

(resourceful) ***SWAP* gate** can be implemented via a **'SWAP-gadget'** (teleportation protocol) using the magic state $|M\rangle = \frac{1}{2}\left(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle\right)$

Characterization of **magic states** for matchgates:

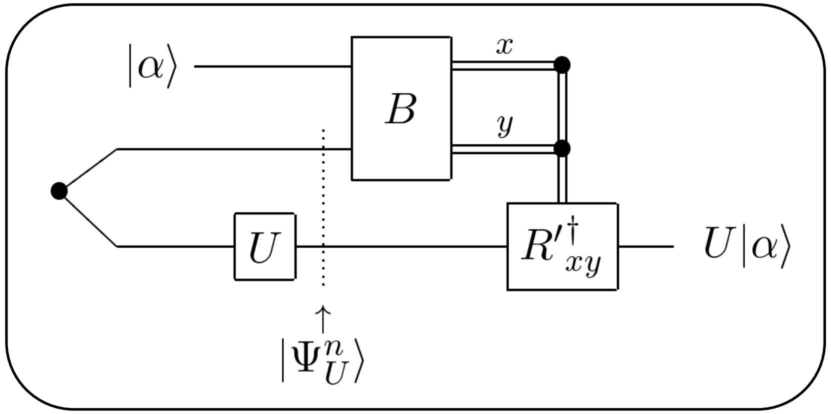
Theorem. *Every pure fermionic (eigenstate of* $Z^{\otimes n}$*) state which is non-Gaussian (cannot be generated by a matchgate circuit from a computational basis state) is a ´matchgate-magic´ state.*



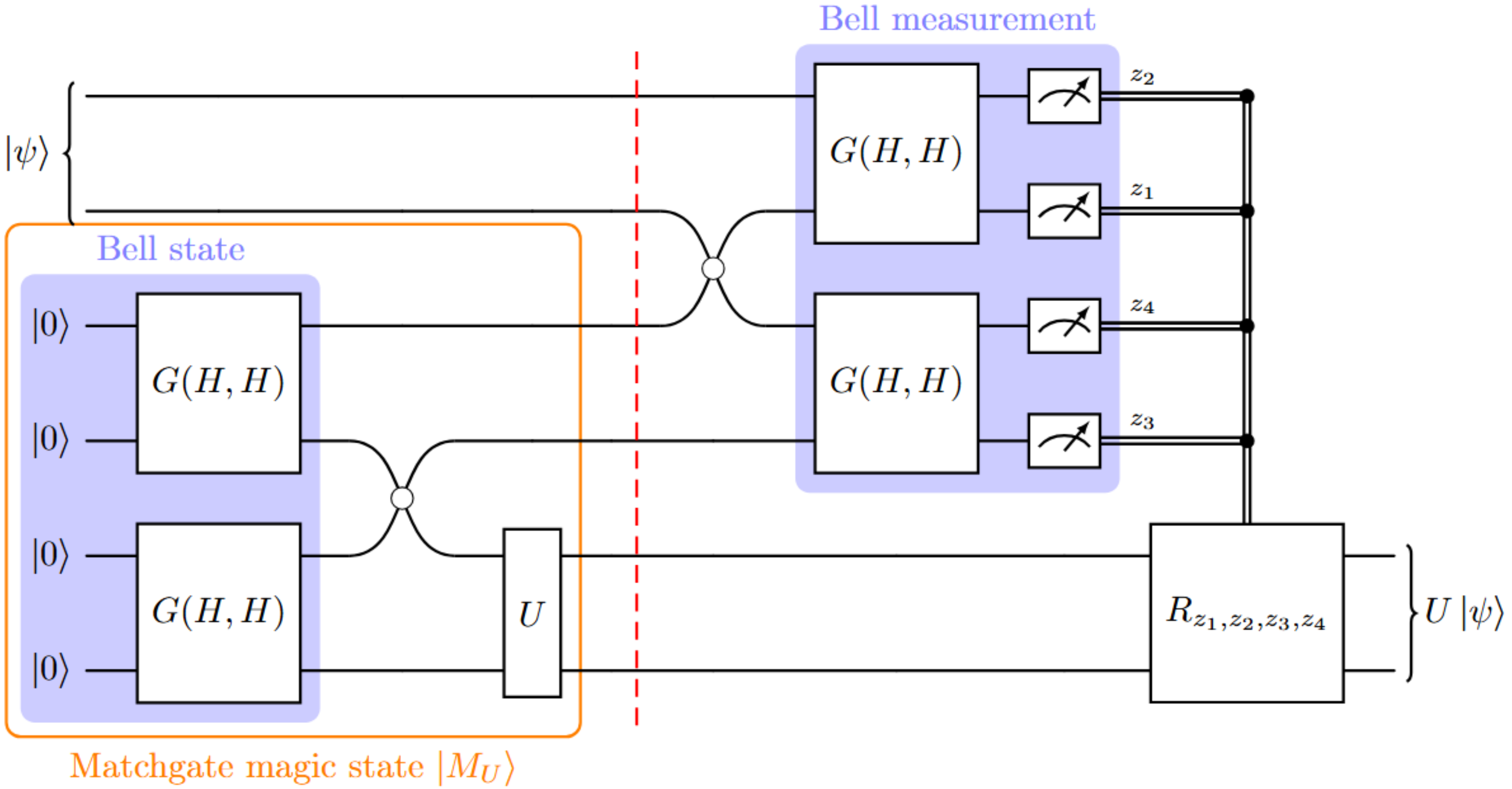
Gadget that consumes single copy of $|M\rangle$ state to *deterministically* implement a *SWAP* gate

Matchgate hierarchy gate teleportation protocol

Gottesman and Chuang 1999:

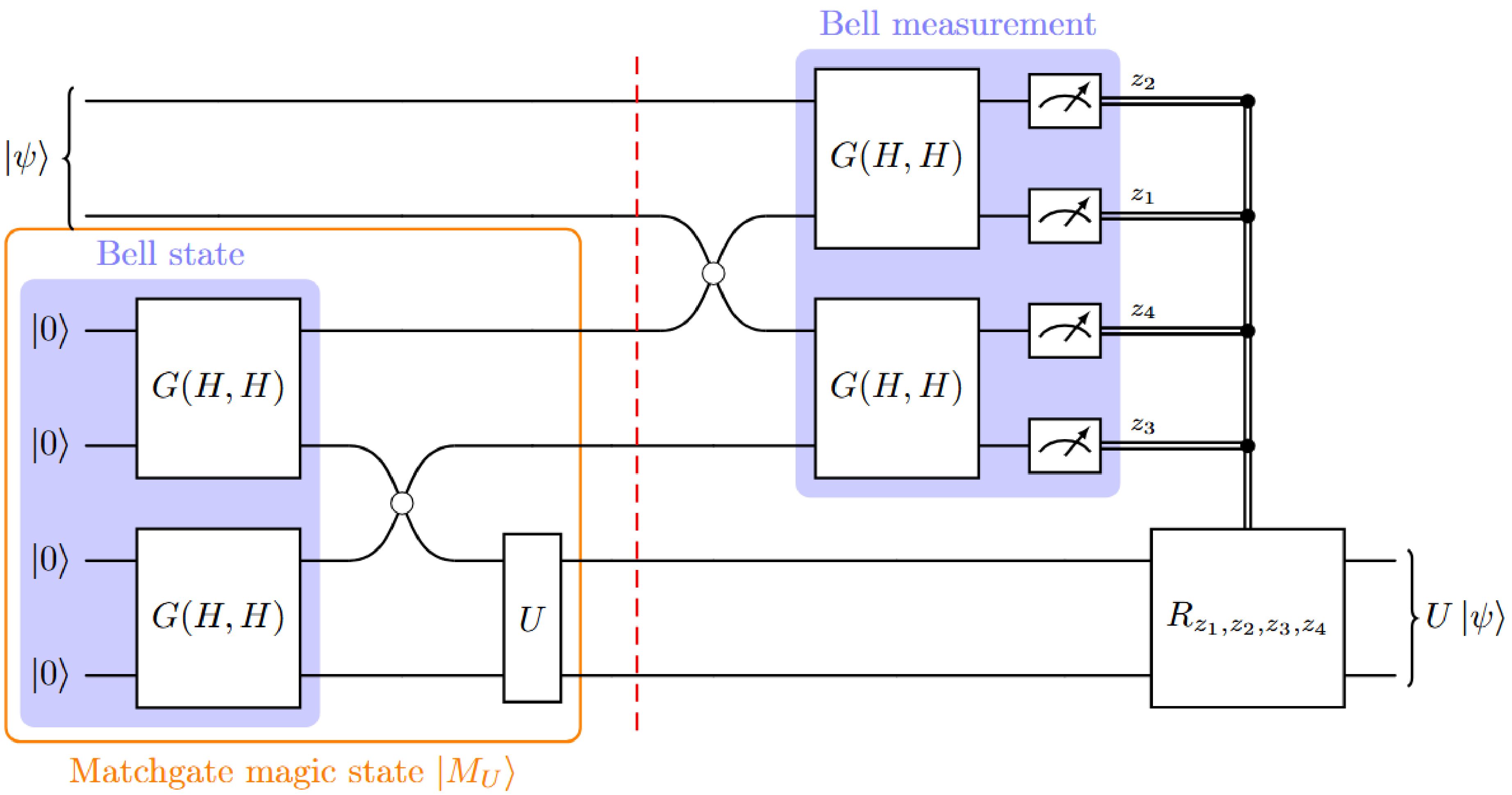


Teleportation protocol that performs any *U* in the Clifford hierarchy fault-tolerantly.

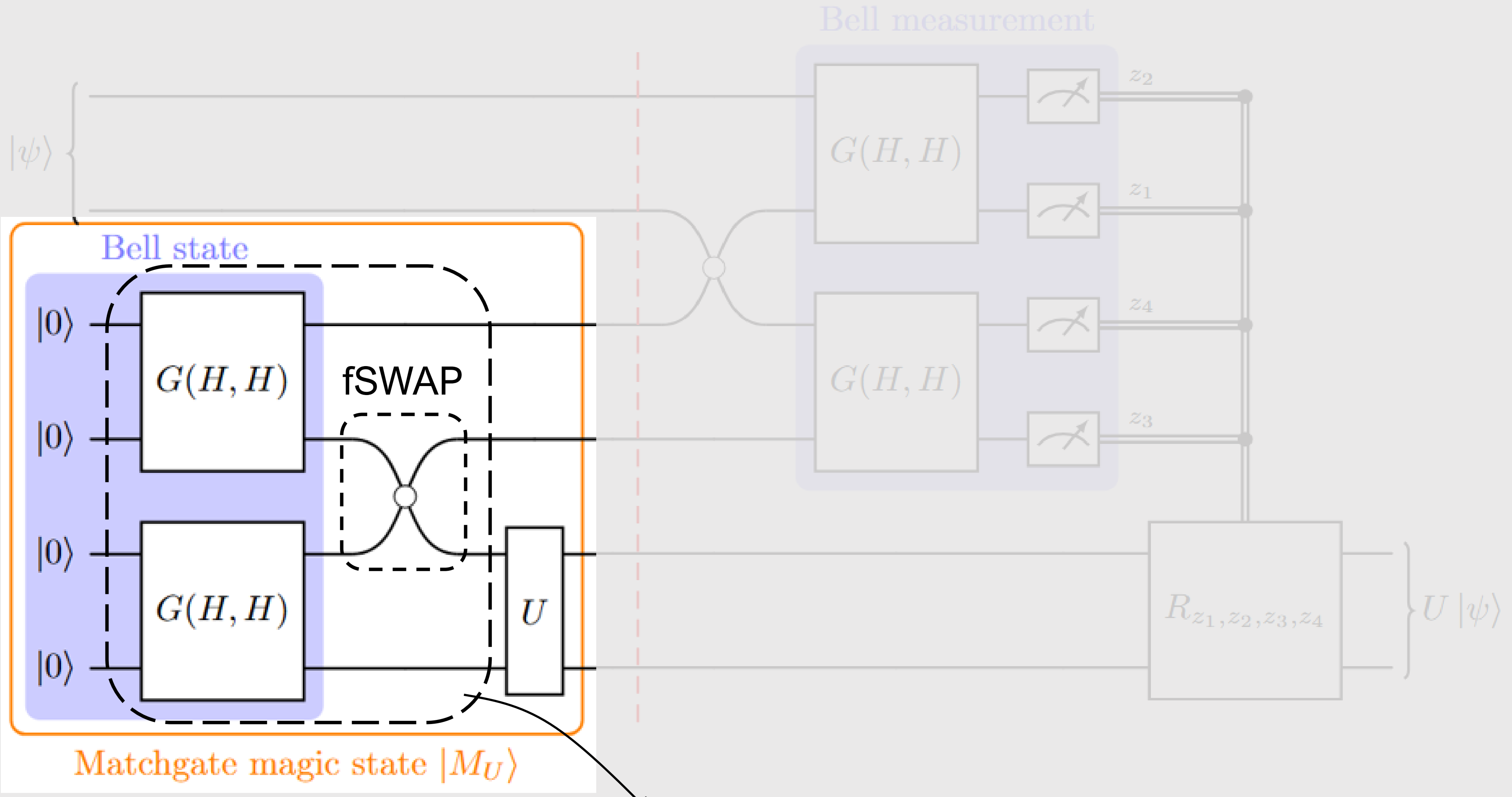


Circuit to implement *deterministically* a gate $U \in \mathcal{G}^{(n)}_{k+1}$ by gate teleportation protocol using n.n. matchgates fSWAP and *G(H,H),* a pre-prepared ´matchgate-magic´ state and a ´correction´ operator $R_{\boldsymbol{z}} := i^{\alpha(z)} U \left(\prod_{\mu} c_{\mu}^{z_{\mu}}\right) U^{\dagger}$, indexed by measurement outcomes $\boldsymbol{z} \in \{0,1\}^{2n}$.

Protocol for 2 qubits



Protocol for 2 qubits



$$|M_U\rangle = (\mathbb{1}\otimes U)\, B\, |0000\rangle$$

$$B = \mathrm{fSWAP}_{[2,3]} G(H,H)_{[1,2]} G(H,H)_{[3,4]}$$

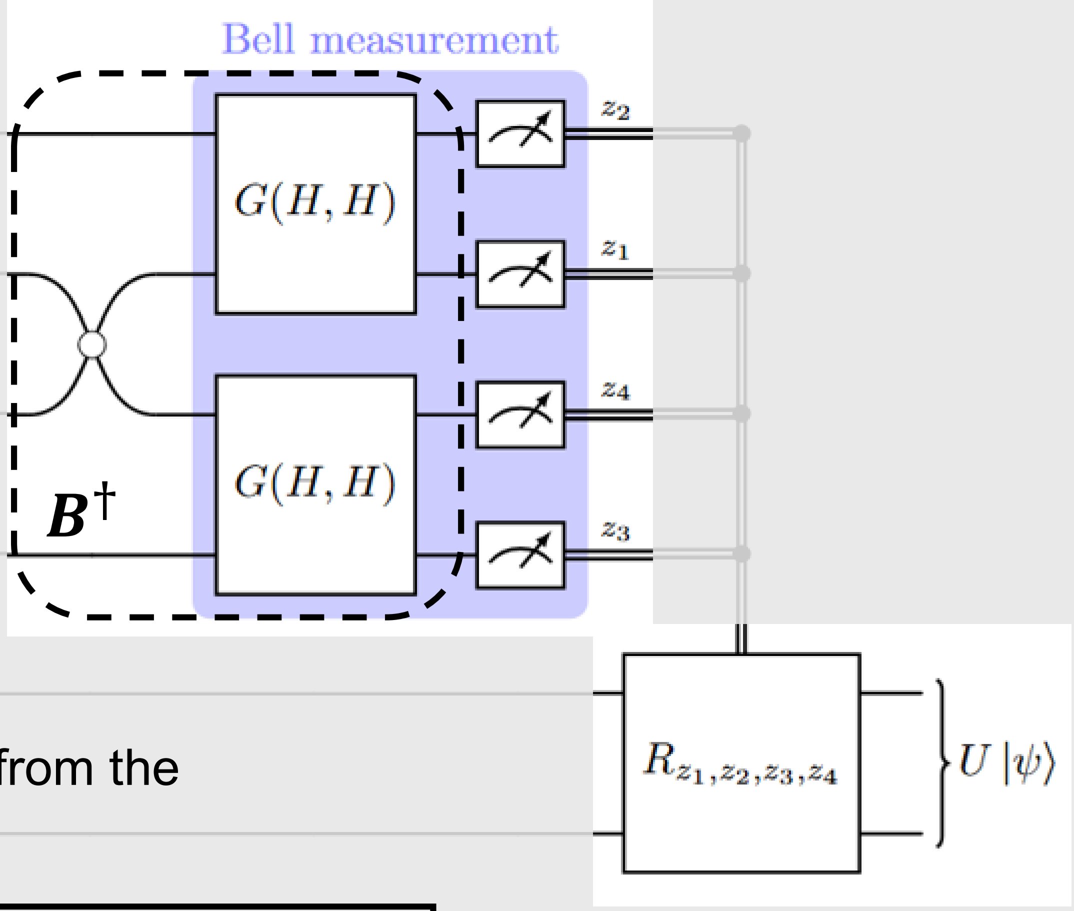
Protocol for 2 qubits

The two inputs wires and one half of the magic state are measured in the computational basis state after applying $\boldsymbol{B}^\dagger$.

It gives a method for *probabilistically* implement any unitary gate U (not necessarily in the hierarchy).

For U being in the hierarchy it's guaranteed that the output can be *corrected* based on the outcomes.

The necessary correction is a sequence of gates from the lower level of the hierarchy.



Recursive application of the protocol allows one to *deterministically* implement ***any*** gate in the hierarchy.

Important remarks about the protocol

Magic states can be prepared offline beforehand, i.e. independently of the input of the circuit.

Crucially, magic states need to be used in matchgate circuits built out of **nearest-neighbour** qubit lines, and **SWAP** is **not** a *free* gate.

⬇

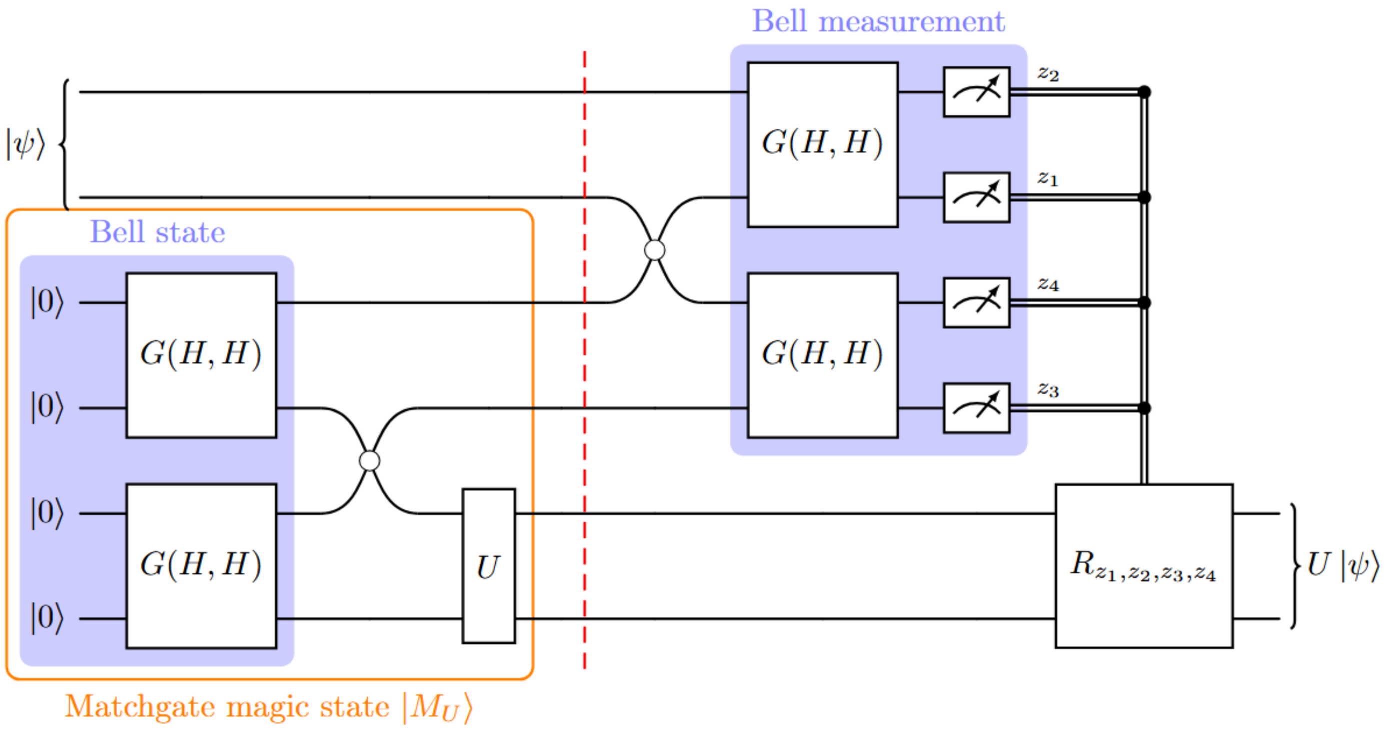
For implementing *U* the magic state needs to be moved next to the gate being implemented. For that the state must be swapped through arbitrary states using ***only free*** gates.

⬇

Magic states can be freely swapped if and only if they are **fermionic**.

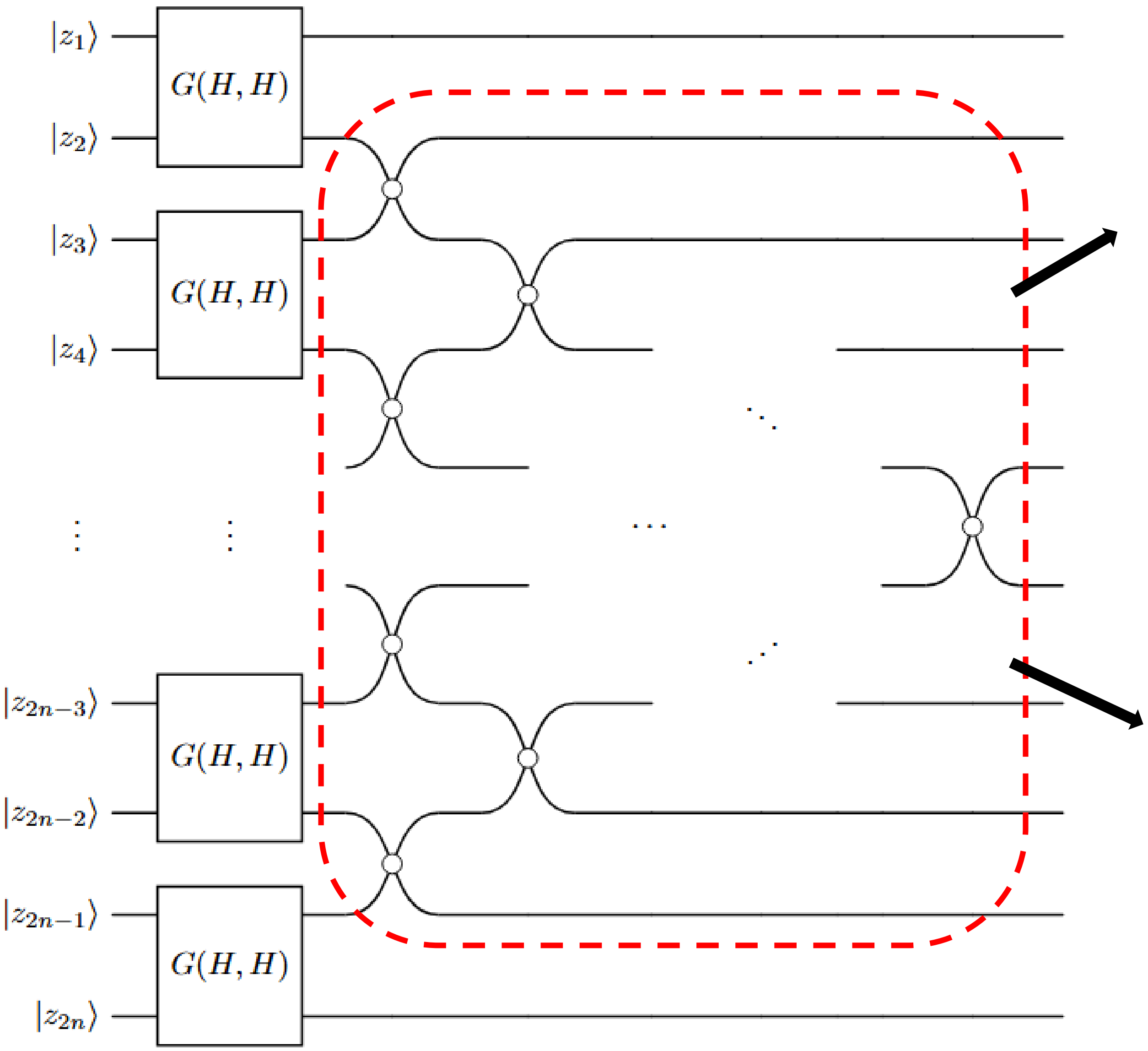
➡

The magic state is **fermionic** when the gate ***U*** is itself **fermionic.**

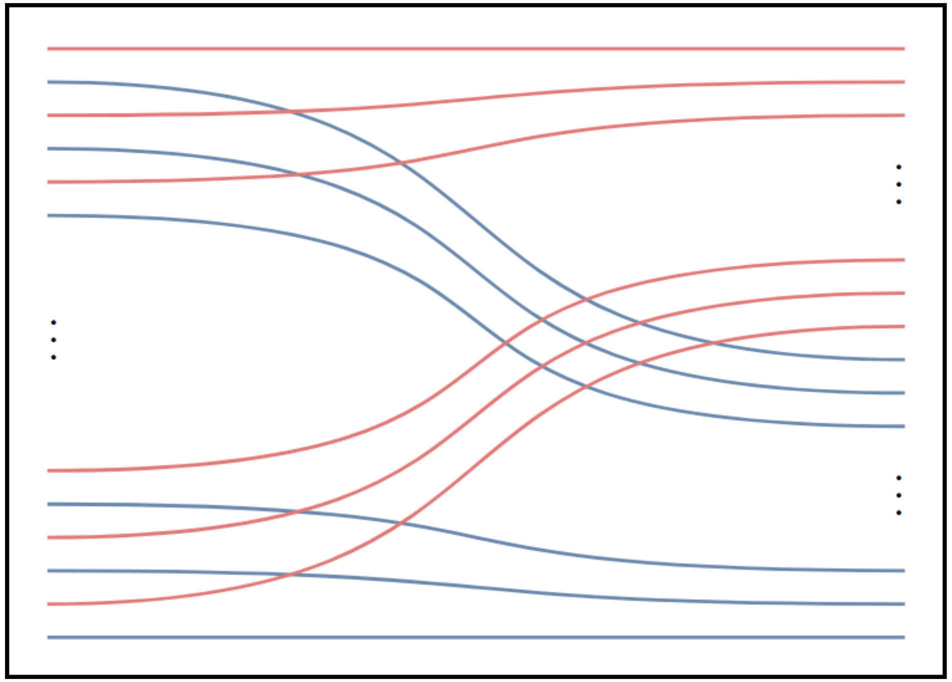


$$|M_U\rangle = (\mathbb{1} \otimes U)\, B\, |0000\rangle$$

Protocol for *n*-qubit gates



- $n-1$ layers of fermionic swaps, fSWAP = G(Z,X)
- Implements a **permutation** of the wires whereby odd numbered qubits are listed first followed by even numbered qubits
- Transposition of neighbouring qubits picks up a -1 phase when both modes are occupied ($|11\rangle$).



Characterisation of 2-qubit gates in the hierarchy

All gates in the hierarchy are **fermionic** (i.e. either **even** or **odd**)

Even gates

$$G(A,B) := \begin{pmatrix} A_{11} & 0 & 0 & A_{12} \\ 0 & B_{11} & B_{12} & 0 \\ 0 & B_{21} & B_{22} & 0 \\ A_{21} & 0 & 0 & A_{22} \end{pmatrix} \in \mathcal{U}(4)$$

Odd gates

$$J(A,B) := \begin{pmatrix} 0 & A_{11} & A_{12} & 0 \\ B_{11} & 0 & 0 & B_{12} \\ B_{21} & 0 & 0 & B_{22} \\ 0 & A_{21} & A_{22} & 0 \end{pmatrix} \in \mathcal{U}(4)$$

They are **matchgates** only when $|\boldsymbol{A}| = |\boldsymbol{B}|$.

Proposition. *$\mathcal{G}_1^{(2)}$ is the set of unitaries of the form $J(A, A^\dagger)$ with $|A| = -1$.*

Proposition. *For any $k \geq 2$, $\mathcal{G}_k^{(2)}$ is the set of unitaries of the form $G(A,B)$ or $J(A,B)$ with $|A|^{2^{k-2}} = |B|^{2^{k-2}}$.*

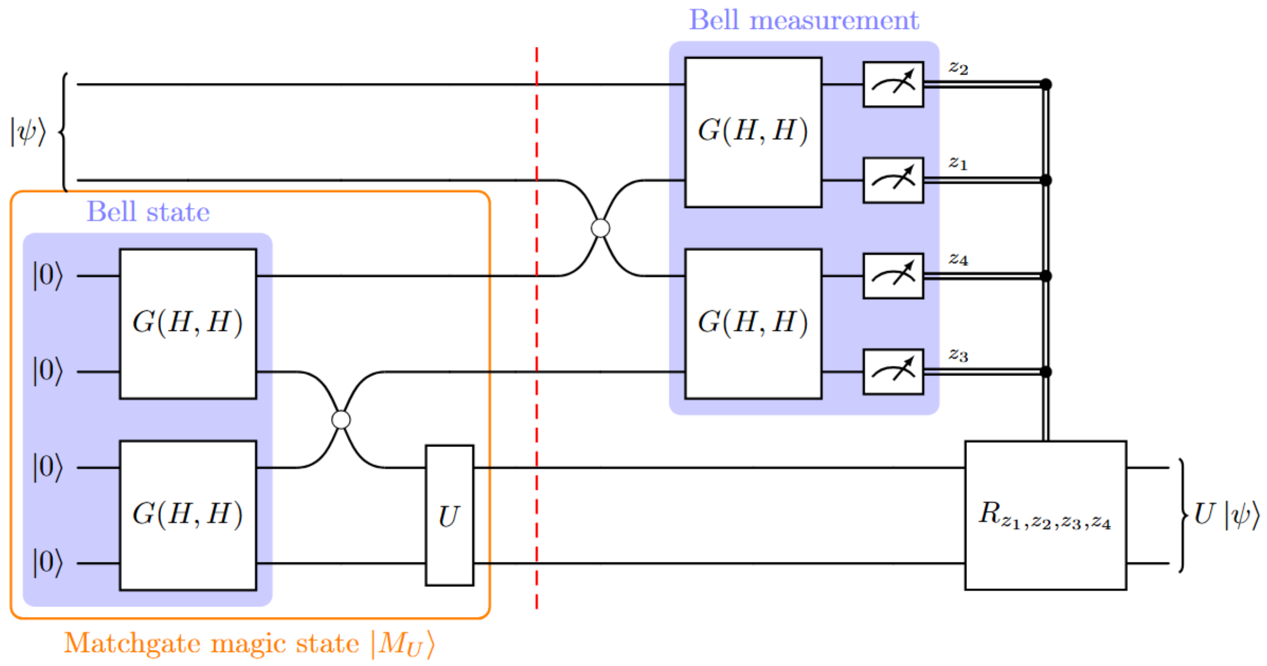
Characteristic examples: $\mathrm{SWAP} = G(\mathbb{I}, X)$ and $CZ = G(Z, \mathbb{I})$, which are both **third level** gates.

Efficiency of 2-qubit protocol

Proposition. *For any* $k \geq 2$, $\mathcal{G}_k^{(2)}$ *is the set of unitaries of the form* $G(A,B)$ *or* $J(A,B)$ *with* $|A|^{2^{k-2}} = |B|^{2^{k-2}}$.

$\Downarrow$

$\boxed{\text{Each of the sets } \mathcal{G}_k^{(2)} \text{ is a subgroup of } \mathcal{U}(2^n)\text{ .}}$



The correction operator is: $R_{\boldsymbol{z}} \coloneqq i^{\alpha(z)} U \left(\prod_\mu c_\mu^{z_\mu}\right) U^\dagger$, indexed by measurement outcomes $\boldsymbol{z} \in \{0,1\}^4$.

The correction is a product of **up to four** gates that belong *strictly* in the lower level

$\Downarrow$

This four-way branching causes the overall efficiency of implementing a gate to **scale exponentially in** $\boldsymbol{k}$**.**

$\Downarrow$

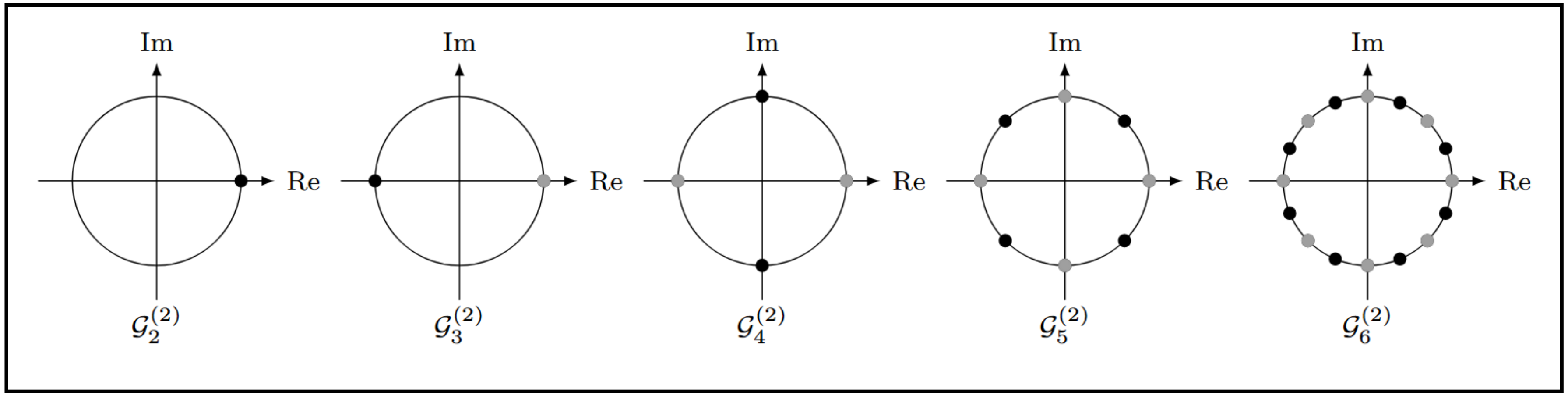
The fact that $\mathcal{G}_k^{(2)}$ is a group makes the product in the correction (now a single gate) be in a strictly lower level than k. Therefore, the protocol scales **linearly** with the level k.

Matchgate-equivalence classes

Two gates are said to be (**generalised-**)**matchgate-equivalent** if they can be obtained from one another by multiplying on both sides with (generalised) matchgate circuits.

Proposition. *Any two-qubit even unitary gate is matchgate-equivalent to the controlled-phase* $C_\phi := G(P_\phi, \mathbb{1})$, *where* $P_\varphi = \mathrm{diag}(1, e^{i\varphi})$, *for a unique phase* $\varphi \in [0,2\pi)$.

Corollary. *There are* 2^{k-2} *classes of even two-qubit k-level gates. Representatives of each class are given by the gates* C_φ, *so that* $e^{i\varphi} = |A|/|B|$ *are the* 2^{k-2} *roots of unity*.



Matchgate-equivalence classes

An odd gate $J(A,B)$ can be decomposed as $G(A,B)\,J(\mathbb{1},\mathbb{1})$

$\Downarrow$

Bijection between even and odd gate $\Rightarrow$ Equivalence classes of **odd** *k*-level gates

$$J(A,B) = G(A,B)\,J(\mathbb{1},\mathbb{1}) = J(\mathbb{1},\mathbb{1})\,G(B,A) \Rightarrow \text{Multiplication by an } \textbf{odd} \text{ gate} \begin{cases} |A|/|B| \\ |B|/|A| \end{cases}$$

The equivalence classes of C_φ and $C_{-\varphi}$ **collapse!**

Proposition. *Any two-qubit fermionic unitary gate is matchgate-equivalent to the controlled-phase* $C_\phi := G(P_\phi, \mathbb{1})$*, where* $P_\varphi = \mathrm{diag}(1, e^{i\varphi})$*, for a unique phase* $\varphi \in [0,\pi]$.

Corollary. *There are* $2^{k-3}+1$ *classes of* ***fermionic*** *two-qubit k-level gates, i.e. gates in* $\mathcal{G}_k^{(2)}$*, under generalised matchgate equivalence. Representatives of each class are given by the gates* C_φ*, for* $\varphi \in \left\{\frac{2j\pi}{2^{k-2}} \mid j = 0, \dots, 2^{k-3}\right\}$ *so that* $e^{i\varphi} = |A|/|B|$ *are the* 2^{k-2}*-th roots of unity nonnegative imaginary part*.

Stone-von Neumann theorem

The generators X, Z of the one-qubit Pauli group satisfy $ZX = -XZ$ and $X^2 = Z^2 = \mathbb{I}$.

For multiple qubits, we have $[\square_i, \square_j] = 0$ for $i \neq j$.

These are the finite-dimensional ***canonical commutation relations***.

⬇

Analogue of the more familiar relation between two canonical conjugate quantities: $[x, p] = i\hbar$

Stone-von Neumann theorem:

- Foundational result in quantum theory that was originally proved to unify the *matrix* and the *wave mechanics* pictures of quantum theory.

- Roughly, it asserts that two representations of canonical commutation relations are **unitarily equivalent**.

fermionic Stone-von Neumann theorem

Consider a set $\{c_\mu\}_{\mu=1}^{2n}$ (Majorana operators) of 2*n* Hermitian operators that satisfy the canonical anticommutation relations (CAR): $\{c_\mu, c_\nu\} = 2\delta_{\mu\nu}\,\mathbb{I}$.

$\Downarrow$

presentation of an algebra over $\mathbb{C}$ $\Rightarrow$ elements: $\sum_{\mathrm{m}=0}^{2n} \sum_{\mu_1<\cdots<\mu_{\mathrm{m}}} \alpha_{\mu_1,\ldots,\mu_{\mathrm{m}}} c_{\mu_1}\cdots c_{\mu_{\mathrm{m}}}$

Jordan-Wigner representation of Majorana operators:

$$c_{2k-1} = \left(\prod_{i=1}^{k-1} Z_i\right) X_k, \qquad c_{2k} = \left(\prod_{i=1}^{k-1} Z_i\right) Y_k \qquad (k = 1, \ldots, n)$$

The monomials $c_{\mu_1}\cdots c_{\mu_{\mathrm{m}}}$ with $\mu_1 < \cdots < \mu_{\mathrm{m}}$ form a **basis** of $M_{2^n}(\mathbb{C})$ of *n*-qubits

Theorem *(fermionic Stone-von Neumann theorem).* *Given two sets $\{c_\mu\}_{\mu=1}^{2n}$ and $\{d_\mu\}_{\mu=1}^{2n}$ of n-qubit operators which satisfy CARs, there is an n-qubit unitary $U = \mathcal{U}(2^n)$, unique up to a phase s.t. $Uc_\mu U^\dagger = d_\mu$.*

fermionic Stone-von Neumann theorem

Theorem *(fermionic Stone-von Neumann theorem).* *Given two sets* $\{c_\mu\}_{\mu=1}^{2n}$ *and* $\{d_\mu\}_{\mu=1}^{2n}$ *of n-qubit operators which satisfy CARs, there is an n-qubit unitary* $U = \mathcal{U}(2^n)$*, unique up to a phase s.t.* $U c_\mu U^\dagger = d_\mu$.

Sketch proof.

We set $\varphi(c_{\mu_1} \dots c_{\mu_{\mathfrak{m}}}) = d_{\mu_1} \dots d_{\mu_{\mathfrak{m}}}$ for $1 \le \mu_1 \le \cdots \le \mu_{\mathfrak{m}} \le 2n$. This sends a basis of $M_{2^n}(\mathbb{C})$ *to* another, so it extends to a linear automorphism $\varphi: M_{2^n}(\mathbb{C}) \to M_{2^n}(\mathbb{C})$.

Afterwards, we check that it preserves adjoints and multiplications. Therefore φ is a *-automorphism. By Skolem-Noether theorem it is an inner automorphism induced by a unitary *U.*

fermionic Stone-von Neumann theorem

The unitary *U* can be given explicitly as follows: for each $\boldsymbol{z} \in \mathbb{Z}_2^n$, the action of *U* on the corresponding computational basis vector satisfies

$$U\,|\mathbf{z}\rangle = \tilde{c}_1^{z_1}\tilde{c}_3^{z_2}\cdots\tilde{c}_{2n-1}^{z_n}\;U\,|\mathbf{0}\rangle = \left(\prod_{k=1}^{n}\tilde{c}_{2k-1}^{z_k}\right)\;U\,|\mathbf{0}\rangle\,,$$

and $U|\mathbf{0}\rangle$ is the simultaneous $+1$-eigenvector of the operators $(-i\tilde{c}_{2k-1}\tilde{c}_{2k})$ for all $k \in [n]$.

Corollary. There is a bijective *correspondence between n-qubit **fermionic** unitary maps up to a phase and tuples of n-qubit odd operators $(\tilde{c}_\mu)_{\mu=1}^{2n}$ satisfying CAR.*

Sketch proof.

Given a unitary *U,* take $\tilde{c}_\mu = U^\dagger c_\mu U$. These relations satisfy the CAR, an (inner) automorphism.

Its conjugation action on the Majoranas since they generate the whole matrix space they fully determine the action on the matrix algebra. This shows that the map from unitaries up to a phase to CAR tuples is **injective**.

The previous theorem shows that the map is **surjective.**

fermionic Stone-von Neumann theorem

Restricting to **fermionic (even** or **odd)** unitaries (recall they preserve matrix parity under conjugation) we have the following Corollaries:

Corollary. *Given two sets* $\{c_\mu\}_{\mu=1}^{2n}$ *and* $\{d_\mu\}_{\mu=1}^{2n}$ *of* ***odd*** *n-qubit operators which satisfy CARs, there is an n-qubit* ***fermionic*** *unitary* $U = \mathcal{U}(2^n)$*, unique up to a phase, s.t.* $U c_\mu U^\dagger = d_\mu$.

Corollary. *There is a bijective correspondence between n-qubit* ***fermionic*** *unitary maps up to a phase and tuples of n-qubit* ***odd*** *operators* $(\tilde{c}_\mu)_{\mu=1}^{2n}$ *satisfying CAR.*

Hierarchical fermionic Stone-von Neumann theorem

The following version follows from the definition of the **matchgate hierarchy**:

Corollary (*Hierarchical Stone-von Neumann theorem*). *Given a set* $\{\tilde{c}_\mu\}_{\mu=1}^{2n}$ *of odd operators in* $\mathcal{G}_k^{(n)}$ *that satisfy the CARs, there exists a unitary* $U \in \mathcal{G}_{k+1}^{(n)}$*, up to a phase, s.t.* $U c_\mu U^\dagger = \tilde{c}_\mu$ *for all μ. This U is given explicitly as follows: for each* $\mathbf{z} \in \mathbb{Z}_2^n$*, the action of U on the computational basis vector satisfies*

$$U\left|\mathbf{z}\right\rangle = \tilde{c}_1^{z_1}\tilde{c}_3^{z_2}\cdots\tilde{c}_{2n-1}^{z_n}\,U\left|\mathbf{0}\right\rangle = \left(\prod_{k=1}^{n}\tilde{c}_{2k-1}^{z_k}\right)\,U\left|\mathbf{0}\right\rangle,$$

and $U|\mathbf{0}\rangle$ *is the simultaneous* $+1$*-eigenvector of the operators* $(-i\tilde{c}_{2k-1}\tilde{c}_{2k})$ *for all* $k \in [n]$.

Corollary. *There is a bijection between* $\mathcal{G}_{k+1}^{(n)}$ *and 2n-tuples in* $\mathcal{G}_k^{(n)} \cap \mathcal{O}^{(n)}$ *that satisfy the CAR. Consequently* $\mathcal{G}_k^{(n)}$ *spans a subspace of* $2^n \times 2^n$ *matrices of dimension* $(2n)^k$.

Outlook

- Based on the concept of Clifford hierarchy we introduced an analogous hierarchy in the context of matchgate circuits, the (***generalised-) matchgate hierarchy*** of fermionic unitary gates.

- Presented a **gate teleportation protocol** where *any n*-qubit gate in the hierarchy can be **deterministically** implemented using adaptive matchgate circuits and magic states.

- Gave a **complete characterisation** of two-qubit gates in the hierarchy, whereby **matchgate-equivalence classes of even *k*-level gates** correspond to the $\mathbf{2^{k-2}}$**-th roots of unity**.

- For an arbitrary number of qubits we showed a **´hierarchy-aware´ fermionic Stone-von Neumann theorem** which may help understanding the structure of the matchgate hierarchy

Open questions

- How to **characterise** the gates in the matchgate hierarchy for **higher number** of qubits? One possible way to do it is by relaxing the matchgate identities.

- Search for finer analysis of the **resource complexity** of the gate teleportation protocol for *n*-qubit gates as one climbs up the hierarchy.

- Looking at the structure of the matchgate hierarchy using the **Majorana expansion** of unitaries into linear combinations of monomials of Majoranas.

- **Parafermions** are a generalisation of Majoranas which satisfy the generalised version of CAR: $c_\mu^d = \mathbb{I}$ and $c_\mu c_\nu = \omega c_\nu c_\mu$ for $\mu < \nu$ where $\omega = e^{2\pi i/d}$. It would be interesting to explore how the story carries over that case.

- In the Clifford hierarchy **semi-Clifford gates** (of the form $C_1 D C_2$ for C_i Clifford and *D* diagonal) admit a more efficient gate teleportation protocol. A semi-Clifford gate consumes an *n*-qubit state rather than 2*n*-qubit state.

- Clifford hierarchy plays essential role in **fault-tolerance** of quantum computing with stabiliser circuits. Can we find a fault tolerant model for matchgate circuits?

Thank you for your attention!

Questions?