Characterizing the projective-unitary invariant properties of a set of states, and applications

Work in progress

Ernesto F. Galvão
INL, UFF (on leave)

Daniel J. Brod
UFF, Brazil

Michał Oszmaniec
Center for Theoretical Physics of the Polish Academy of Sciences, Warsaw

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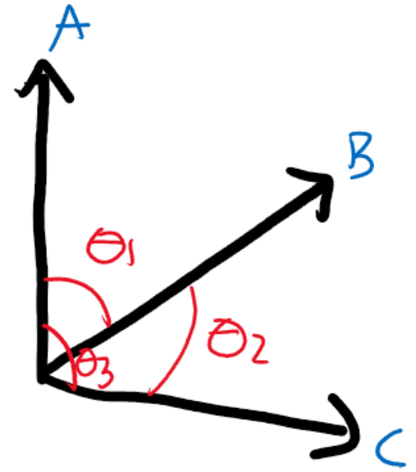
Outline

- What are the projective-unitary invariant properties of a set of states?
  - Geometrical in nature
  - Invariant under unitaries & gauge choices
- Mathematical characterization
- Measuring these properties
- Applications
  - Linear independence test
  - Basis-independent tests of: imaginarity, coherence
  - Characterizing multi-photon indistinguishability

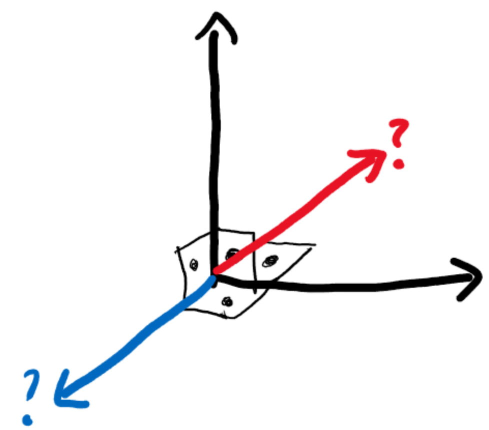
Projective-unitary invariant properties of a set of quantum states

- Properties that are invariant under:
  - unitary transformations
  - physically meaningless choice of global phases (gauge degree of freedom in QM)

- Geometrical in character – pertain to the **relative orientation of the states**



- For **real-valued vectors**, pairwise angles/inner products gives complete characterization (up to +- signs). Example:



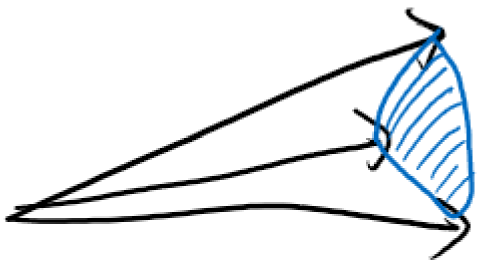
- Quantum-mechanical states are rays in a **complex** vector space – how does the characterization change in this case?

Bargmann invariants

- Mathematical result: projective-unitary invariant properties only depend on **k-vertex Bargmann invariants:**

$$\Delta_{ABC\cdots K} = \langle A|B\rangle\langle B|C\rangle\langle C|D\rangle\cdots\langle K|A\rangle$$

[Chien, Waldron. SIAM J. DISCRETE MATH. 30 (2), 976 (2016)]



Analogy: area of spherical triangle whose vertices are given by 3 unit vectors

- Bargmann invariants related to geometric phases, photonic indistinguishability

[Bargmann, J. Math. Phys. 5, 862 (1964)] [Simon, Mukunda, Phys. Rev. Lett. 70, 880 (1993)]

[A. J. Menssen, A. E. Jones, B. J. Metcalf, M. C. Tichy, S. Barz, W. S. Kolthammer, and I. A. Walmsley, Phys. Rev. Lett. 118, 153603 (2017)]

- For a set with *N* vectors, we may need to know up to *N*-vertex invariants
- If there’s no pair of orthogonal states, 3-vertex invariants are sufficient

Bargmann invariants

- **k-vertex Bargmann invariants:**

$$\Delta_{ABC\cdots K} = \langle A|B\rangle\langle B|C\rangle\langle C|D\rangle\cdots\langle K|A\rangle$$

- Notes:
  - Overlaps are special cases of 3-invariants (repeated indices)

$$\Delta_{AAB} = \langle A|A\rangle\langle A|B\rangle\langle B|A\rangle = |\langle A|B\rangle|^2\langle A|A\rangle = \Delta_{AB}$$

  - Phase of any single inner product is a gauge d.o.f., but cyclic products of inner products are <u>gauge-invariant</u>. See:

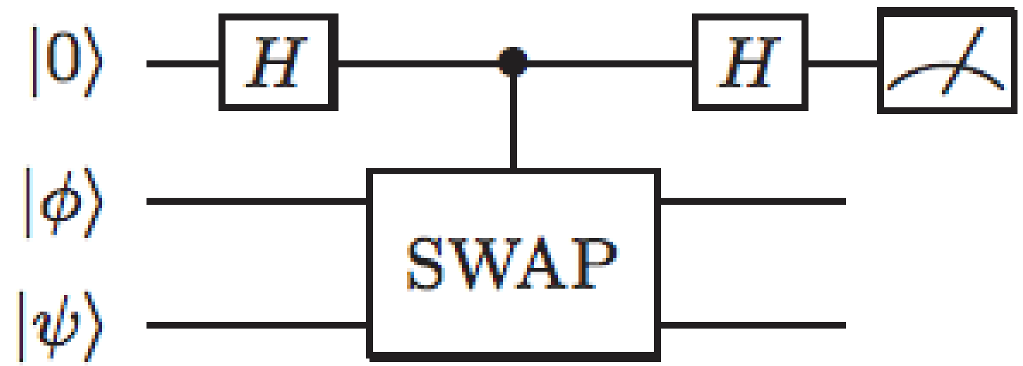
$$\Delta(e^{i\phi_A}U|A\rangle, e^{i\phi_B}U|B\rangle, e^{i\phi_C}U|C\rangle)$$
$$= e^{i(\phi_B-\phi_A)}\langle A|U^\dagger U|B\rangle \cdot e^{i(\phi_C-\phi_B)}\langle B|U^\dagger U|C\rangle \cdot e^{i(\phi_A-\phi_C)}\langle C|U^\dagger U|A\rangle$$
$$= \langle A|B\rangle\langle B|C\rangle\langle C|A\rangle = \Delta(|A\rangle, |B\rangle, |C\rangle)$$

Measuring Bargmann invariants: cycle test

[Brod, Oszmaniec, Galvão, in preparation]

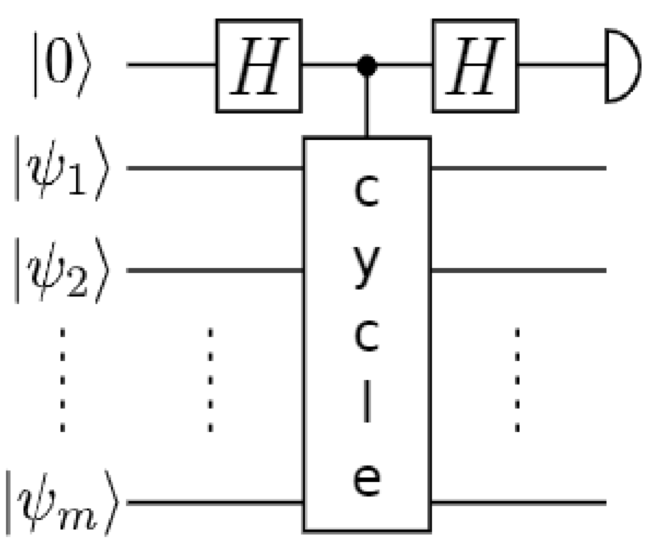
- Known result: SWAP test circuit enables estimate of the two-state overlap:

$$\Delta_{AB} = |\langle A|B\rangle|^2 = \mathrm{Tr}(\rho_A \rho_B)$$

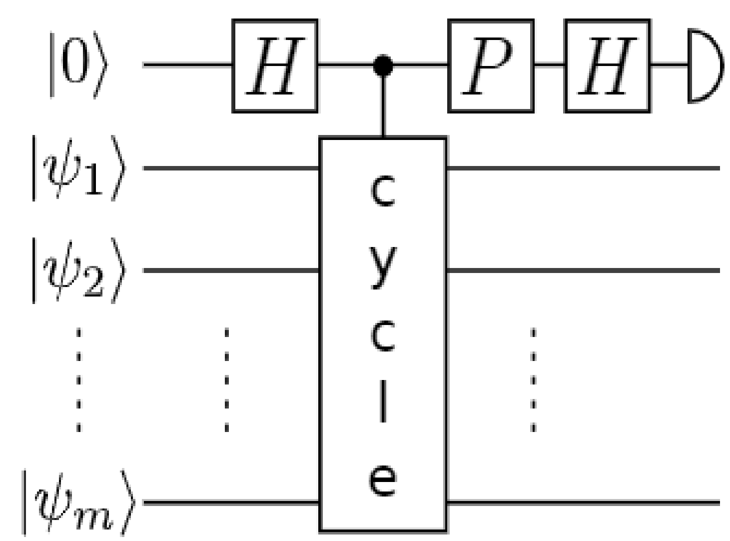


$$p(0) = \frac{1+|\langle\phi|\psi\rangle|^2}{2}$$

- Our result: **cycle test** circuits measure real and imaginary parts of any *m*-vertex Bargmann invariant:



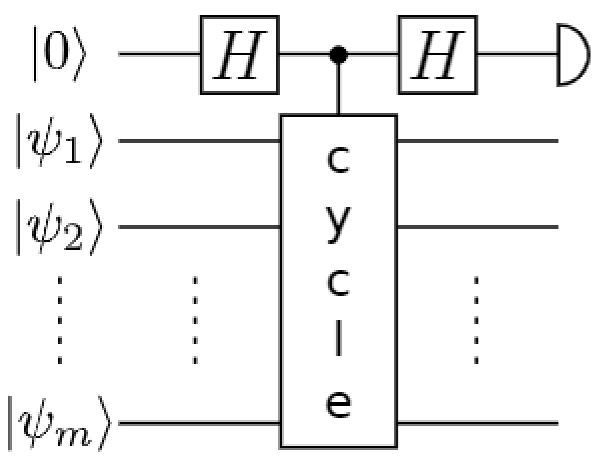
$$p(0) = \frac{1+\mathrm{Re}(\Delta_{12\ldots m})}{2}$$



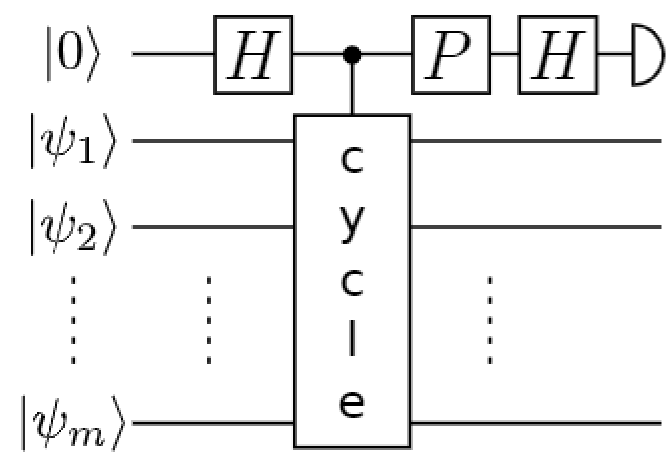
$$p(0) = \frac{1+\mathrm{Im}(\Delta_{12\ldots m})}{2}$$

Cycle test circuits

[Brod, Oszmaniec, Galvão, in preparation]

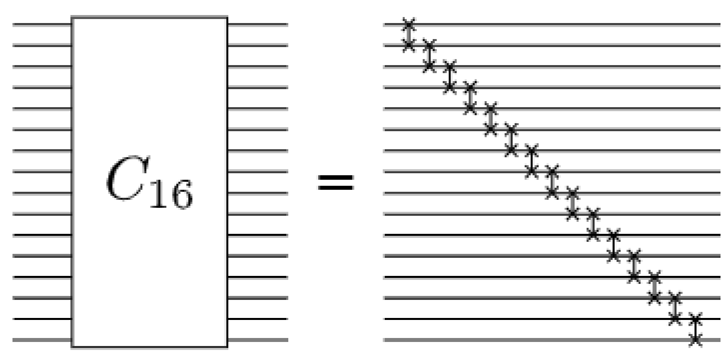


$$p(0) = \frac{1 + \mathrm{Re}(\Delta_{12\ldots m})}{2}$$

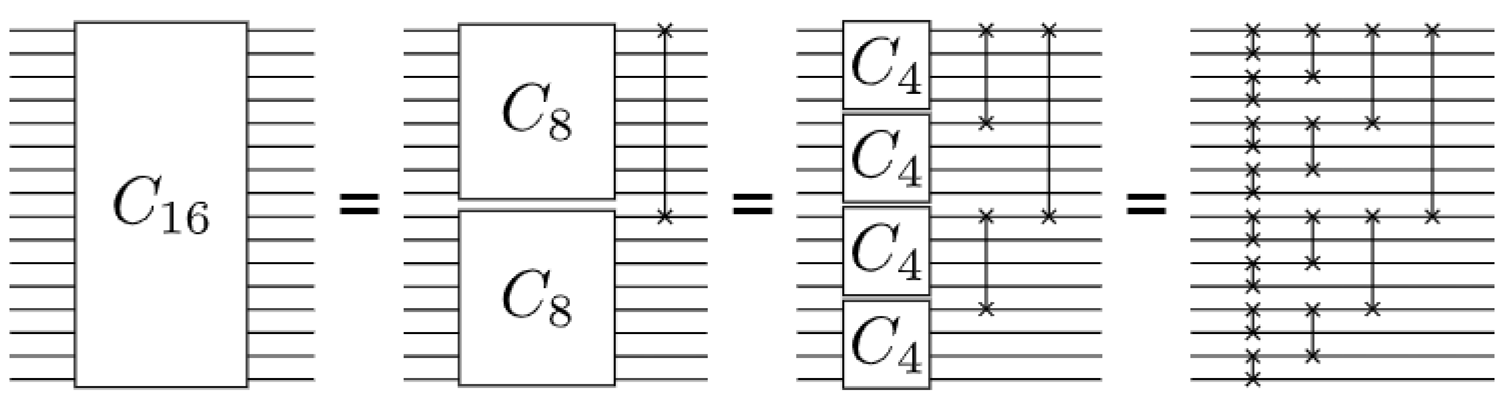


$$p(0) = \frac{1 + \mathrm{Im}(\Delta_{12\ldots m})}{2}$$

- The cycle test circuit can have linear depth (with local C-SWAP/Fredkin gates):



… or log-depth with non-local C-SWAP gates:

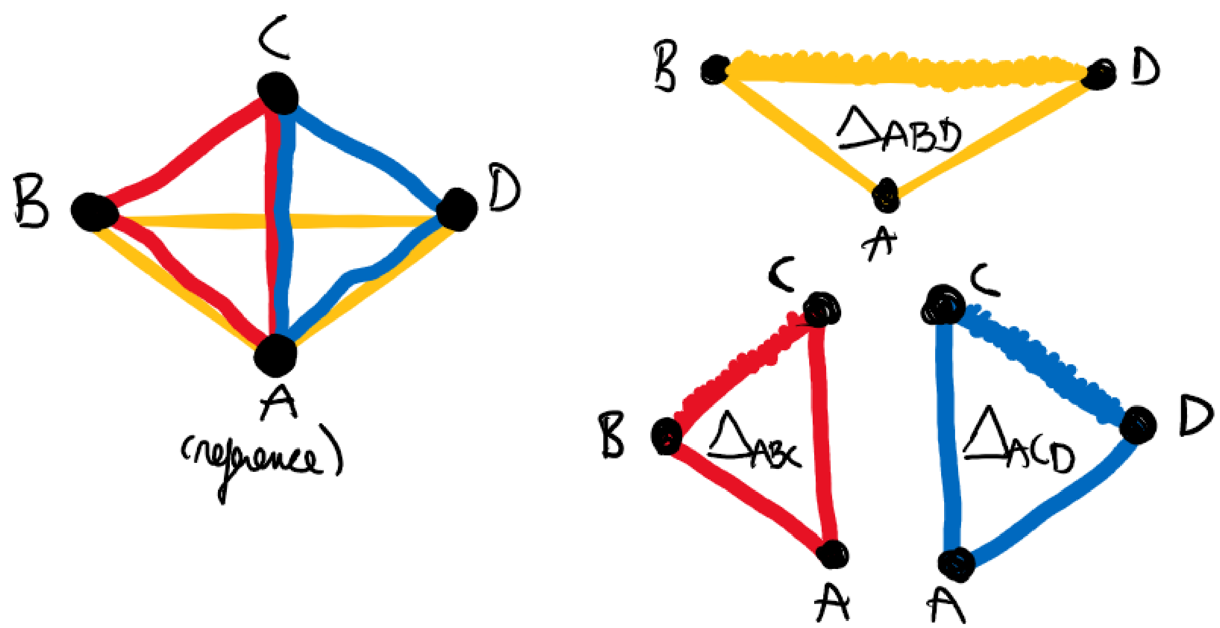


Gram matrix encodes all PU-invariant properties

[Brod, Oszmaniec, Galvão, in preparation]

- Complete knowledge of PU-invariant properties enables applications we will describe next.

- In the case of no null overlaps, characterization is simple:
  - Use all 3-invariants of a reference state with all pairs – “triangulating” the set



$$G = \begin{pmatrix} 1 & \sqrt{\Delta_{12}} & \sqrt{\Delta_{13}} & \sqrt{\Delta_{14}} \\ \sqrt{\Delta_{12}} & 1 & \sqrt{\Delta_{23}}e^{i\phi_{23}} & \sqrt{\Delta_{24}}e^{i\phi_{24}} \\ \sqrt{\Delta_{13}} & \sqrt{\Delta_{23}}e^{-i\phi_{23}} & 1 & \sqrt{\Delta_{34}}e^{i\phi_{34}} \\ \sqrt{\Delta_{14}} & \sqrt{\Delta_{24}}e^{-i\phi_{24}} & \sqrt{\Delta_{34}}e^{-i\phi_{34}} & 1 \end{pmatrix}$$

- Information is neatly encoded in the Gram matrix: $G_{ij} \equiv \langle i | j \rangle$
  - Phase of G_{kl} = phase of Δ_{rkl} reference state

- All parameters in G are gauge-invariant and can be measured with cycle tests

Application: basis-independent linear-independence test

[Brod, Oszmaniec, Galvão, in preparation]

- Recognizing dimension of spanning space useful e.g. in machine learning
- Volume of parallelepiped created by a set of vectors is V=sqrt(det(G))
- N states are **linearly independent iff det(G)>0**
  - Example with N=3:

$$G = \begin{pmatrix} 1 & |\langle\psi_1|\psi_2\rangle| & |\langle\psi_1|\psi_3\rangle| \\ |\langle\psi_1|\psi_2\rangle| & 1 & \langle\psi_2|\psi_3\rangle \\ |\langle\psi_1|\psi_3\rangle| & \langle\psi_2|\psi_3\rangle^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{\Delta_{12}} & \sqrt{\Delta_{13}} \\ \sqrt{\Delta_{12}} & 1 & \sqrt{\Delta_{23}}e^{i\phi_{23}} \\ \sqrt{\Delta_{13}} & \sqrt{\Delta_{23}}e^{-i\phi_{23}} & 1 \end{pmatrix}$$

$$\det(G) > 0 \Leftrightarrow 1 - (\Delta_{12} + \Delta_{13} + \Delta_{23}) + 2\sqrt{\Delta_{12}\Delta_{13}\Delta_{23}}\cos(\phi_{23}) > 0.$$

  - N=4:

$$G = \begin{pmatrix} 1 & \sqrt{\Delta_{12}} & \sqrt{\Delta_{13}} & \sqrt{\Delta_{14}} \\ \sqrt{\Delta_{12}} & 1 & \sqrt{\Delta_{23}}e^{i\phi_{23}} & \sqrt{\Delta_{24}}e^{i\phi_{24}} \\ \sqrt{\Delta_{13}} & \sqrt{\Delta_{23}}e^{-i\phi_{23}} & 1 & \sqrt{\Delta_{34}}e^{i\phi_{34}} \\ \sqrt{\Delta_{14}} & \sqrt{\Delta_{24}}e^{-i\phi_{24}} & \sqrt{\Delta_{34}}e^{-i\phi_{34}} & 1 \end{pmatrix}$$

$$\det(G) > 0 \Leftrightarrow 1 - (\Delta_{12} + \Delta_{13} + \Delta_{14} + \Delta_{23} + \Delta_{24} + \Delta_{34}) + (\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23}) + \quad \text{(E3)}$$

$$+2\left[\sqrt{\Delta_{12}\Delta_{13}\Delta_{23}}\cos(\phi_{23}) + \sqrt{\Delta_{12}\Delta_{14}\Delta_{24}}\cos(\phi_{24}) + \sqrt{\Delta_{13}\Delta_{14}\Delta_{34}}\cos(\phi_{34}) + \sqrt{\Delta_{23}\Delta_{24}\Delta_{34}}\cos(\phi_{34} - \phi_{24})\right] \quad \text{(E4)}$$

$$-2\left[\sqrt{\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{34}}\cos(\phi_{34}) + \sqrt{\Delta_{12}\Delta_{14}\Delta_{23}\Delta_{34}}\cos(\phi_{34} - \phi_{23}) + \sqrt{\Delta_{13}\Delta_{14}\Delta_{23}\Delta_{24}}\cos(\phi_{24} - \phi_{23})\right] > 0 \quad \text{(E5)}$$

Application: imaginarity witness

[Brod, Oszmaniec, Galvão, in preparation]

- "Imaginarity": resource provided by complex numbers in quantum theory

- Recent results show imaginarity is unavoidable in QT (for certain Bell nonlocality scenarios)

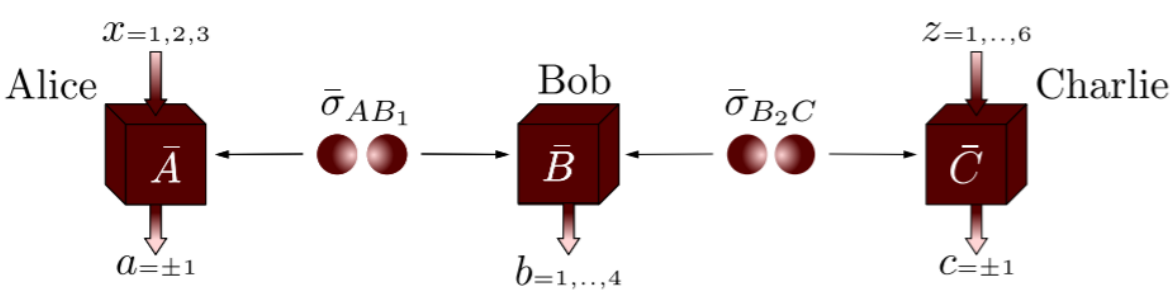
Operational Resource Theory of Imaginarity

Kang-Da Wu, Tulja Varun Kondra, Swapan Rana, Carlo Maria Scandolo, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo, and Alexander Streltsov

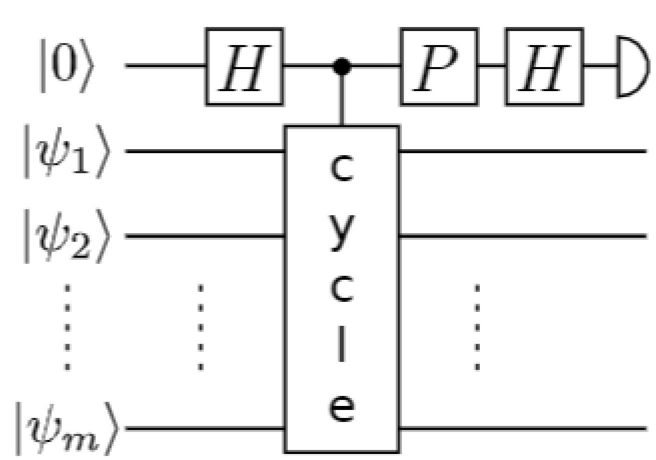
Phys. Rev. Lett. **126**, 090401 – Published 1 March 2021

  - Quantum physics needs complex numbers. M.-O. Renou et al., arXiv:2101.10873v1 (theory)

  - Ruling out real-number description of quantum mechanics- arXiv:arXiv:2103.08123v1 (Experiment by Jian-Wei Pan group)



- Cycle test can be used to witness imaginarity:



$$p(0) = \frac{1 + \mathrm{Im}(\Delta_{12\ldots m})}{2}$$

Application: basis-independent coherence witness

[Brod, Oszmaniec, Galvão, in preparation]

- Previously: basis-independent coherence witness using overlaps:

  [Galvão, Brod, Phys. Rev. A 101, 062110 (2020)] [Giordani et al. Phys. Rev. Res. , 3, 023031 (2021)]

- We can now do the same, but with the complete PU characterization of a set of states

- Imaginarity $\Rightarrow$ coherence

- Classical, incoherent states: diagonal states in a fixed reference basis. Overlaps and higher-order invariants have an interpretation as a probability. Example:

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 \\ 0 & \rho_{22} & 0 \\ 0 & 0 & \rho_{33} \end{pmatrix} \qquad \sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

Then the two-state overlap

$$r_{\rho\sigma} = Tr(\rho\sigma) = \sum_i \rho_{ii}\sigma_{ii}$$

= probability of getting same outcomes from independent measurements of reference observables on the two states

- For 3 states: 3-invariant must be real, and besides:

$$\begin{aligned}
&\Delta_{123} \geq 0, \\
&\Delta_{123} \leq \Delta_{ij}, \text{for } ij = 12, 13, 23, \\
&\Delta_{123} \geq \frac{1}{2}\left(\Delta_{12} + \Delta_{13} + \Delta_{23} - 1\right)
\end{aligned}$$

Application: characterizing multi-photon indistinguishability

[Brod, Oszmaniec, Galvão, in preparation]

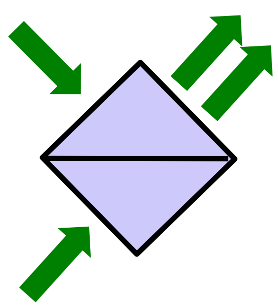
- Previously: overlaps for multi-photon indistinguishability tests

Brod et al., Witnessing genuine multiphoton indistinguishability. *Phys. Rev. Lett.* 122, 063602 (2019)

Giordani et al., Experimental quantification of genuine four-photon indistinguishability. *N. J. Phys.* 22 043001 (2020)

- Higher order invariants may help in the certification of multi-photon indistinguishability

- Example: Single 3-vertex invariant gives lower bound for the 3 overlaps/HOM visibilities:

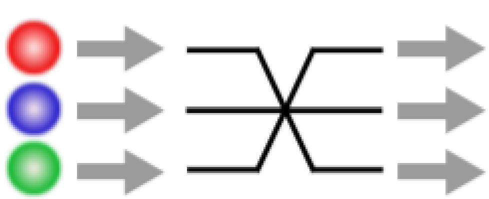


$$p_b^{AB} = \frac{1+|\langle A|B\rangle|^2}{2}$$

$$\Delta_{ABC} = \langle A|B\rangle\langle B|C\rangle\langle C|A\rangle$$

$$\Rightarrow \Delta_{ij} \geq |\Delta_{ABC}|^2, \; ij = \begin{cases} AB \\ AC \\ BC \end{cases}$$

- Higher-order invariants can be directly measured using multimode interferometers, e.g. 3-mode balanced tritters (QFT):



$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{4\pi}{3}} & e^{i\frac{2\pi}{3}} \\ 1 & e^{i\frac{2\pi}{3}} & e^{i\frac{4\pi}{3}} \end{pmatrix}$$

$$P_{120} = P_{012} = P_{201} = \frac{1}{9}(1 - 2r_{12}r_{23}r_{31}\cos(\varphi + \pi/3))$$

$$P_{021} = P_{210} = P_{102} = \frac{1}{9}(1 - 2r_{12}r_{23}r_{31}\cos(\varphi - \pi/3))$$

[Menssen et al., PRL **118**, 153603 (2017)]

Conclusion

[Brod, Oszmaniec, Galvão, in preparation]

- k-vertex Bargmann invariants encode all physically sound, relational information about the geometrical arrangement of a set of quantum states

- Cycle tests can be used to measure all invariants

- We've found many applications of these results, generalizing uses of overlaps:

  - Basis-independent coherence and imaginarity witnesses
  - Tests for linear independence
  - Multiphoton indistinguishability

- What other algorithmic/cryptographic applications can we propose?

Thank you for your attention!