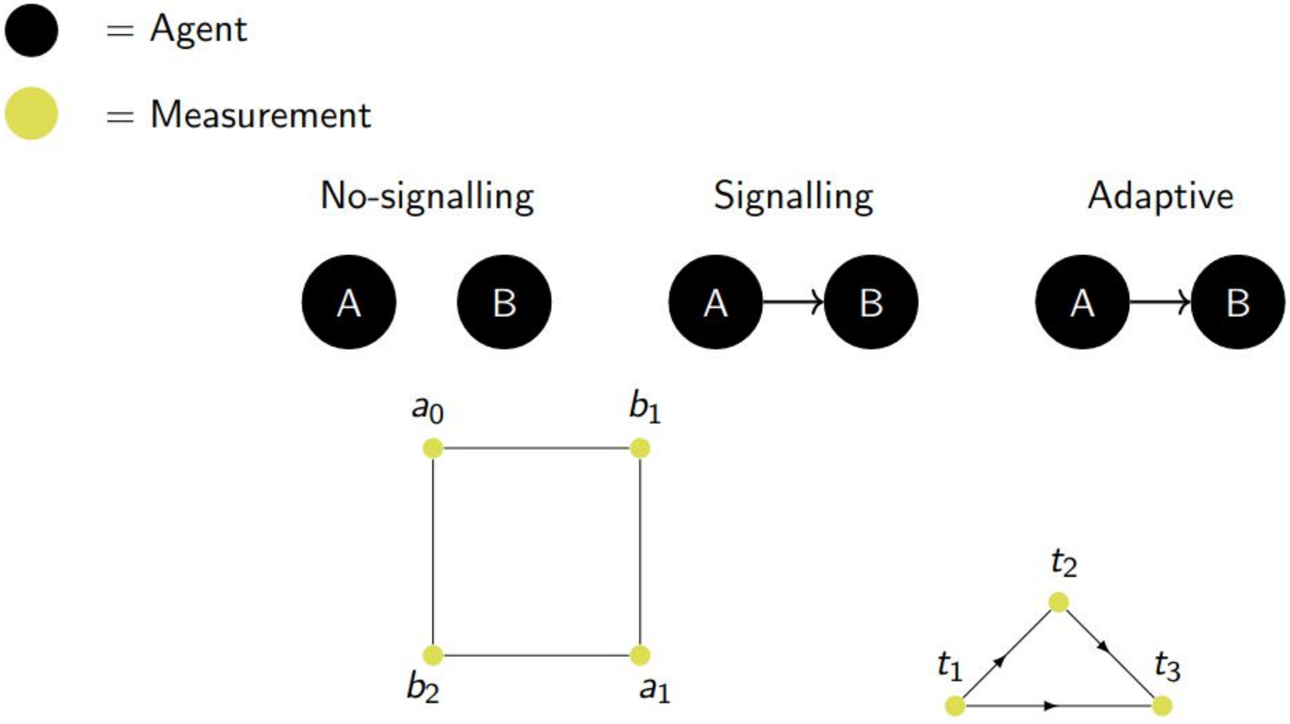
Causal Models of Contextuality

Amy Searle (joint with Samson Abramsky and Rui Soares Barbosa)

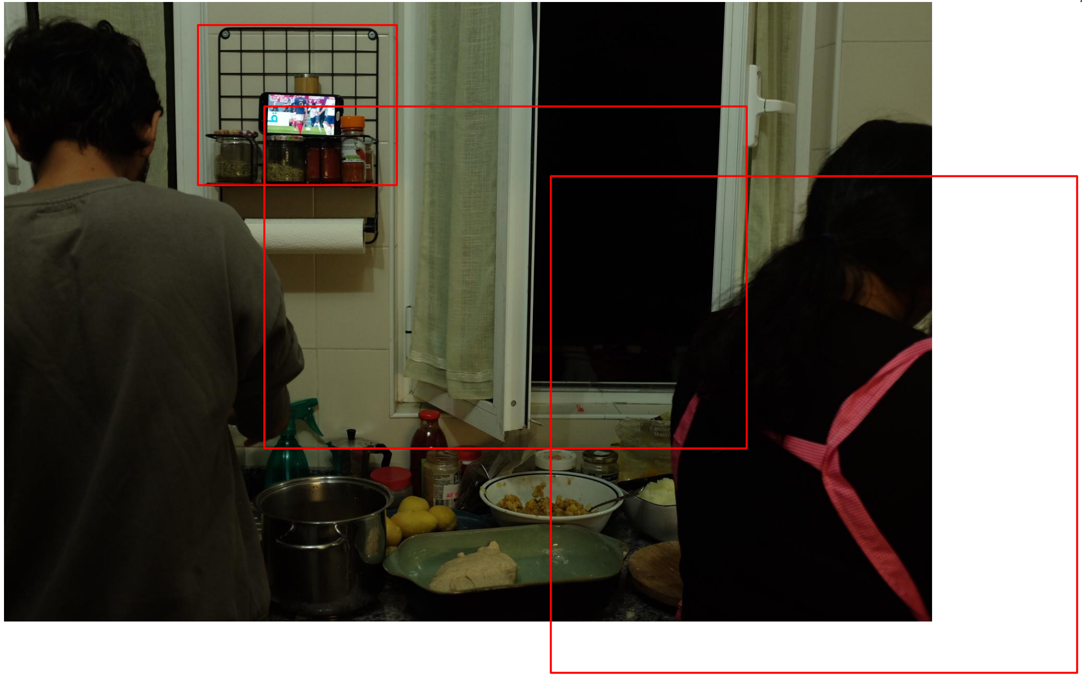
Overview

I. Motivation
II. Sheaf contextuality
III. Game Trees
IV. Causal Contextuality

Motivation


= Agent
= Measurement
No-signalling
Signalling
Adaptive
A
B
A
B
A
B
a_0
b_1
b_2
a_1
t_2
t_1
t_3

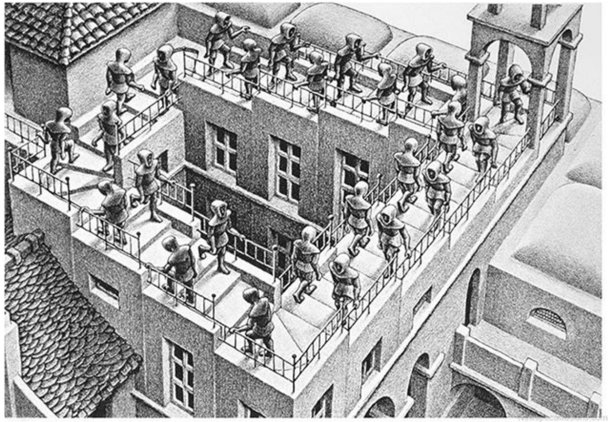
What is a Sheaf?



A sheaf assigns data to parts of a space such that

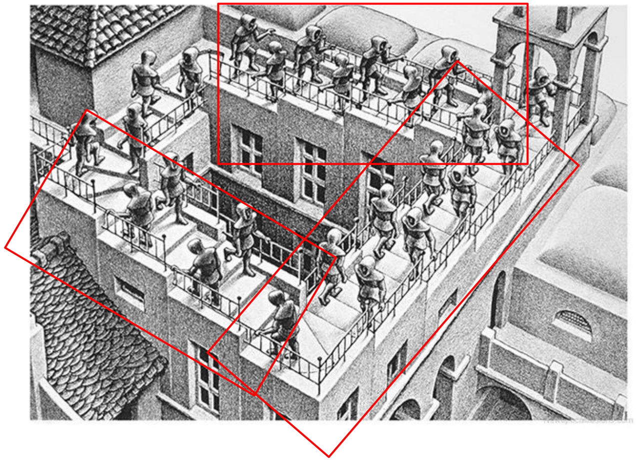
1. The data is consistent on overlaps
2. There exists a global set of data which restricts down to give the local ‘snapshots’

Contextuality



Escher, Ascending and Descending

Contextuality



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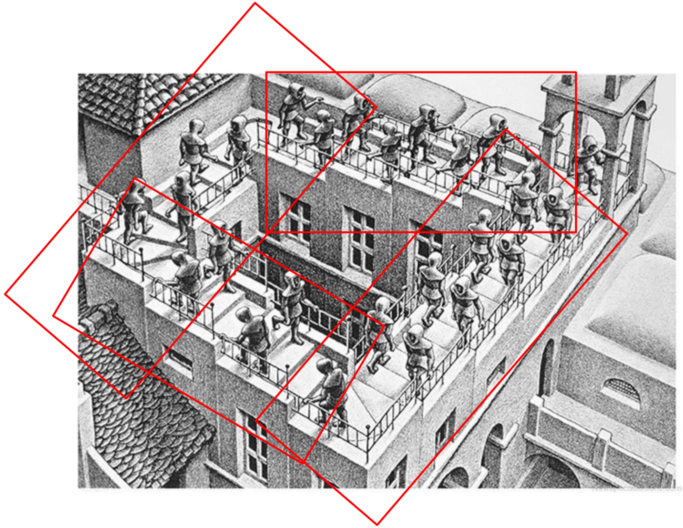
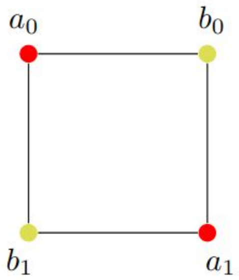


Escher, Ascending and Descending

Sheaf Contextuality: Measurement Scenarios

Measurement Scenario $\langle X, \{O_i\}_{i \in X}, \Sigma \rangle$

1. X : set of measurements
2. O_i: outcome set for each measurement
3. Σ : compatibility of measurements



Sheaf Contextuality: Behaviours

Deterministic Systems

$$\{a_0, b_0\} \longrightarrow \boxed{S} \longrightarrow s :: \{a_0 \mapsto 0, b_0 \mapsto 0\}$$

Sheaf Contextuality: Behaviours

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Probabilistic Systems

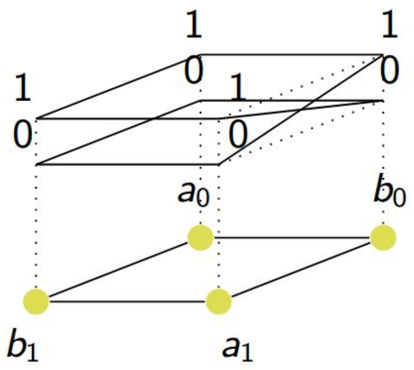
$$\{a_0, b_0\} \longrightarrow \boxed{\mathcal{S}} \longrightarrow$$

Prob. distribution on

$$\left[\begin{array}{l} \{s_0 :: \{a_0 \mapsto 0, b_0 \mapsto 0\}, \\ \ s_1 :: \{a_0 \mapsto 0, b_0 \mapsto 1\}, \\ \ s_2 :: \{a_0 \mapsto 1, b_0 \mapsto 0\}, \\ \ s_3 :: \{a_0 \mapsto 1, b_0 \mapsto 1\}\} \end{array}\right]$$

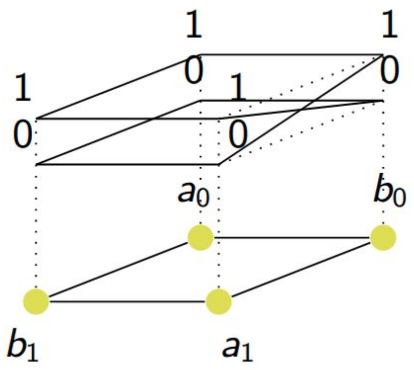
Example: $p(s_0) = 0.2, \quad p(s_1) = 0.1, \quad p(s_2) = 0.7, \quad p(s_3) = 0$

The Bundle Diagram



How to think of it: assigning (observed) data to spaces.

The Bundle Diagram



A choice of probability distribution for each face of the simplicial complex is called an **empirical model**

How to think of it: assigning (observed) data to spaces.

Classical behaviours

What would we expect to happen classically?

1. No-Signalling:

$$\sum_{j \in \{0,1\}} p(a_0 \mapsto i, b_0 \mapsto j) = \sum_{j \in \{0,1\}} p(a_0 \mapsto i, b_1 \mapsto j)$$

Classical behaviours

What would we expect to happen classically?

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2. Noncontextuality:

Hidden variable model: Before measurement begins, a definite value exists for all measurements and this does not change across space/time.

$\longleftrightarrow$ Prob. distribution on

$$\left[\begin{array}{l} \{h_0 :: a_0 \mapsto 0, a_1 \mapsto 0, b_0 \mapsto 0, b_1 \mapsto 0, \\ h_1 :: a_0 \mapsto 0, a_1 \mapsto 0, b_0 \mapsto 0, b_1 \mapsto 1, \\ \qquad \ldots \\ h_{15} :: a_0 \mapsto 1, a_1 \mapsto 1, b_0 \mapsto 1, b_1 \mapsto 1\} \end{array}\right.$$

Contextuality

Given an empirical model, is there a distribution on global sections which marginalises to reproduce the empirical model data?

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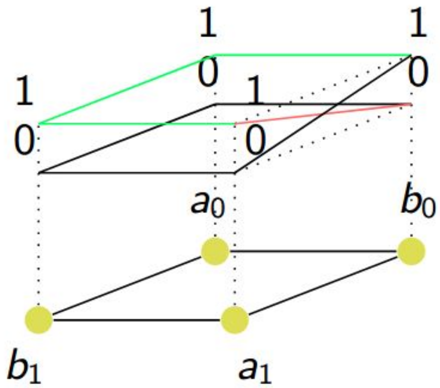
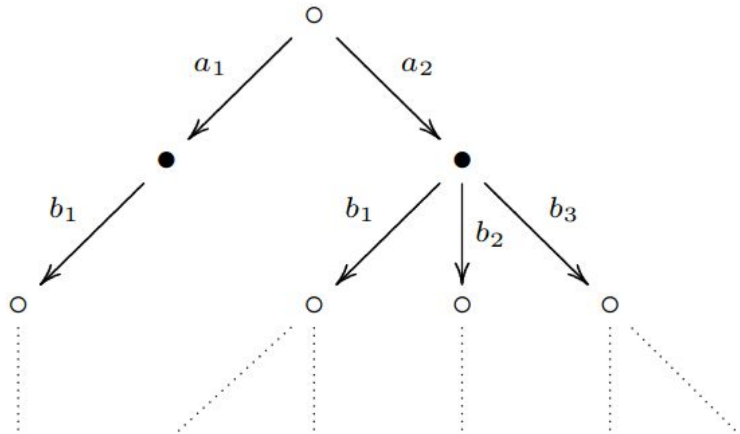
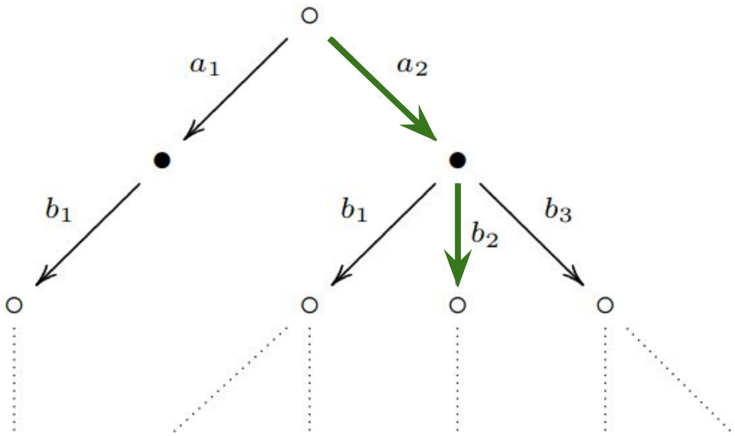


Game Semantics

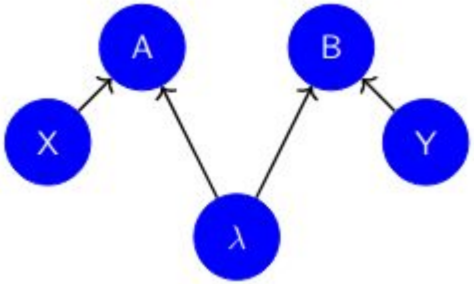
- Player (P), Opponent (O)
- Each player makes moves alternatively
- Rules of the game determine how each player can move
- 

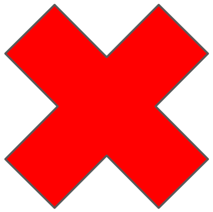
Example of a Strategy



○	●	○
a_2	b_2	…

N-Strategies and E-Strategies





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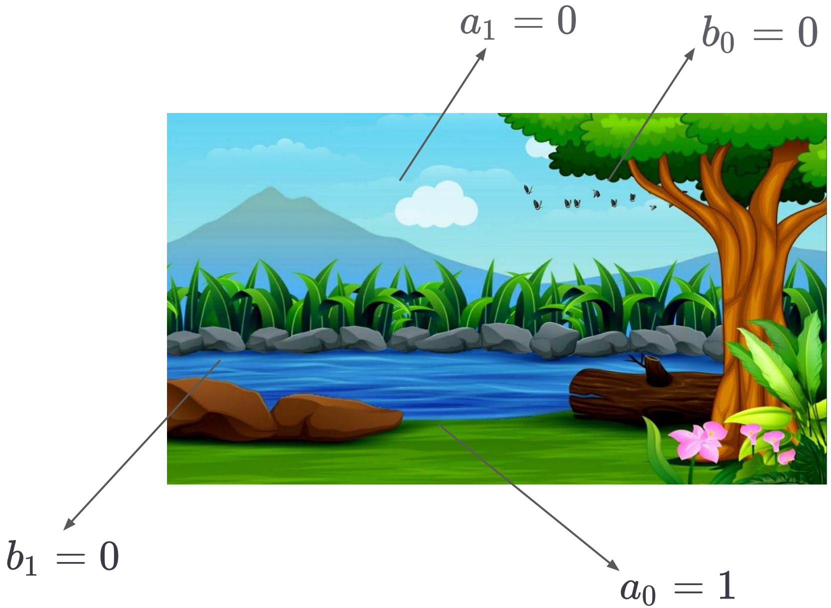


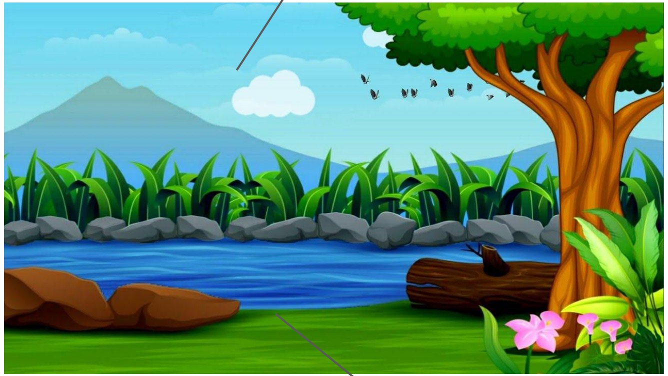
$$a_1 ?(a_1 = 0) b_0 ?(b_0 = 0)$$

New types of N-strategies: Alice signals to Bob

measurement

T=0

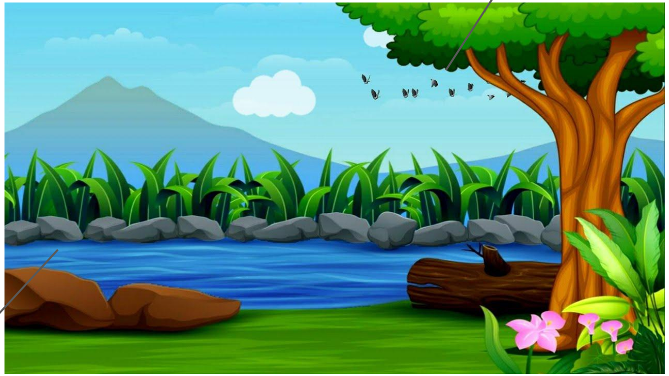
$a_1 = 0$



$a_0 = 1$

T=1

$b_0 = 0$



$b_1 = 0$

Causal Contextuality: Causal Measurement Scenarios

Causal Measurement Scenario $\mathcal{M} = \langle X, \{O_i\}_{i \in X}, \vdash \rangle$

1. X : set of measurements
2. O_i: outcome set for each measurements

   A tuple (m, o_m) is called an **event**.

3. $\vdash$: Enabling relation. $s \vdash m$ expresses that s must happen before m can happen

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Examples: Alice and Bob (no-signalling)

$X = \{a_0, a_1, b_0, b_1\} \quad O = \{0,1\} \quad \emptyset \vdash a_0, \quad \emptyset \vdash a_1, \quad \emptyset \vdash b_0, \quad \emptyset \vdash b_1$

Alice and Bob (A -> B)

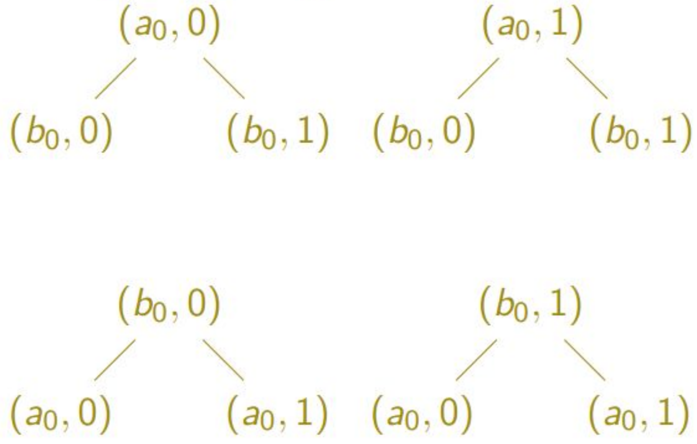
$X = \{a_0, a_1, b_0, b_1\} \quad O = \{0,1\} \quad \emptyset \vdash a_0, \quad \emptyset \vdash a_1,$

$\{a_j = i\} \vdash b_0, \quad \{a_j = i\} \vdash b_1$

Histories

Given $\mathcal{M} = \langle X, \{O_i\}_{i \in X}, \vdash \rangle$ we can generate the set of histories $H(\mathcal{M})$:

Non-signalling setups: over $\{a_0, b_0\}$



Histories of a Measurement Scenario

Given $\mathcal{M} = \langle X, \{O_i\}_{i \in X}, \vdash \rangle$ we can generate the set of histories $H(\mathcal{M})$:

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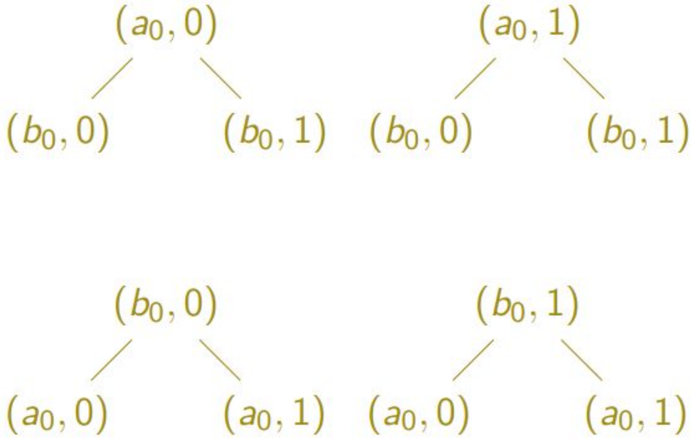
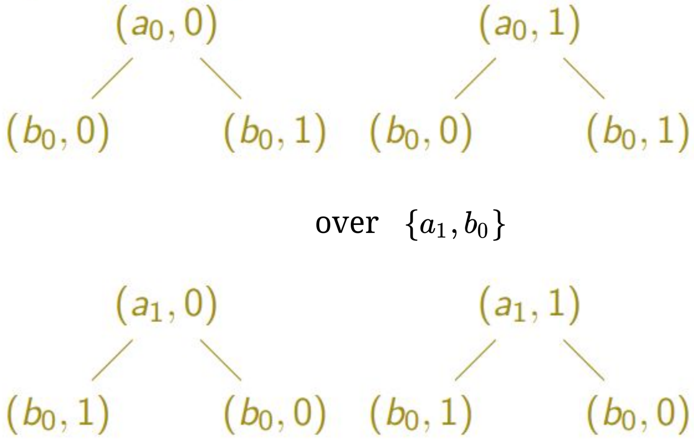


Signalling setups: over $\{a_0, b_0\}$

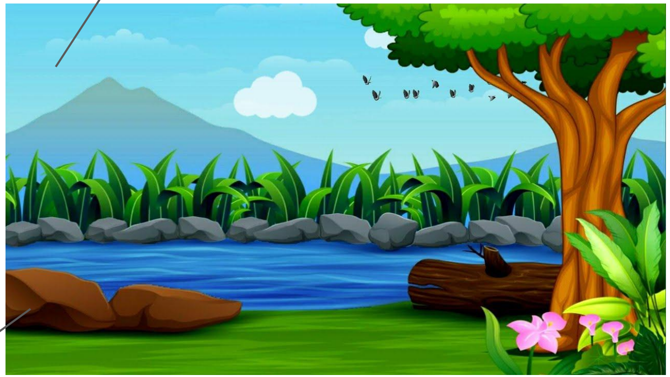
over $\{a_1, b_0\}$



N- Strategies

$a_1 = 0$

$\xrightarrow{T}$



$a_0 = 1$

An N-strategy $\sigma \subseteq \mathcal{H}(\mathcal{M})$ is:

1. Downwards closed.

Non-signalling A to B

$$\{(a_0,0),(b_0,0)\} \in \sigma \Rightarrow \{(a_0,0)\} \in \sigma \text{ and } \{(b_0,0)\} \in \sigma$$

Signalling A to B

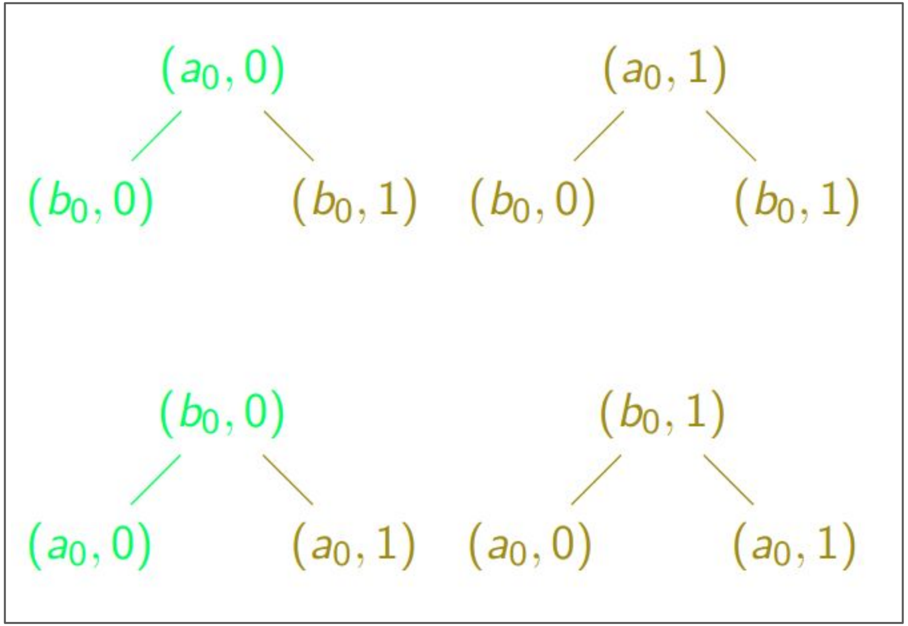
$$\{(a_0,0),(b_0,0)\} \in \sigma \Rightarrow \{(a_0,0)\} \in \sigma$$

2. Deterministic and total.

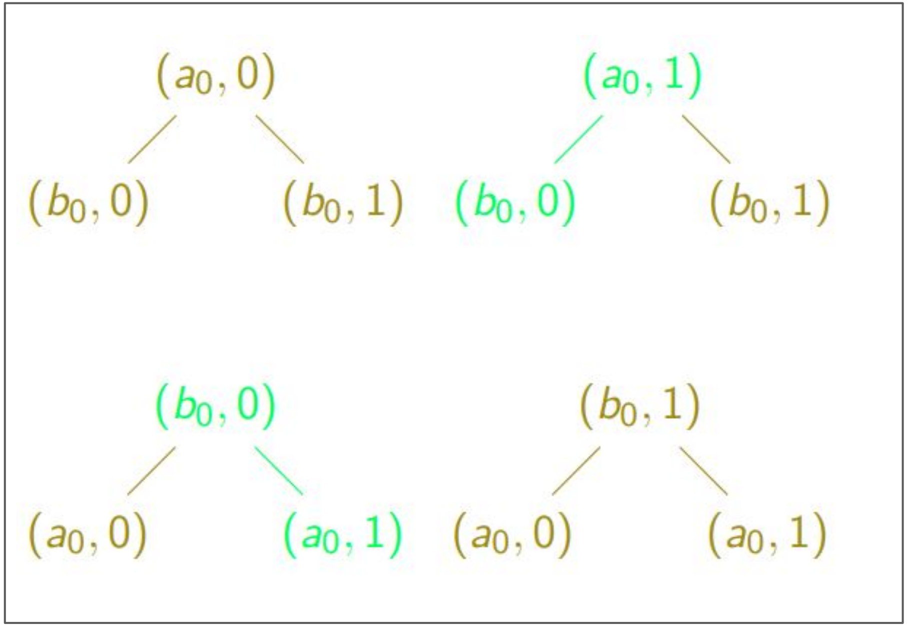
If $\{(a_0,0)\} \in \sigma$ then there exists a unique $j \in \{0,1\}$ such that $\{(a_0,0),(b_0,j)\} \in \sigma$

Example N-strategies for CHSH

Strategy σ_1



Strategy σ_2



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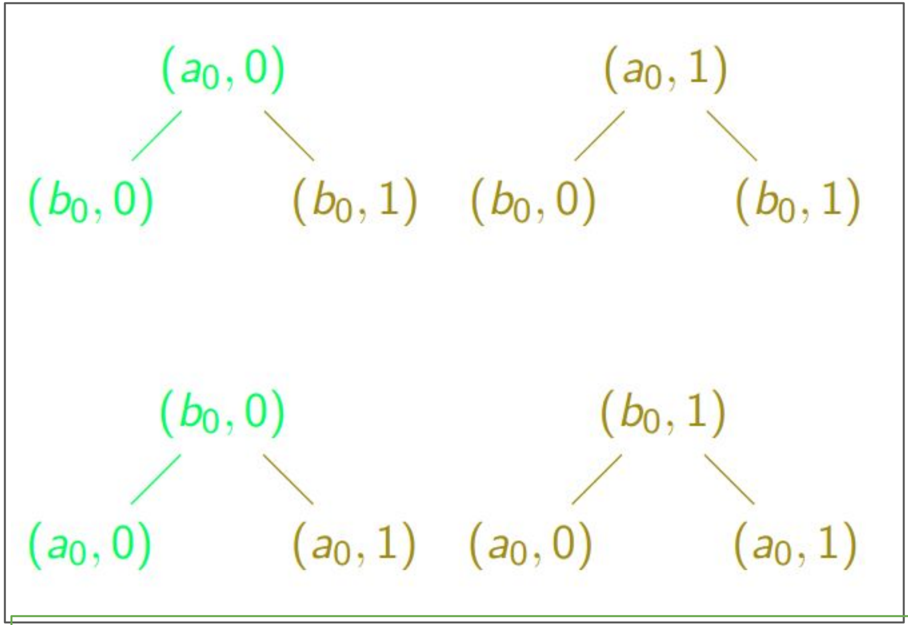
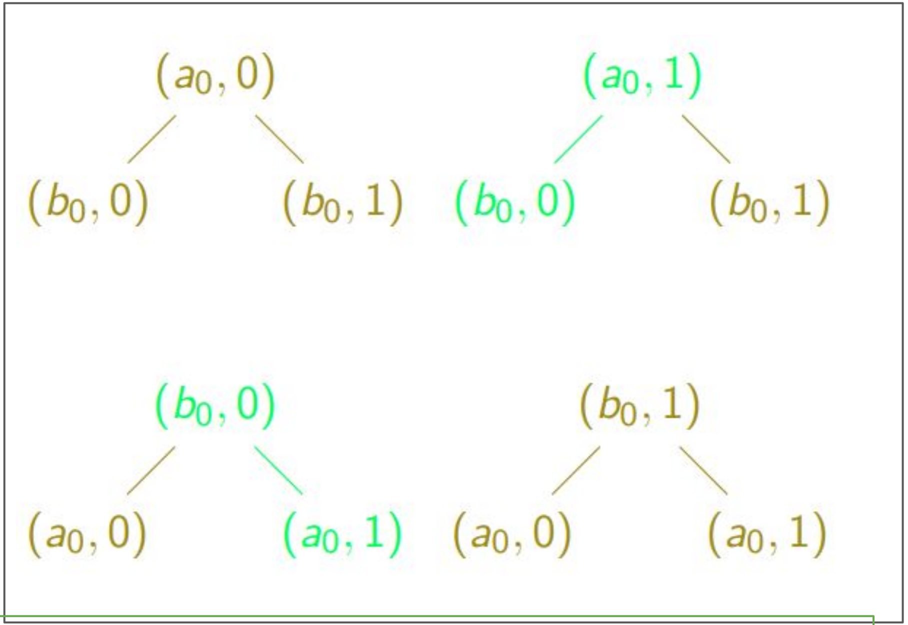


Strategy σ_2



Proposition: N-strategies are isomorphic to hidden variable models for no-signalling strategies.

Example N-strategies for CHSH

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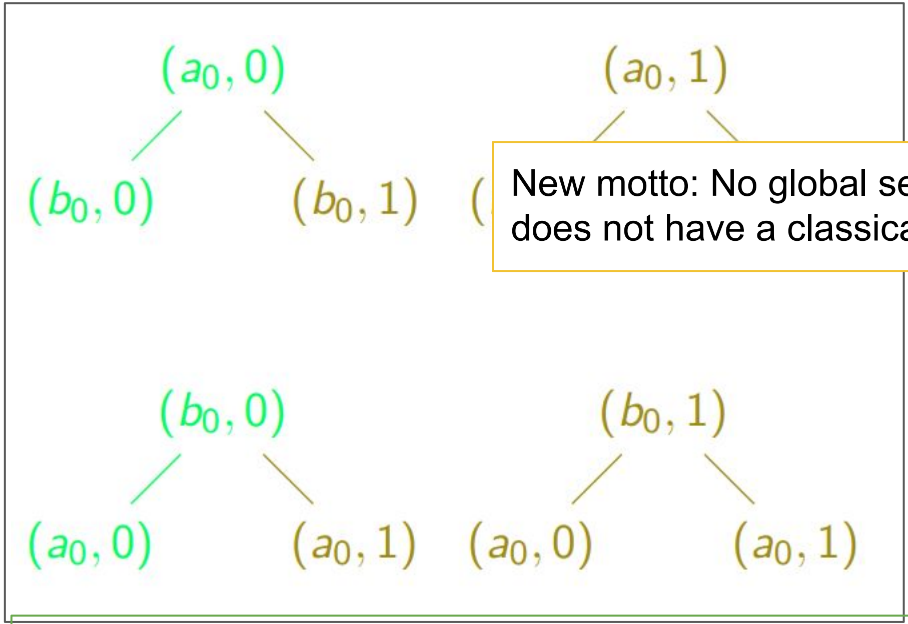
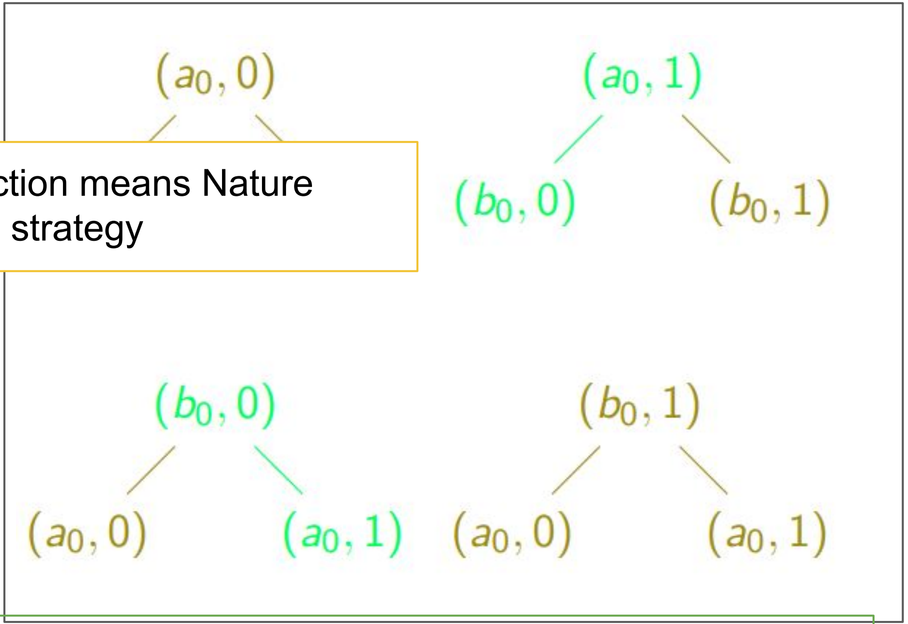


Strategy σ_2

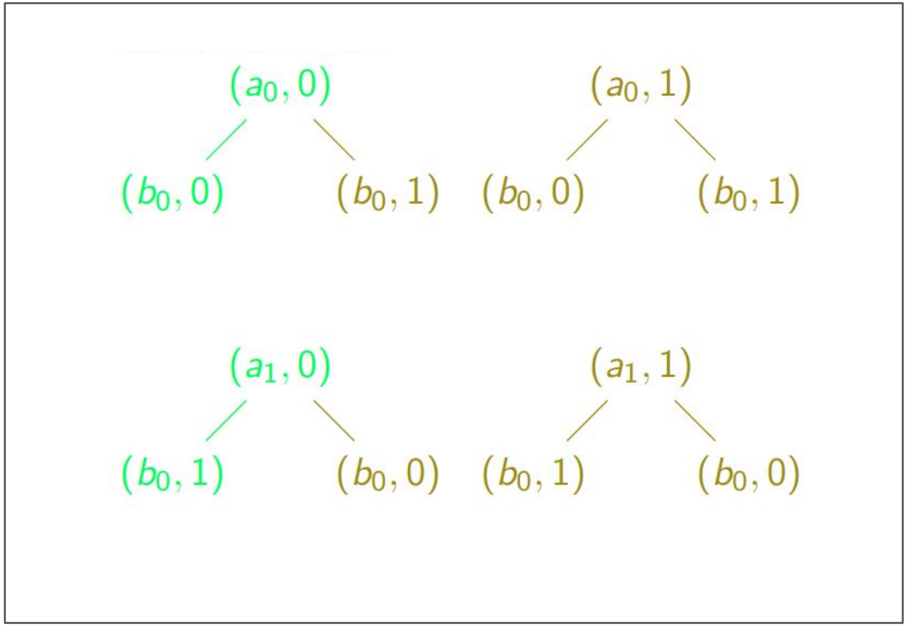


New motto: No global section means Nature does not have a classical strategy

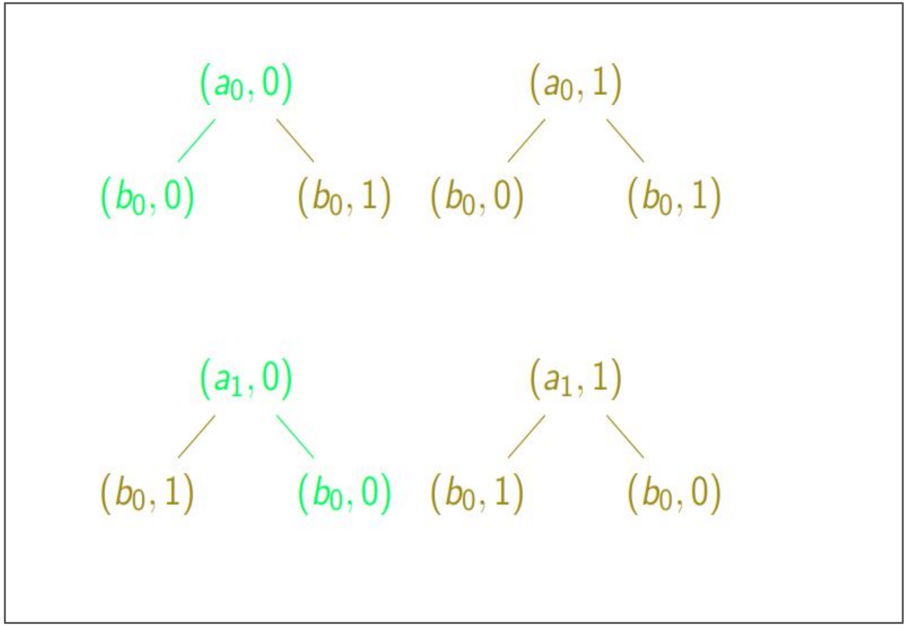
Proposition: N-strategies are isomorphic to hidden variable models for no-signalling strategies.

Example Strategies for Signalling A-> B

Strategy σ_1



Strategy σ_2



E- Strategies

a_1 ?

An E-strategy $\tau \subseteq H(\mathcal{M})$ is deterministic and total: if $s \in \tau \backslash H_{\max}(\mathcal{M})$ then there is a unique x such that $s \cup \{(x, o)\} \in \tau$ for some o. Moreover, $s \cup \{(x, o)\} \in \tau$ for all $o \in O_x$

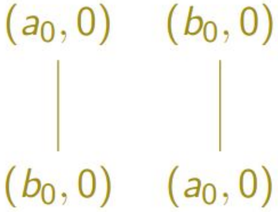
- $(a_0, 0)$
  - $(b_0, 0)$
  - $(b_0, 1)$
- $(a_0, 1)$
  - $(b_0, 0)$
  - $(b_0, 1)$

- $(b_0, 0)$
  - $(a_0, 0)$
  - $(a_0, 1)$
- $(b_0, 1)$
  - $(a_0, 0)$
  - $(a_0, 1)$

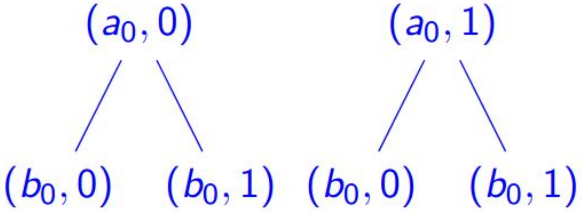
Playing off E-strategies against N-strategies

Restricted to the context $\{a_0, b_0\}$:

N-strategy σ:



E-strategy τ:



The result is a deterministic sequence $\langle \sigma || \tau \rangle = (a_0 = 0)(b_0 = 0)$

Empirical Models

An empirical model consists of

1. A distribution on each context of N-strategies over that context
2. A distribution on each context of E-strategies over that context

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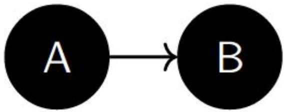
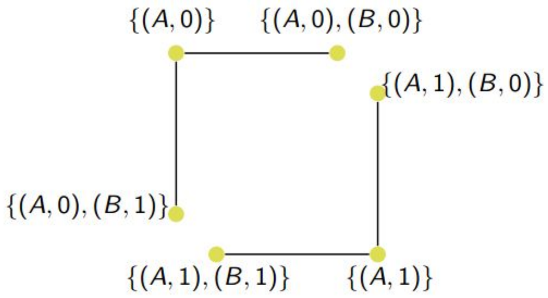
2. A distribution on each context of E-strategies over that context

The result is a distribution on N-strategy–E-strategy pairs for each context.

Contextuality now says something even stronger: There exists no *adaptive* classical strategy for nature (consistent with the causal constraints of the setup) which reproduces the observed statistics.

Mappings between Empirical Models





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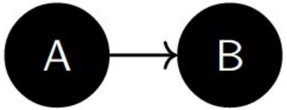
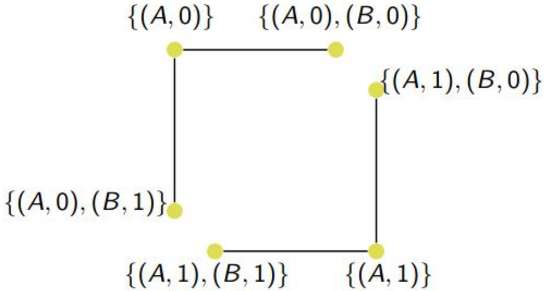
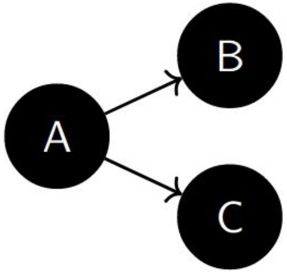
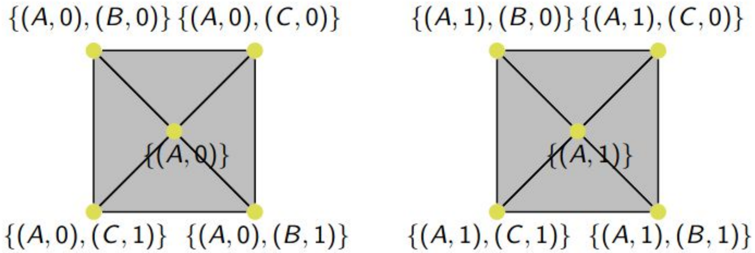
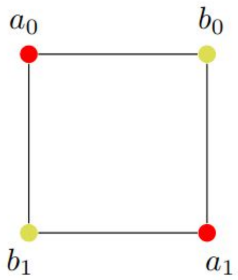








A small detour: Vorobev's theorem



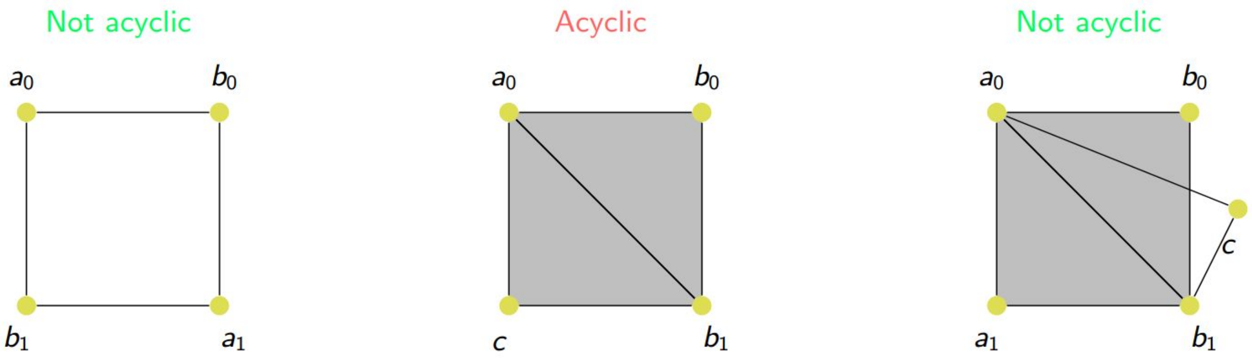
Recipe for attaching data to a space:

1. Define the values each vertex can take on (e.g. 0 or 1)
2. For each face of the simplicial complex, define a probability distribution on mappings on that face.
3. Ensure marginalisation of prob. Distributions agree on overlap

Question: Which simplicial complexes always produce data which is non-contextual?

Answer: The acyclic ones!

A small detour: Vorobev's Theorem



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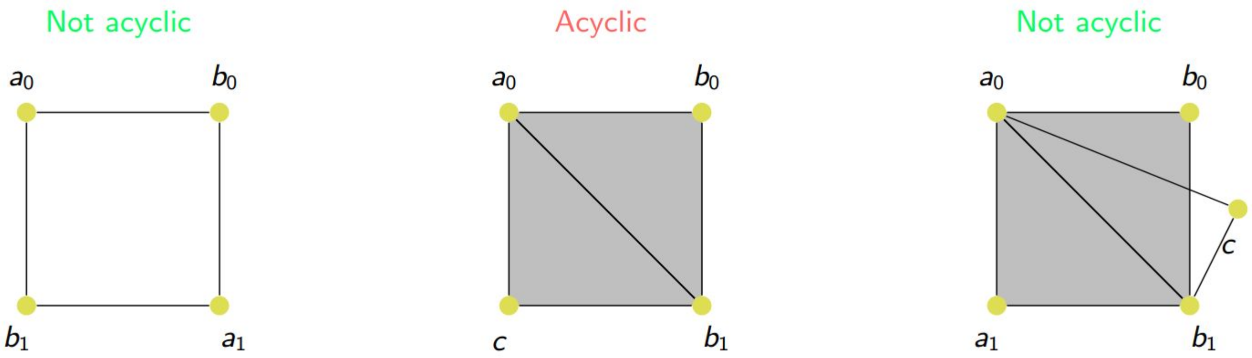
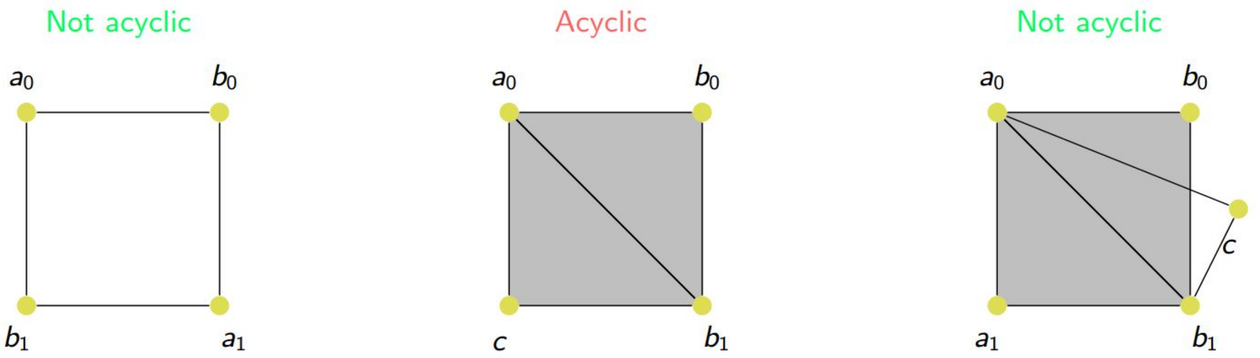


Every empirical model defined on top of the middle simplicial complex is non-contextual.

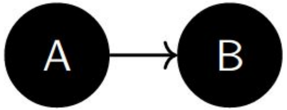
A small detour: Vorobev's Theorem



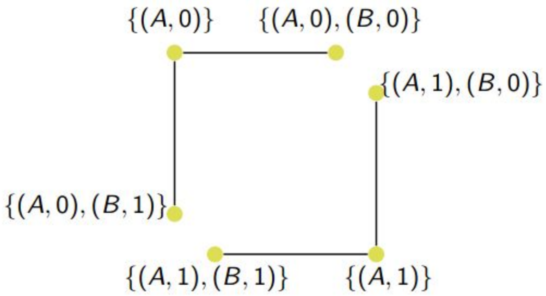
Every empirical model defined on top of the middle simplicial complex is non-contextual.

We can always construct a contextual empirical model on top of the left most and right most simplicial complexes.

Application of Vorob’ev’s Theorem to GP Scenarios



Preserves
contextuality



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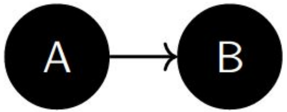

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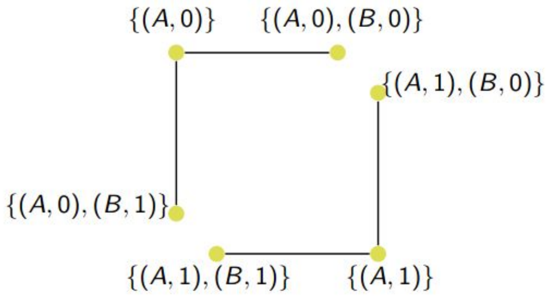

No contextual empirical models

Application of Vorob’ev’s Theorem to GP Scenarios

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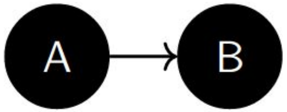
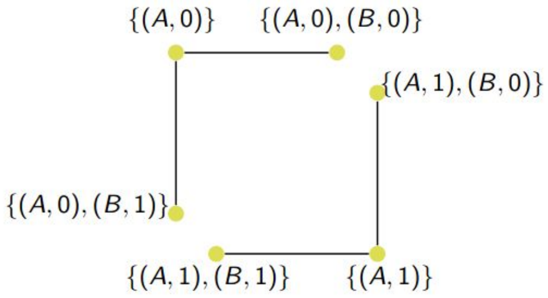


Preserves contextuality →



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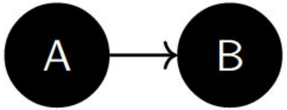

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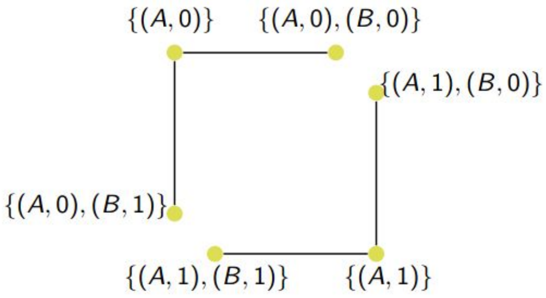

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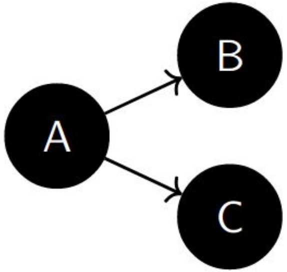
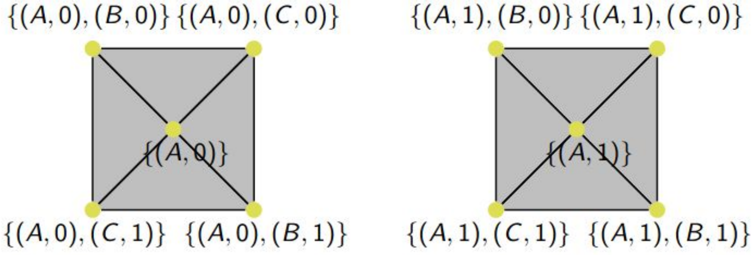
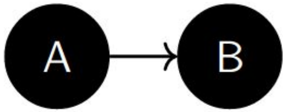


Preserves contextuality

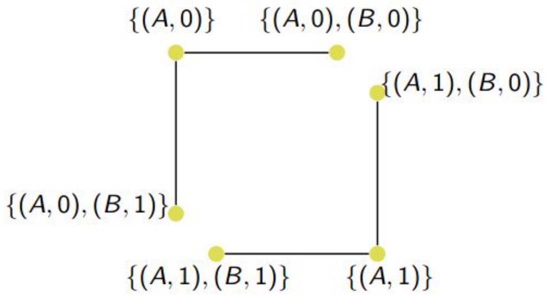


Application of Vorob’ev’s Theorem to GP Scenarios

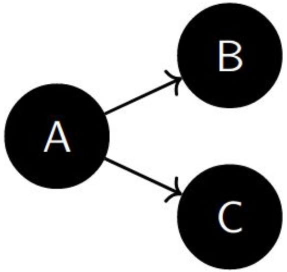
No contextual empirical models



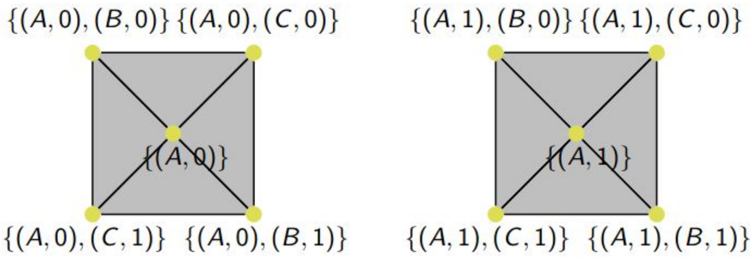
Preserves contextuality



No contextual empirical models



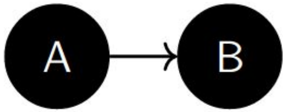
Preserves contextuality



Could be a contextual empirical model

Application of Vorob'ev's Theorem to GP Scenarios

No contextual empirical models



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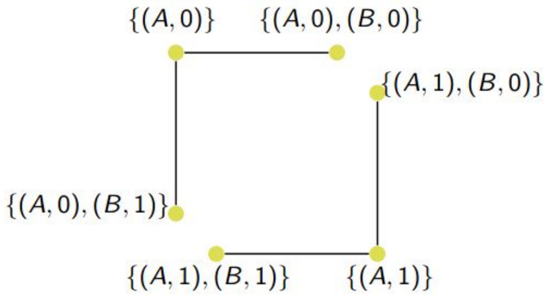

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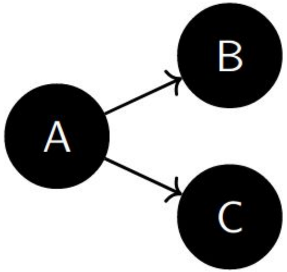

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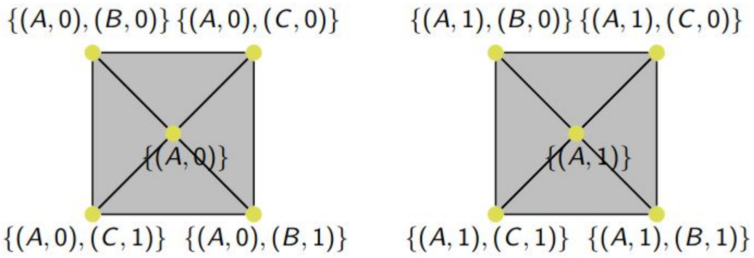

Could be a contextual empirical model

Things you get for ‘free’ using game semantics

1. Monotonicity

   Once a measurement has been enabled, it does not change value after that. This is a property we want in physical systems.

2. Experimenter and Nature strategies are dual.

   As a result it comes out naturally that Nature strategies are **eager**, while experimenter strategies are **lazy**.

Future Directions

1. Understand which examples give rise to contextuality

   For example, we have the result that no empirical model over signalling A->B is contextual. Which other setups does this hold for?

2. Develop a resource theory framework

   This would give a systematic way to answer the above by asking, for example, which operations the contextual fraction cannot increase under

References

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