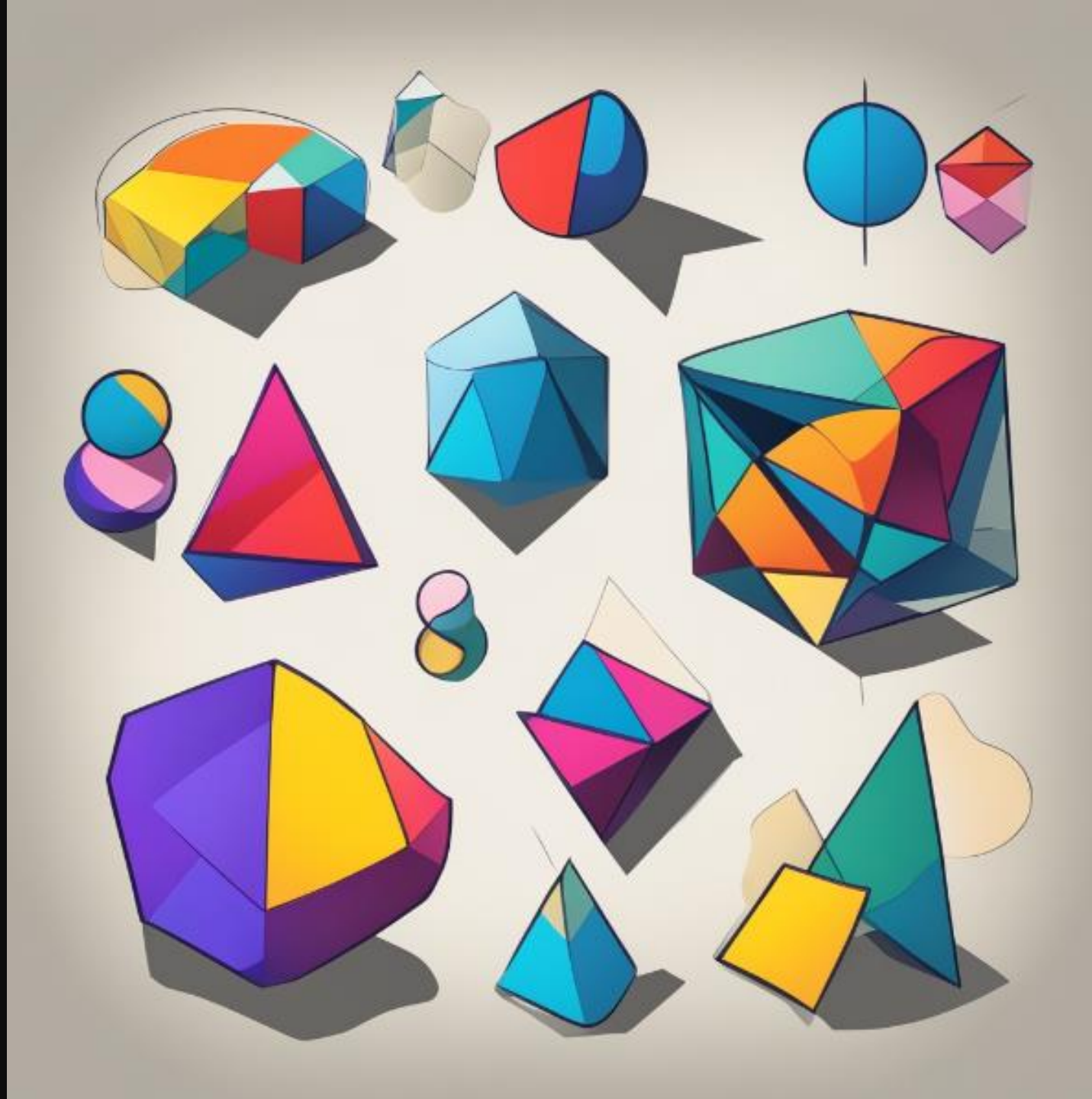
WHICH PHYSICAL THEORIES HAVE TWO STATE OVERLAPS?

Rafael Wagner

International Iberian Nanotechnology Laboratory (INL)

University of Minho (UM)



TWO STATE OVERLAPS

$$r_{\rho\sigma} = \mathrm{Tr}[\rho\sigma]$$

$$r_{p,q} = \sum_{\alpha} p_\alpha q_\alpha$$

TWO STATE OVERLAPS

Quantum theory

$$r_{\rho\sigma} = \mathrm{Tr}[\rho\sigma]$$

Classical probabilistic theory

$$r_{p,q} = \sum_{\alpha} p_\alpha q_\alpha$$

PHYSICAL THEORIES

Physics Reports 1033 (2023) 1–64

Contents lists available at ScienceDirect

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General probabilistic theories: An introduction

Martin Plávala

Naturwissenschaftlich-Technische Fakultät, Universität Siegen, 57068 Siegen, Germany

SciPost Phys. Lect.Notes 28 (2021)

Probabilistic theories and reconstructions of quantum theory

Markus P. Müller[1,2⋆]

1 Institute for Quantum Optics and Quantum Information,
Austrian Academy of Sciences, Boltzmanngasse 3, A-1090 Vienna, Austria
2 Perimeter Institute for Theoretical Physics,
31 Caroline Street North, Waterloo, ON N2L 2Y5, Canada

OUTLINE OF THE TALK

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Roberto D. Baldijão
(Gdansk)

GPTS

- Real finite-dimensional vector space $V \subseteq \mathbb{R}^d$

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- A subset in this vector space that characterize states Ω

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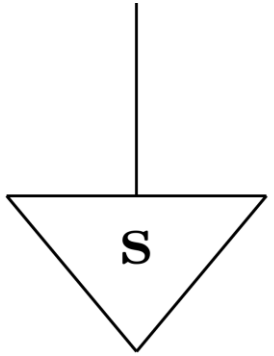
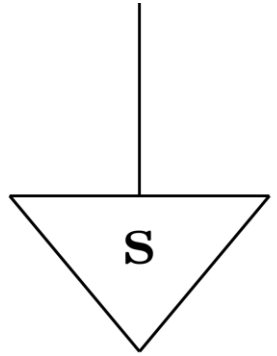


GPTS

- Real finite-dimensional vector space $V \subseteq \mathbb{R}^d$
- A subset in this vector space that characterize states Ω



- The state space defines a (closed convex pointed) cone in the vector space

GPTS

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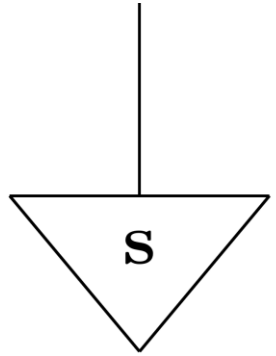


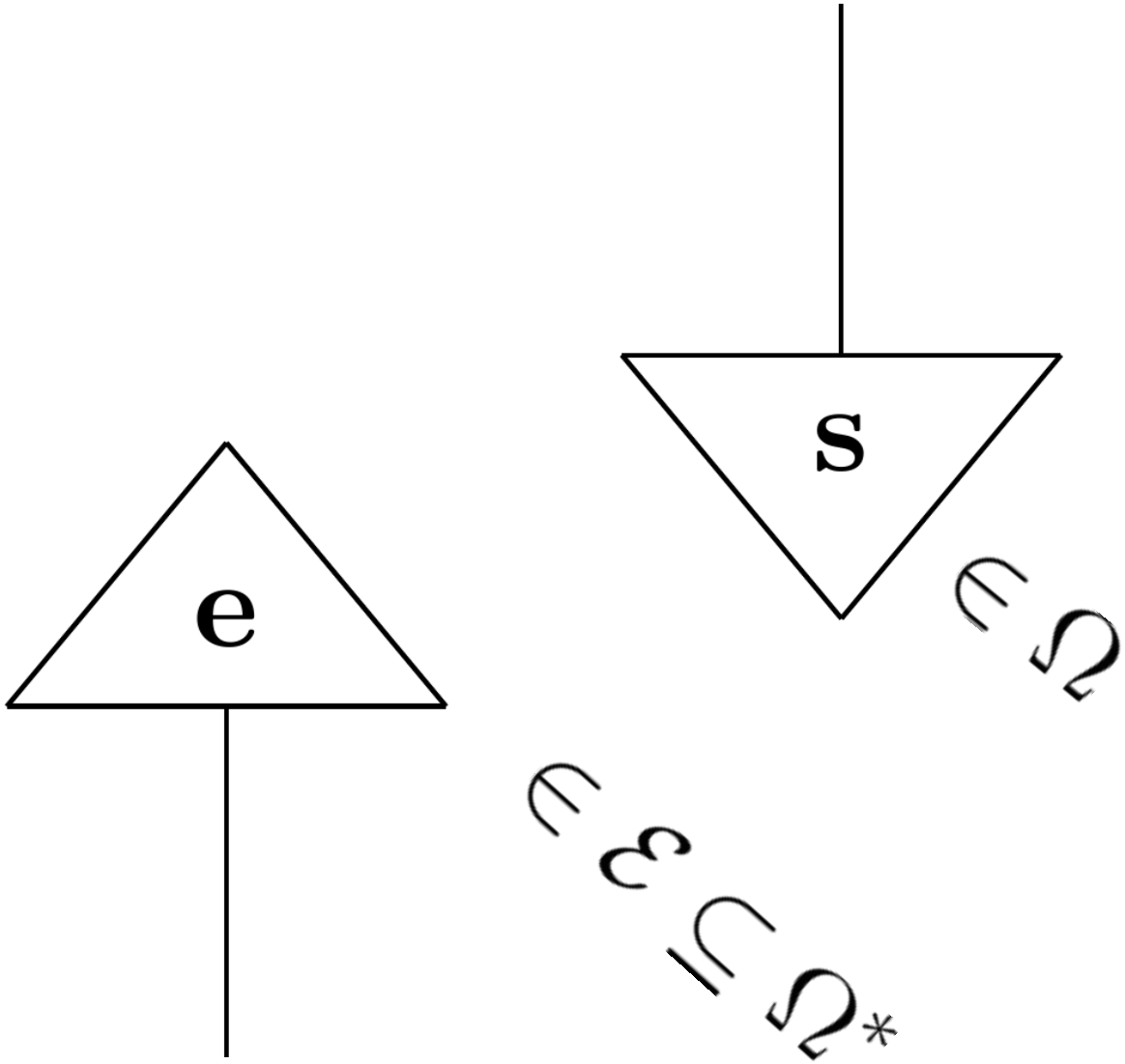
- The state space defines a (closed convex pointed) cone in the vector space

$$\Omega^* := \{\mathbf{e} \in V^* : \mathbf{e}(\mathbf{s}) \geq 0, \forall \mathbf{s} \in \Omega\}$$

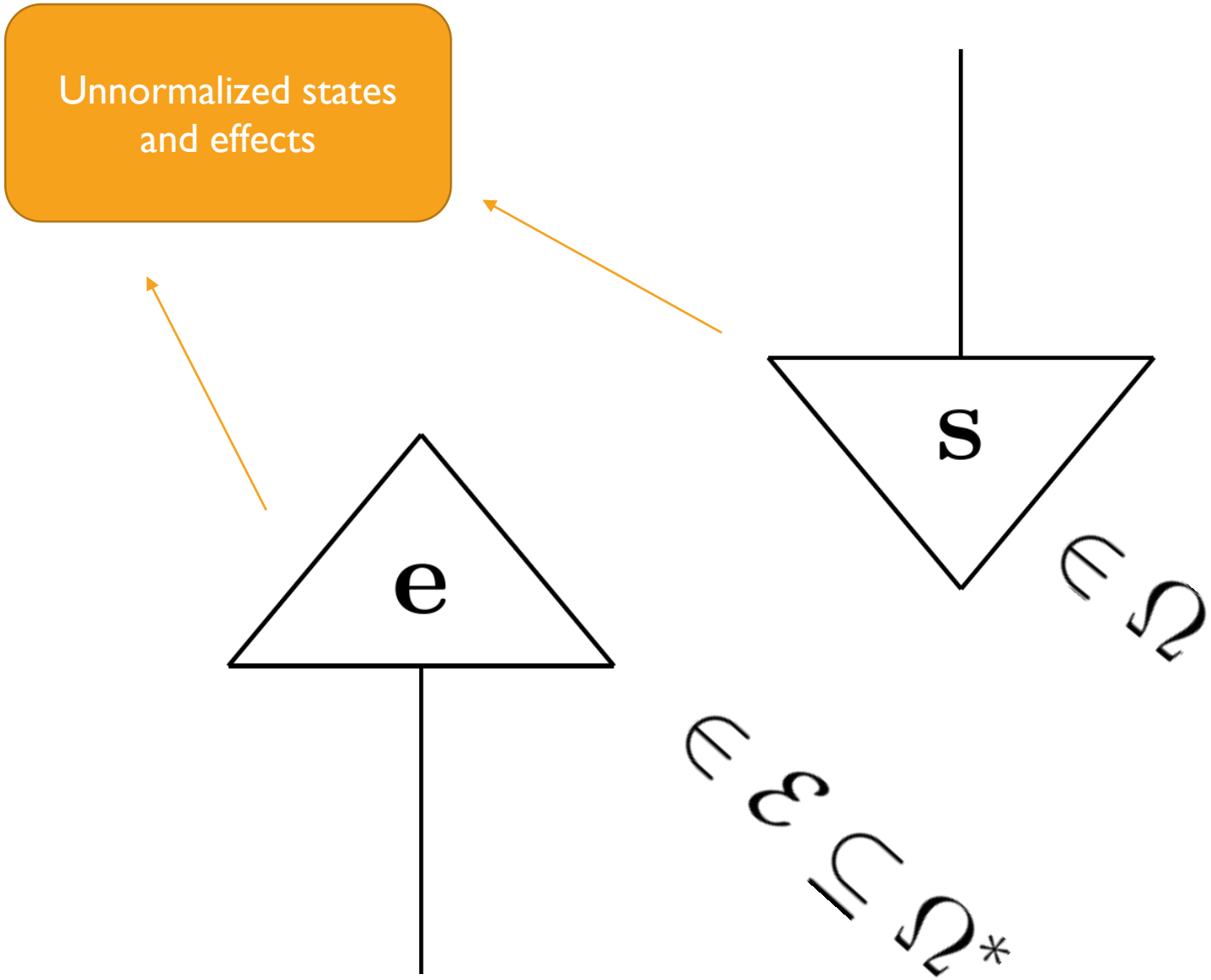
Dual cone

- A subset of the dual cone characterizes the effects

GPTS

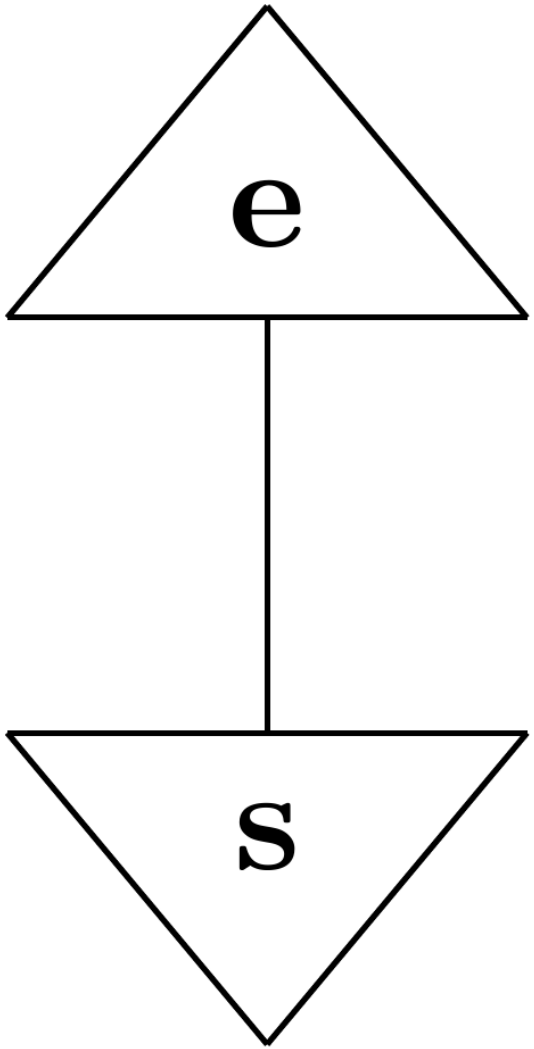


GPTS

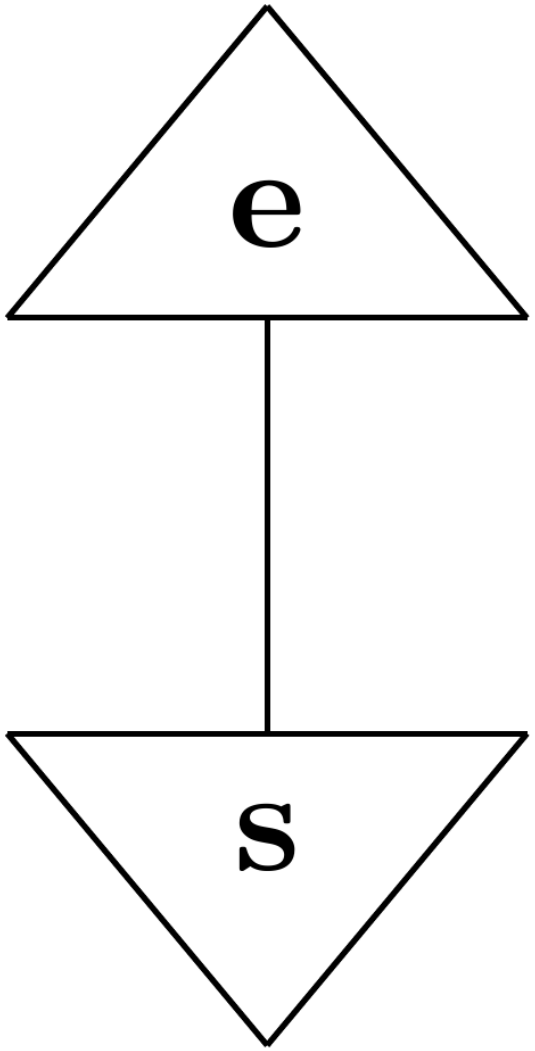

Unnormalized states and effects
s
e
$\in \Omega$
$\in \Omega \times \Omega^*$

GPTS

$$\mathbf{e}(\mathbf{s}) = \in \mathbb{R}_+$$



GPTS

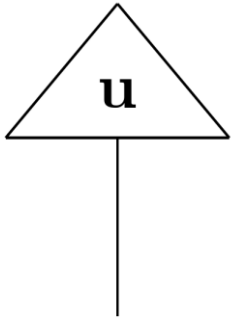


$\in \mathbb{R}_+$

Quantum theory as a GPT: this is not the inner product!

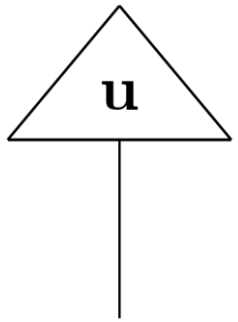
$$\mathrm{Tr}(E\rho)$$

GPTS



The unique ‘unit’ effect

GPTS



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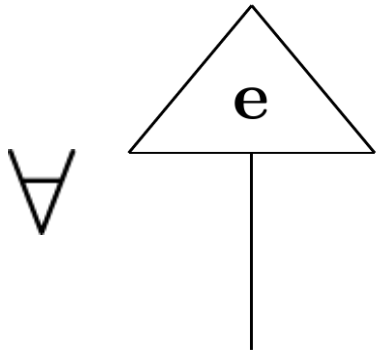
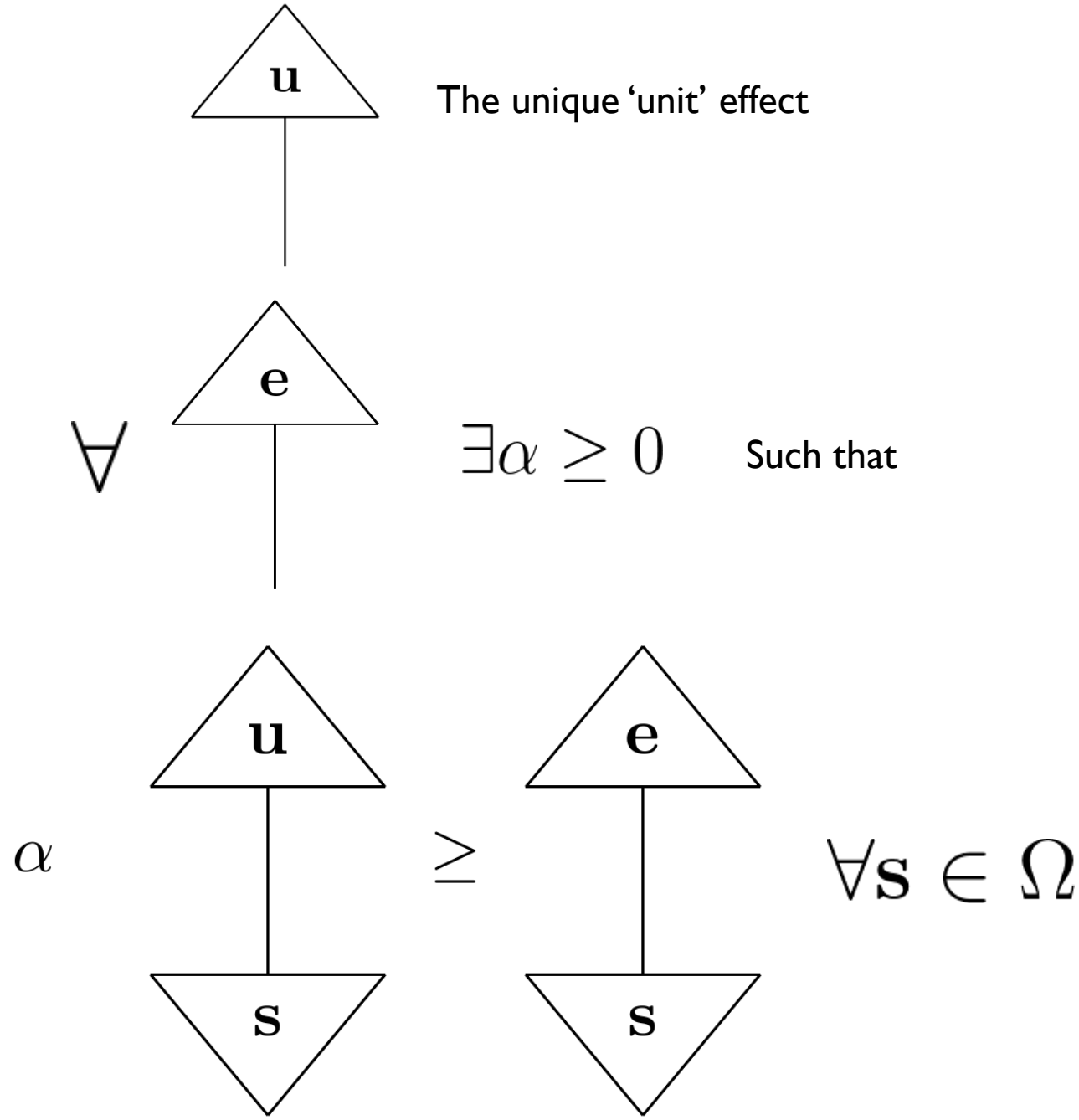


$\forall$

$\exists \alpha \geq 0$

Such that

GPTS


u
The unique ‘unit’ effect
e
$\forall$
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Such that
α
u
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$\forall \mathbf{s} \in \Omega$

GPTS

$$(V, \Omega, \mathcal{E}, u)$$

GPT system

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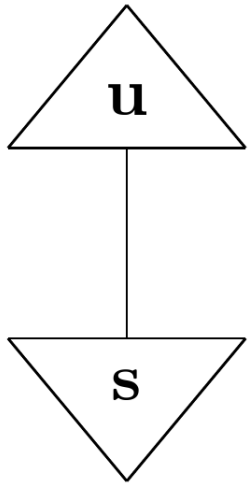
GPT system

Normalized states



$= \quad 1$

GPTS

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GPT system

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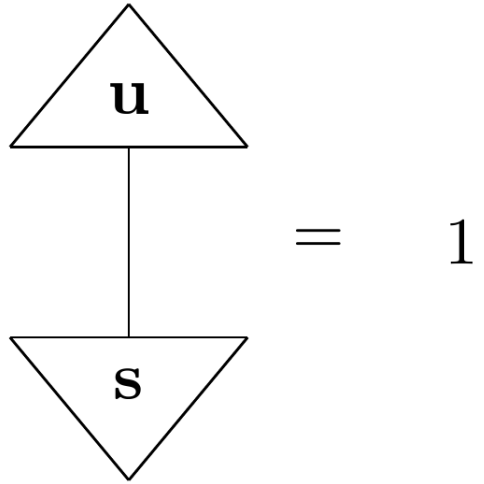
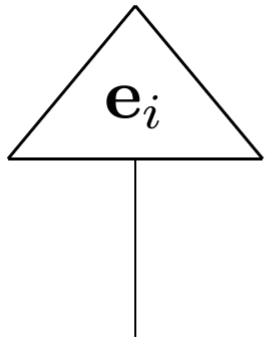
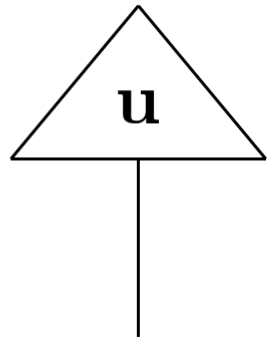


$= \quad 1$

Normalized effects

$\sum_i$ 

$=$ 

WHAT DO WE WANT FROM OVERLAPS?

The Thinker, Auguste Rodin, 1904

WHAT DO WE WANT FROM OVERLAPS?

- 1) It needs to be *a function that takes two states* as an input

$$r : \Omega \times \Omega \to \mathbb{R}$$

$$(\mathbf{s}, \mathbf{t}) \mapsto r_{\mathbf{s},\mathbf{t}}$$

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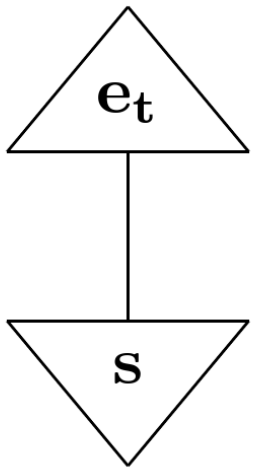
- 2) It needs to be *empirically accessible*

$$\forall \mathbf{s}, \mathbf{t} \in \Omega \qquad \exists \mathbf{e_t} \in \mathcal{E}$$

$r_{\mathbf{s},\mathbf{t}} =$



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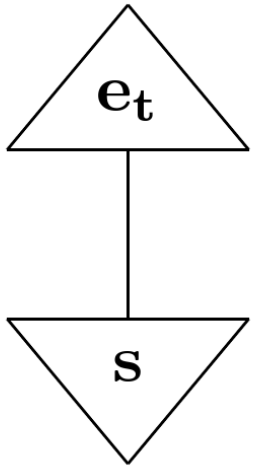

Prediction of the theory

- Remark 1: $\forall \mathbf{s}, \mathbf{t} \in \Omega, r_{\mathbf{s},\mathbf{t}} \geq 0$

WHAT DO WE WANT FROM OVERLAPS?

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- Remark I: $\forall \mathbf{s}, \mathbf{t} \in \Omega, r_{\mathbf{s},\mathbf{t}} \geq 0$

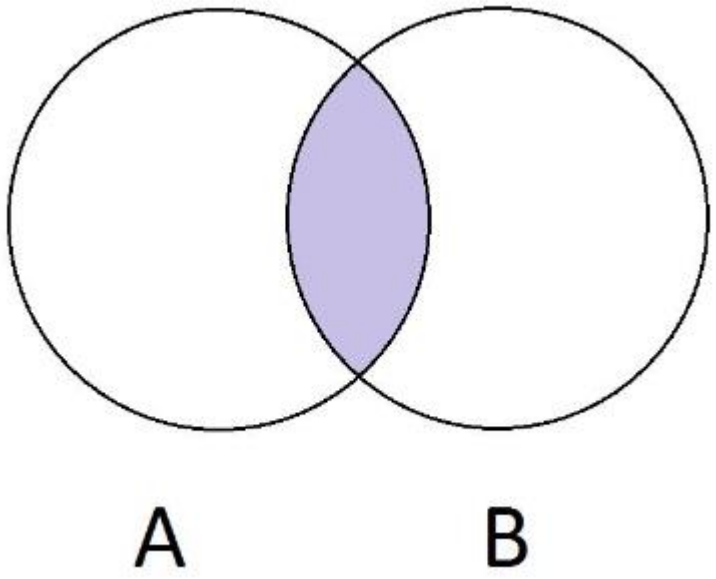
- Remark II: For pure states and pure effects

$$r_{\mathbf{s},\mathbf{t}} \in [0, 1]$$

WHAT DO WE WANT FROM OVERLAPS?

- It needs to be *symmetric*

$$r_{\mathbf{s},\mathbf{t}} = r_{\mathbf{t},\mathbf{s}}$$



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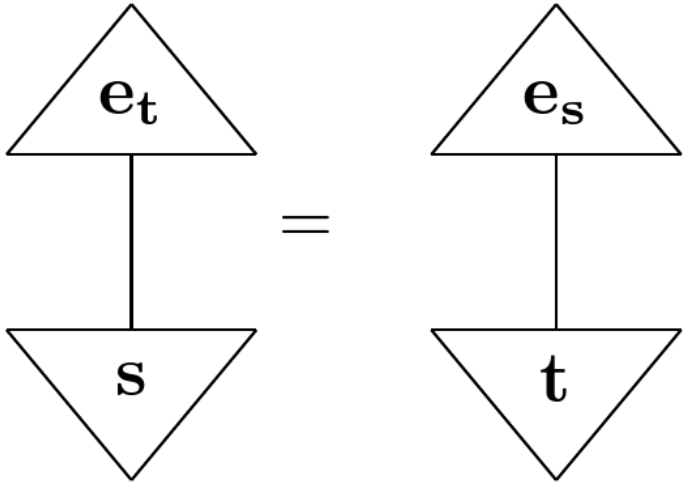
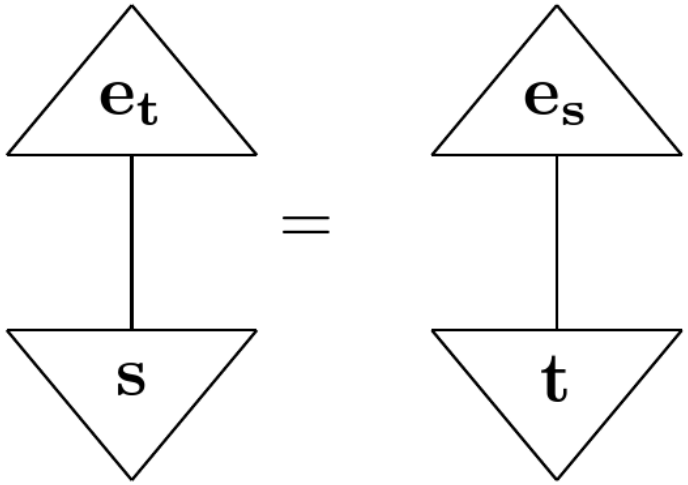
$r_{\mathbf{s},\mathbf{t}} =$  $= r_{\mathbf{t},\mathbf{s}}$

WHAT DO WE WANT FROM OVERLAPS?

- Remark IV: $\forall \mathbf{s}, \mathbf{t} \in \Omega \, \exists \mathbf{e_s}, \mathbf{e_t} \in \mathcal{E}$

$r_{\mathbf{s},\mathbf{t}} =$  $= r_{\mathbf{t},\mathbf{s}}$

These effects are not going to be unique in general

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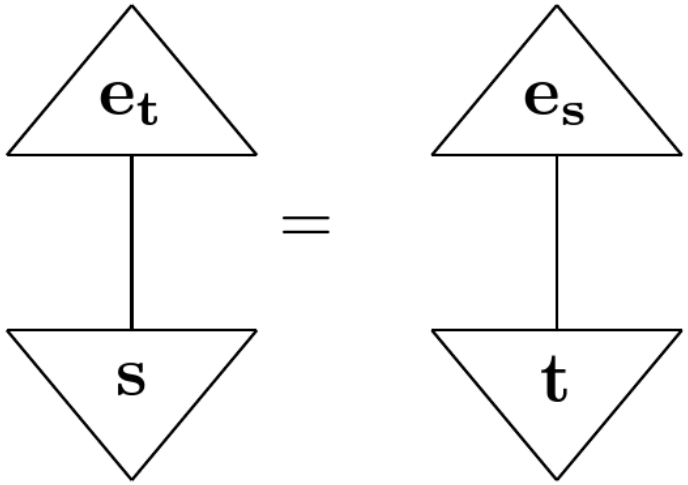
$r_{\mathbf{s},\mathbf{t}} =$  $= r_{\mathbf{t},\mathbf{s}}$

These effects are not going to be unique in general

$$r_{\psi,\psi} = |\langle\psi|\psi\rangle|^2 = 1 = \mathrm{Tr}(\mathbb{I}|\psi\rangle\langle\psi|)$$

Unit effect acting on a state

TWO-STATE OVERLAPS IN PHYSICAL THEORIES

$$(V, \Omega, \mathcal{E}, u)$$

GPT system can have two-state overlaps if

$$\forall \mathbf{s}, \mathbf{t} \in \Omega \; \exists \mathbf{e_s}, \mathbf{e_t} \in \mathcal{E}$$

$$\mathbf{e_s}(\mathbf{t}) = \mathbf{e_t}(\mathbf{s})$$

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In such a case, a two-state overlap is a map $r : \Omega \times \Omega \to \mathbb{R}$

such that for any pair of states there exists a pair of effects for which

$$r(\mathbf{s}, \mathbf{t}) \equiv r_{\mathbf{s},\mathbf{t}} = \mathbf{e_s}(\mathbf{t}) = \mathbf{e_t}(\mathbf{s}) = r_{\mathbf{t},\mathbf{s}}$$

TWO-STATE OVERLAPS IN PHYSICAL THEORIES

From the definition, two-state overlaps are bilinear in the state space.

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NOT ALL THEORIES CAN HAVE TWO-STATE OVERLAPS

Consider any theory where both states and effects can be embedded in the same real inner-product vector space, such that there is a duality between the action of linear functionals and the inner-product

$$\mathbf{e}(\cdot) = \langle \mathbf{e}, \cdot \rangle$$

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$$\Omega^* \subsetneq \Omega$$

Pre-dual theories

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$$\Omega^* \subsetneq \Omega \qquad \mathbf{s}^\star \in \Omega \setminus \mathcal{E}$$

Pre-dual theories

There is at least one state such that

$$r_{\mathbf{s},\mathbf{s}^\star} = \mathbf{e}_\mathbf{s}(\mathbf{s}^\star) = \langle \mathbf{e}_\mathbf{s}, \mathbf{s}^\star \rangle < 0$$

Otherwise this state would define an effect
(real inner-products are symmetric)

NOT ALL THEORIES CAN HAVE TWO-STATE OVERLAPS

PAPER • OPEN ACCESS

Pseudo standard entanglement structure cannot be distinguished from standard entanglement structure

Hayato Arai[1] and Masahito Hayashi[5,1,2,3,4]

Published 9 February 2023 • © 2023 The Author(s). Published by IOP Publishing Ltd on behalf of the Institute of Physics and Deutsche Physikalische Gesellschaft

New Journal of Physics, Volume 25, February 2023

Citation Hayato Arai and Masahito Hayashi 2023 *New J. Phys.* **25** 023009

DOI 10.1088/1367-2630/acb565

They construct a subtheory of quantum theory where the state space contain all quantum states and all entanglement witnesses of trace one. This implies that

$$\exists \rho, \rho_W \in \Omega : \mathrm{Tr}(\rho\rho_W) < 0$$

NOT ALL THEORIES CAN HAVE TWO-STATE OVERLAPS

THEORIES THAT CAN HAVE OVERLAPS

$$\mathcal{E} = \Omega^* = \Omega$$

Non-restricted strongly-self dual

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Non-restricted strongly-self dual

In these theories the cone of states and the cone of effects can be embedded in a finite-dimensional real inner-product vector space

$$r(\cdot, \cdot) := \langle \cdot, \cdot \rangle$$

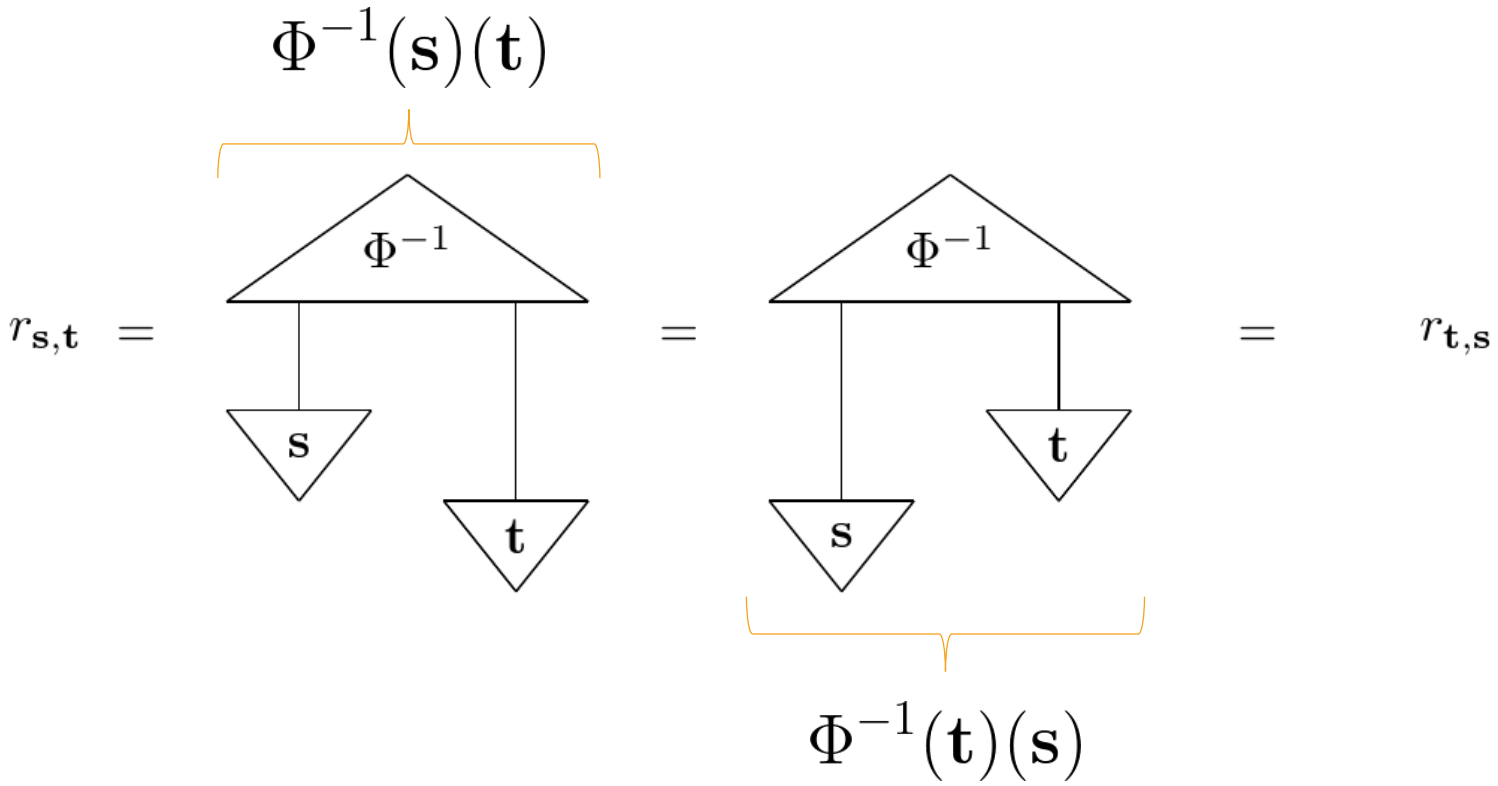
The (self-dualizing) inner-product works as a two-state overlap

THEORIES THAT CAN HAVE OVERLAPS

$$\Phi^{-1} : \Omega \rightarrow \Omega^*$$

Strong selfduality

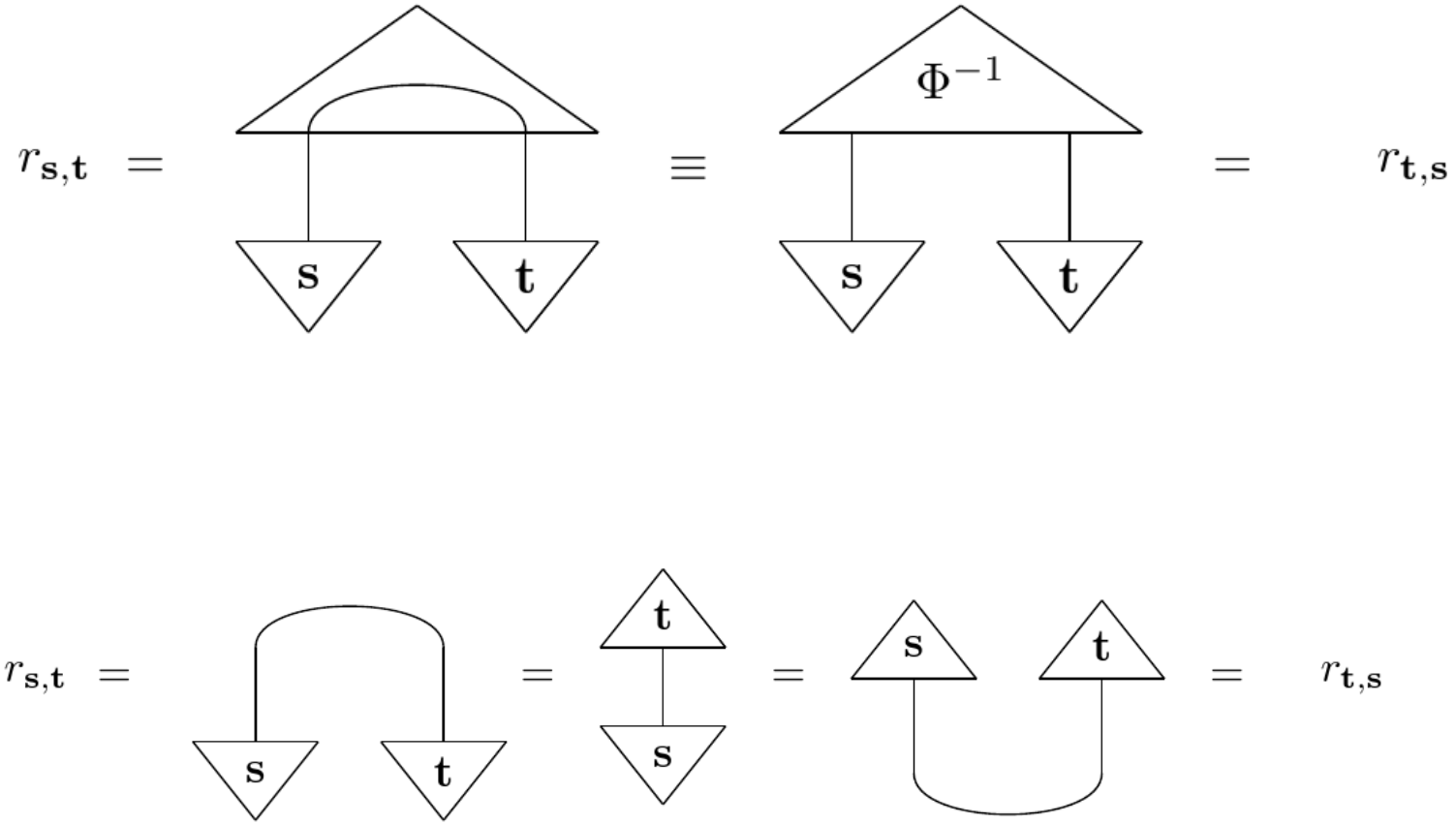
PRA 87, 052131 (2013)



1) Well defined for all pairs of states, II) empirically accessible and inducing a bilinear III) symmetric form (which is taken to be the overlap)

THEORIES THAT CAN HAVE OVERLAPS

GPTs having caps and cups



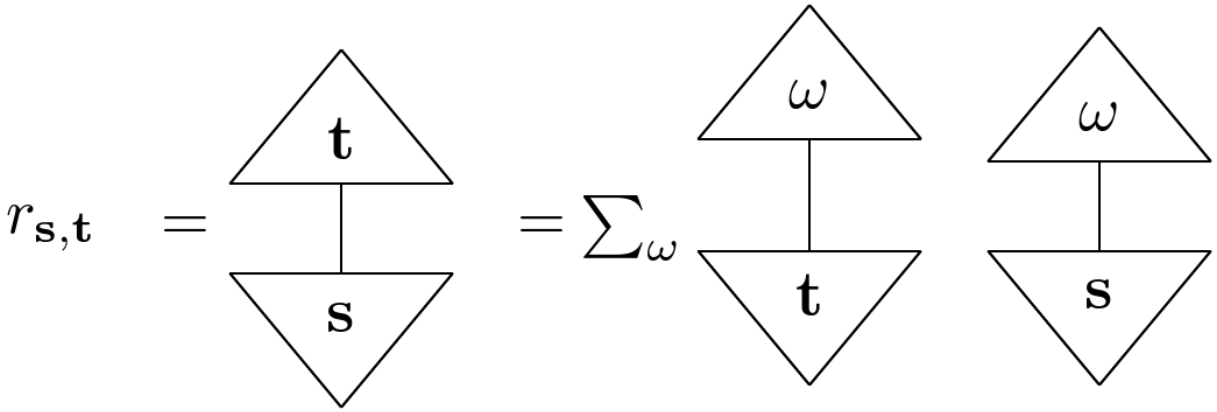
THEORIES THAT CAN HAVE OVERLAPS

APPLICATION

Inequalities witnessing coherence, nonlocality, and contextuality

Rafael Wagner, Rui Soares Barbosa, and Ernesto F. Galvão
Phys. Rev. A **109**, 032220 – Published 21 March 2024

Bounds on classical overlaps



OVERLAP INEQUALITIES

Inequalities witnessing coherence, nonlocality, and contextuality

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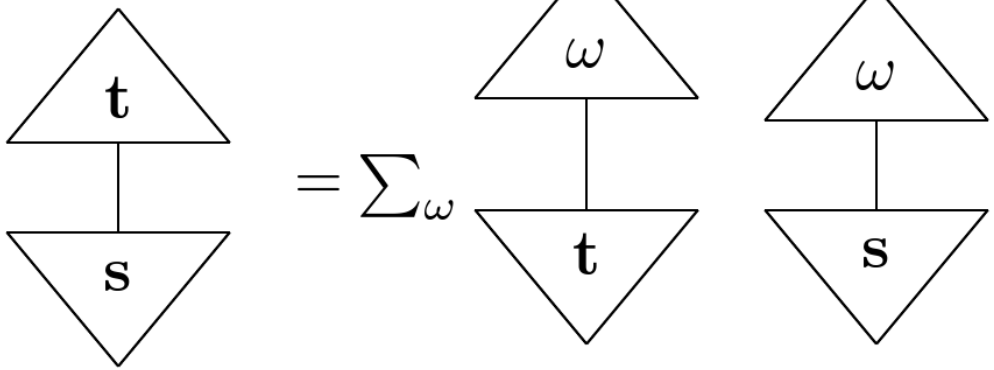
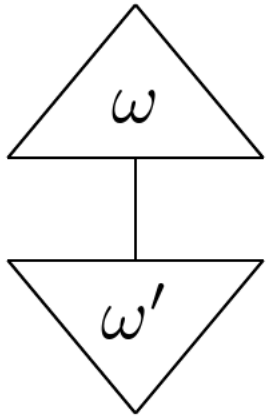
Bounds on classical overlaps

$$r_{\mathbf{s},\mathbf{t}} = \quad = \sum_\omega$$



$$\mathbf{s} = (s_\omega)_\omega, \mathbf{t} = (t_\omega)_\omega$$

Probability vectors



$$= \delta_{\omega,\omega'}$$

OVERLAP INEQUALITIES

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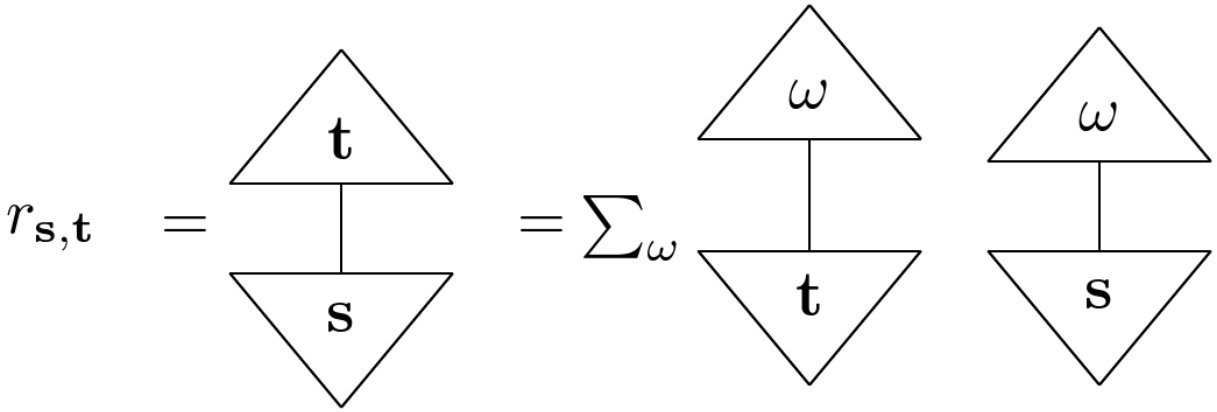

Classical probabilistic theory

$$r_{\mathbf{s},\mathbf{t}} = \sum_{\omega} \mathrm{Prob}(\omega|\mathbf{t})\mathrm{Prob}(\omega|\mathbf{s})$$

Two-state overlaps between normalized states of classical probabilistic theories cannot violate the incoherence inequalities

OVERLAP INEQUALITIES

Theorem. *Empirically adequate, diagram preserving ontological models for GPTs having caps and cups cannot violate two-state overlap inequalities.*

THEOREM

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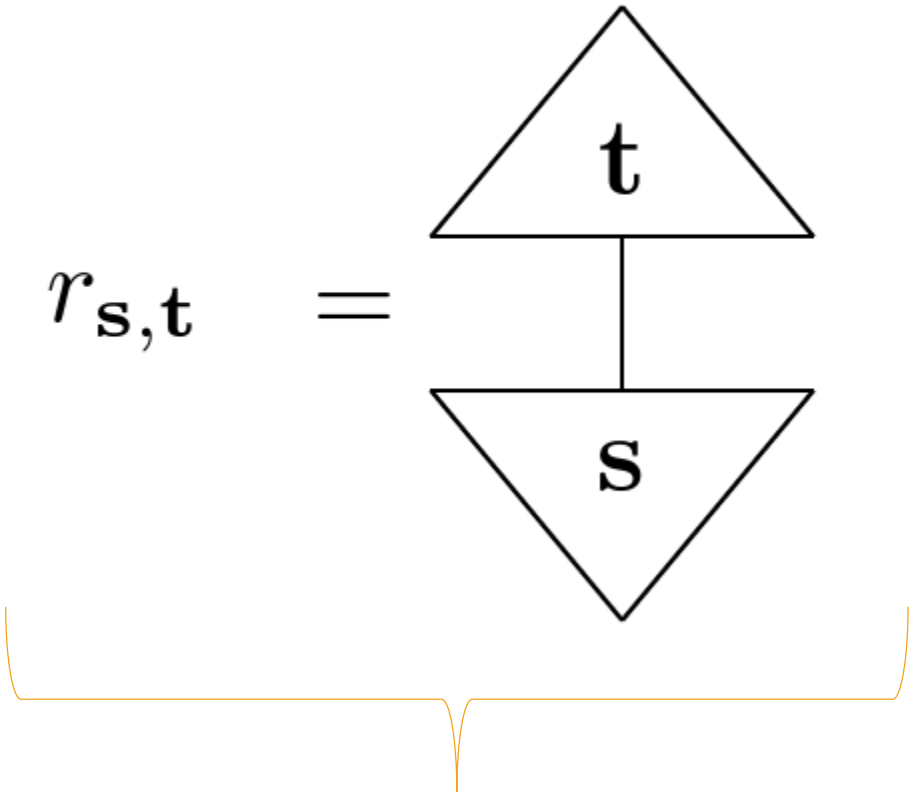
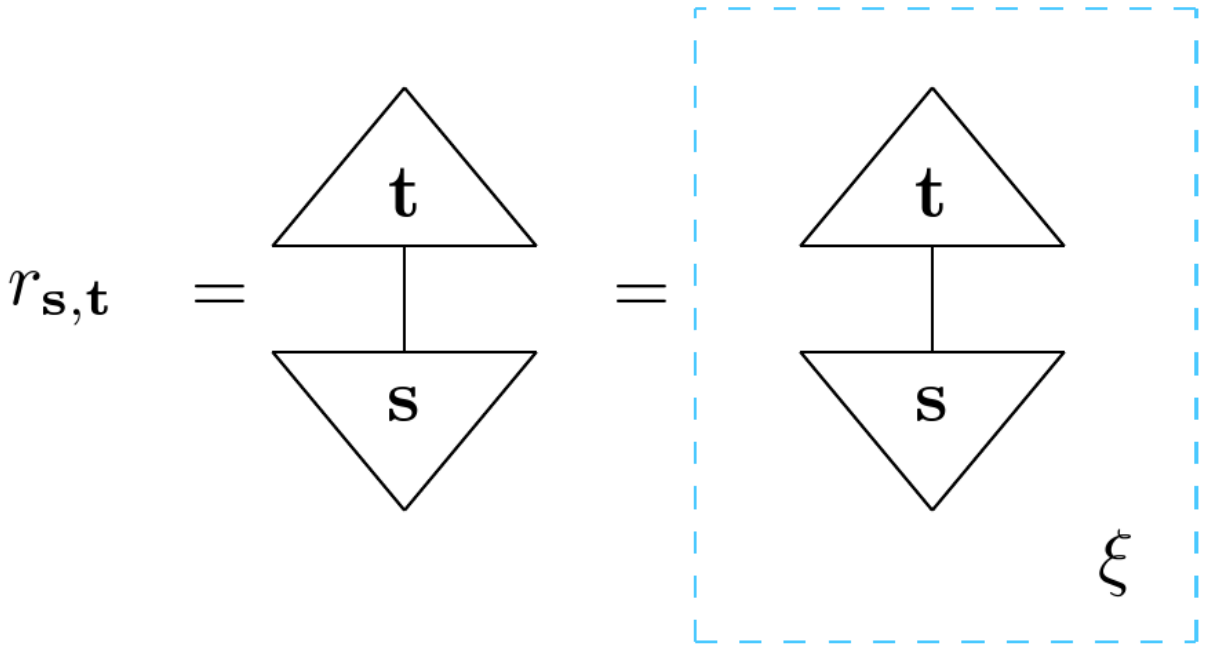


Two-state overlaps can be empirically accessible

THEOREM

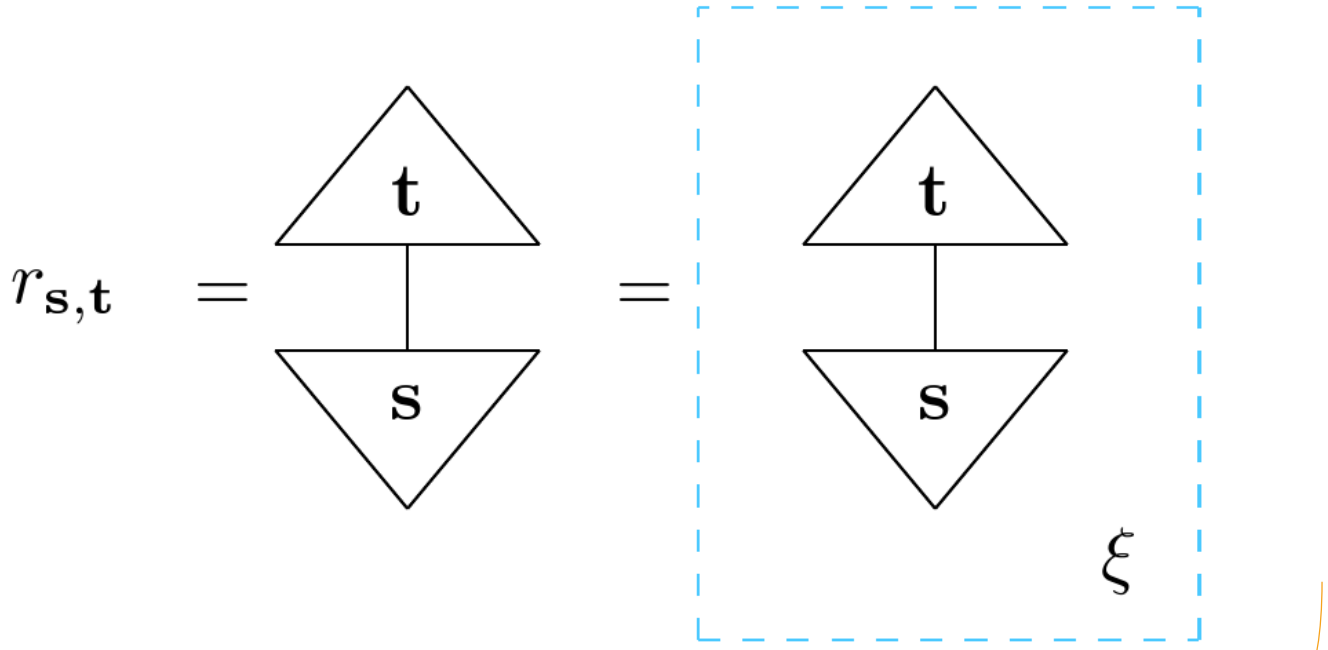
Theorem. *Empirically adequate, diagram preserving ontological models for GPTs having caps and cups cannot violate two-state overlap inequalities.*



Empirical adequacy of the ontological model

THEOREM

Theorem. *Empirically adequate, diagram preserving ontological models for GPTs having caps and cups cannot violate two-state overlap inequalities.*



Empirical adequacy of the ontological model

THEOREM

An ontological model is a map (functor) representing every element of the GPT as a substochastic matrix

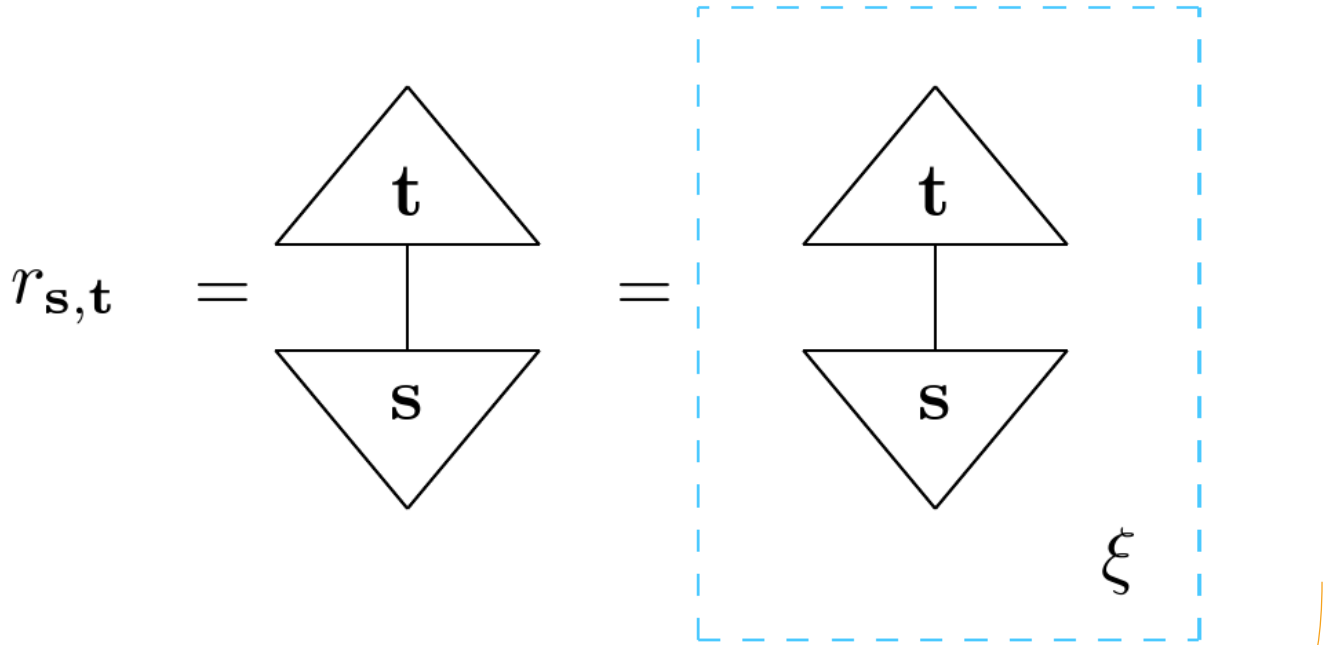
$$\mathbf{s} \mapsto \xi(\mathbf{s})(\lambda)$$

Probability distribution

$$\mathbf{e} \mapsto \xi(\mathbf{e})(\lambda)$$

Response function

Theorem. *Empirically adequate, diagram preserving ontological models for GPTs having caps and cups cannot violate two-state overlap inequalities.*



Empirical adequacy of the ontological model

THEOREM

$$\mathbf{s} \mapsto \xi(\mathbf{s})(\lambda)$$

Probability distribution

$$\mathbf{e} \mapsto \xi(\mathbf{e})(\lambda)$$

Response function

$$\mathbf{T} \mapsto \xi(\mathbf{T})(\lambda'|\lambda)$$

Substochastic matrix

Theorem. *Empirically adequate, diagram preserving ontological models for GPTs having caps and cups cannot violate two-state overlap inequalities.*

THEOREM

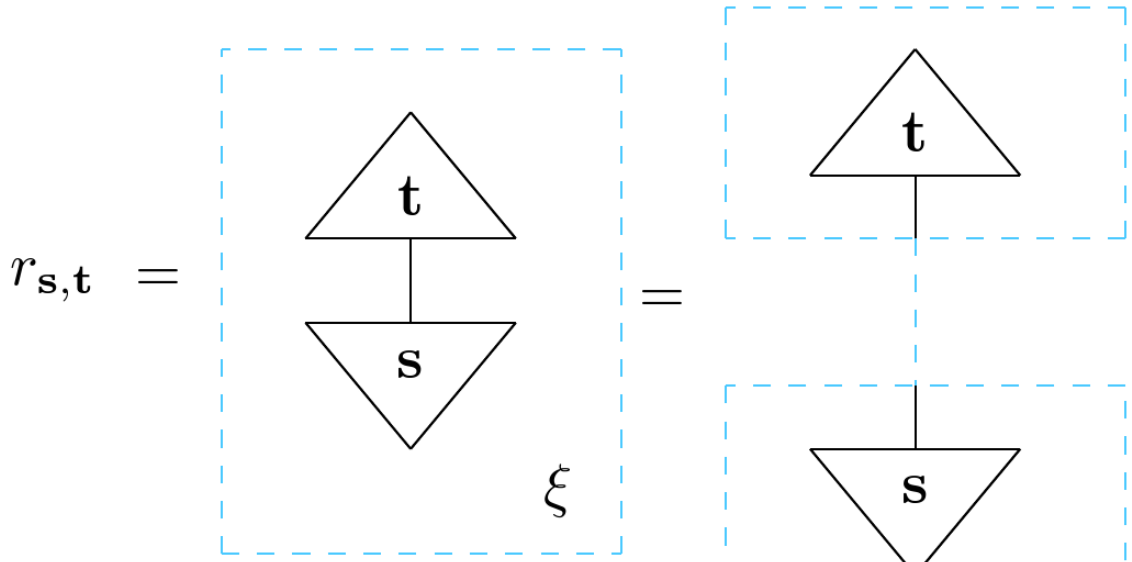
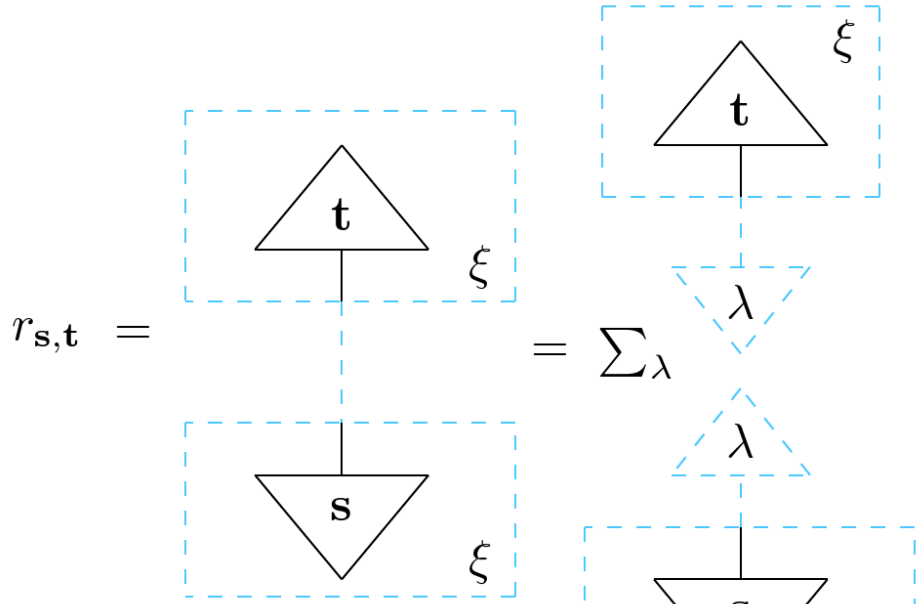


Diagram preservation

Theorem. *Empirically adequate, diagram preserving ontological models for GPTs having caps and cups cannot violate two-state overlap inequalities.*



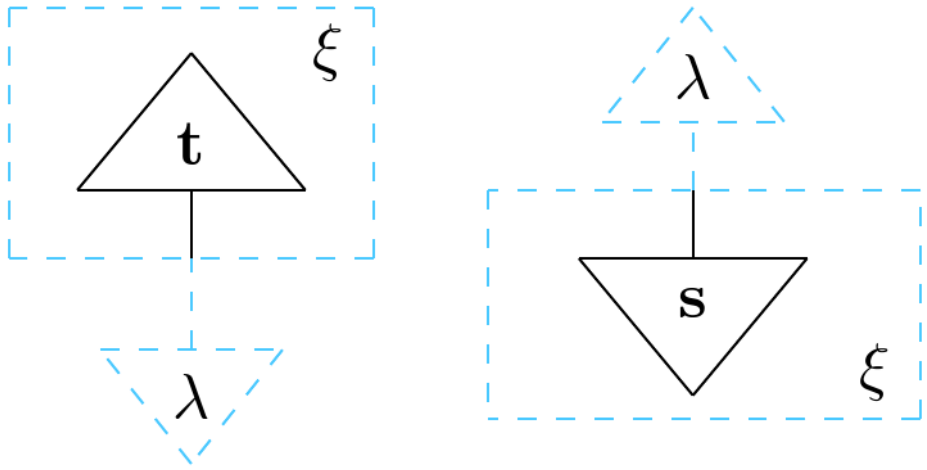
There exists a basis for the space defining the substochastic matrices

THEOREM

Theorem. *Empirically adequate, diagram preserving ontological models for GPTs having caps and cups cannot violate two-state overlap inequalities.*

THEOREM

$$r_{\mathbf{s},\mathbf{t}} = \sum_{\lambda}$$



Theorem. *Empirically adequate, diagram preserving ontological models for GPTs having caps and cups cannot violate two-state overlap inequalities.*

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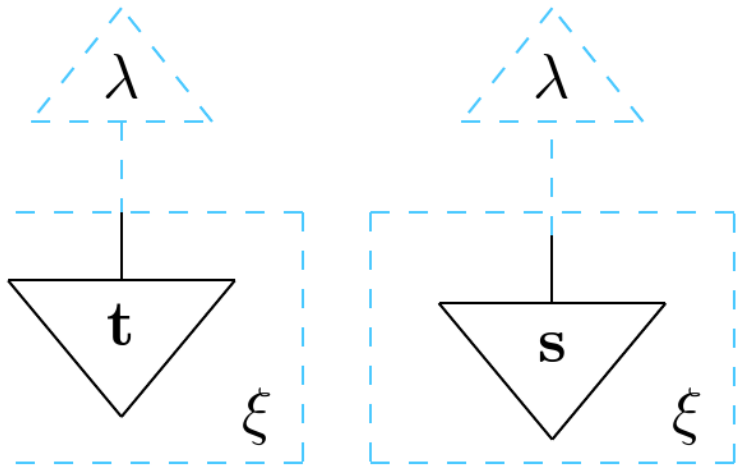
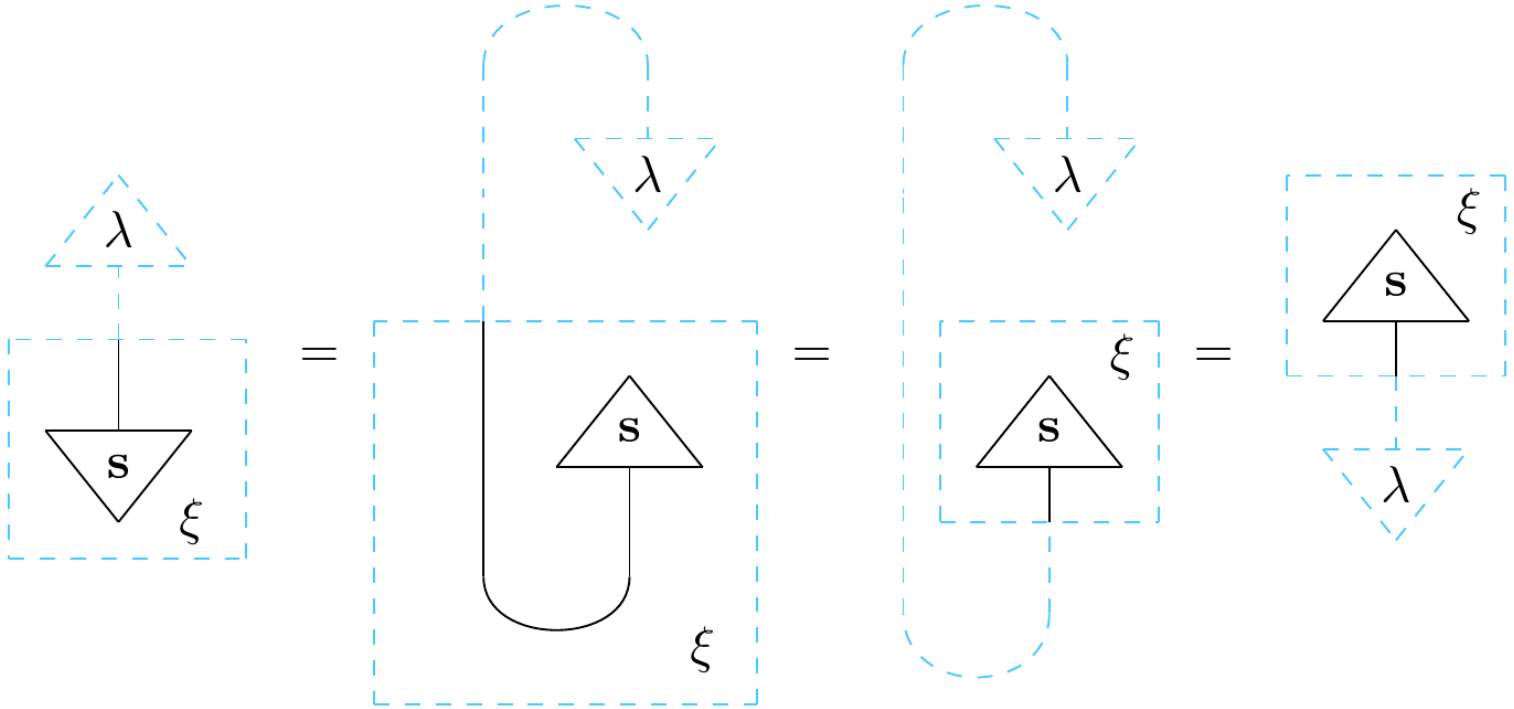
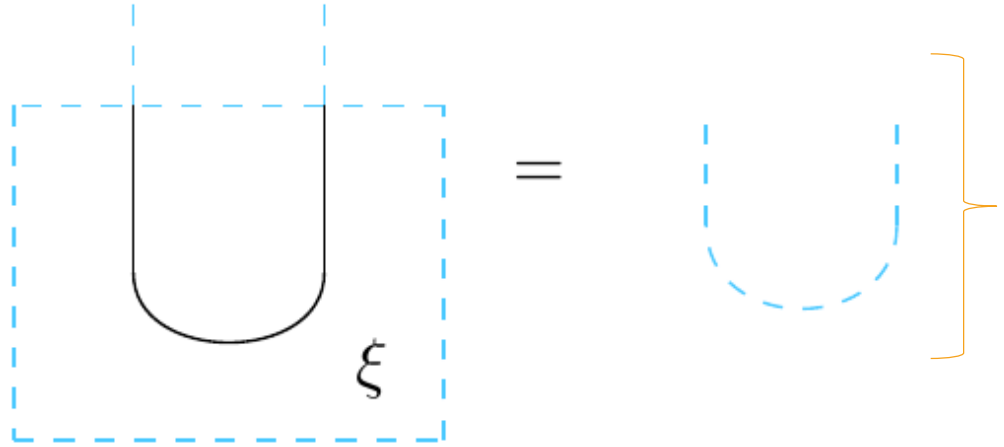


Caps and cups + diagram preservation

EXAMPLE OF DIAGRAMMATIC FLIPPING



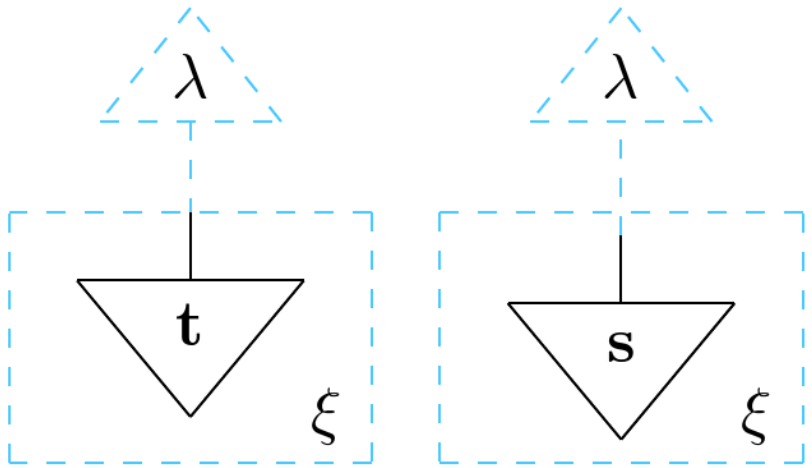


Well defined in Substoch

Theorem. *Empirically adequate, diagram preserving ontological models for GPTs having caps and cups cannot violate two-state overlap inequalities.*

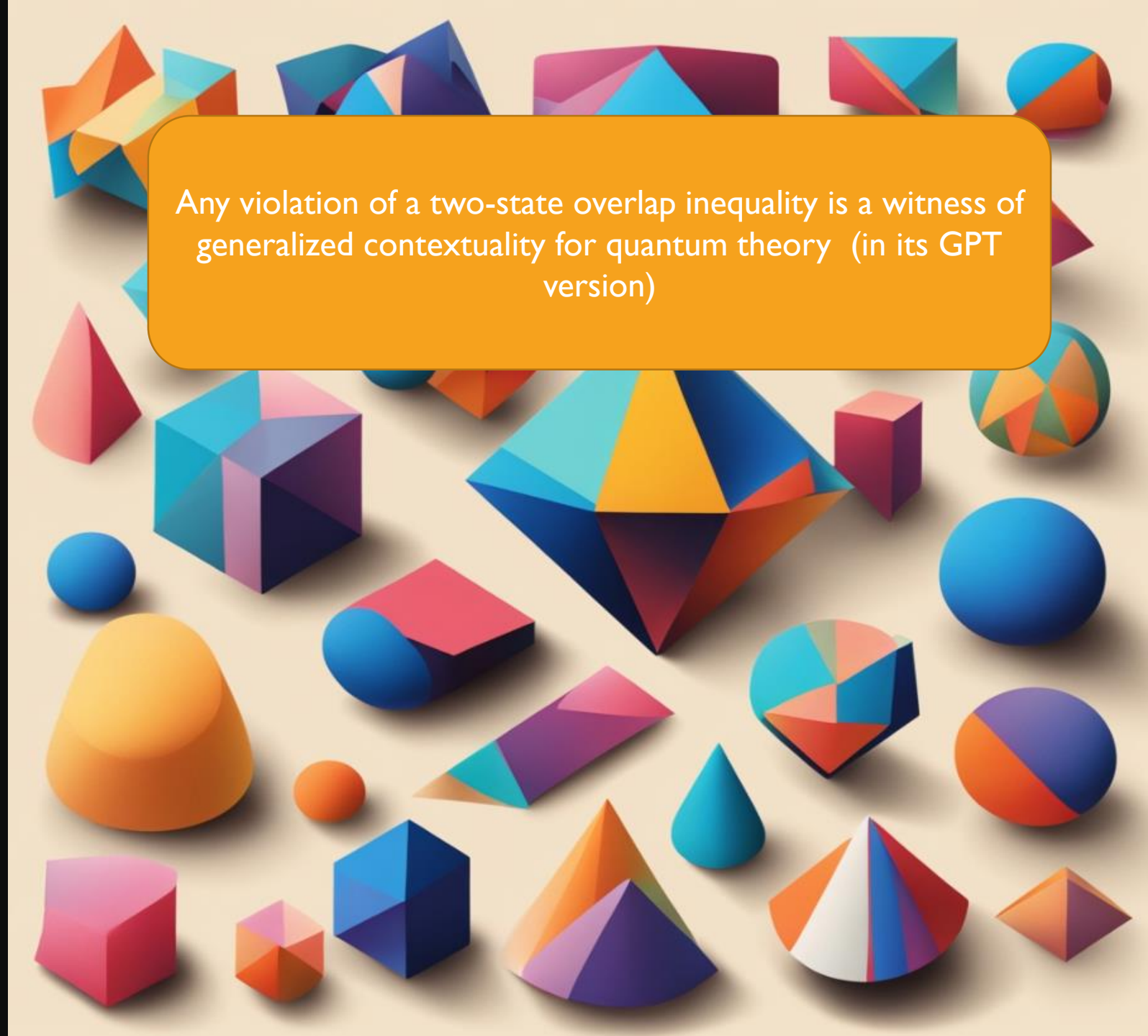
THEOREM

$$r_{\mathbf{s},\mathbf{t}} = \sum_{\lambda}$$



Cannot violate the two-state overlap inequalities by construction

APPLICATION



MORE?

Certifying almost all quantum states with few single-qubit measurements

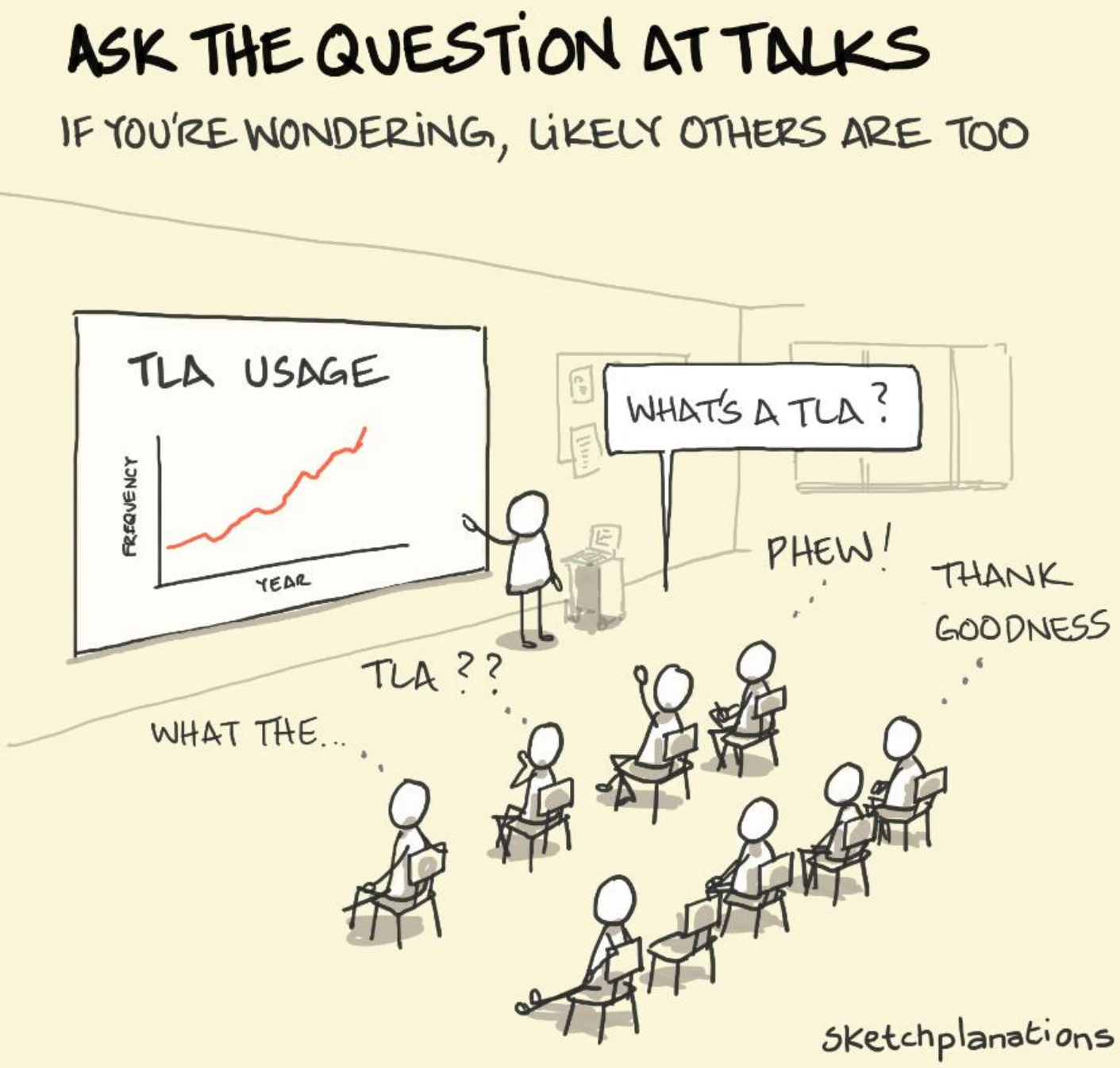
Hsin-Yuan Huang[1,2,3], John Preskill[1,4], and Mehdi Soleimanifar[1]

[1]California Institute of Technology
[2]Google Quantum AI
[3]Massachusetts Institute of Technology
[4]AWS Center for Quantum Computing

Efficient distributed inner product estimation via Pauli sampling

M. Hinsche[*1], M. Ioannou[*1], S. Jerbi[1], L. Leone[1], J. Eisert[1,2], and J. Carrasco[*1]

[1]Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, Berlin, Germany
[2]Helmholtz-Zentrum Berlin für Materialien und Energie, Berlin, Germany



THANK YOU

- rafael.wagner@inl.int
- @QuantumRW

STRONGLY SELFDUAL GPTS HAVE CAPS AND CUPS?

Definition 1. A system is *strongly self-dual* if and only if there exists an isomorphism $\Phi : V_+^* \mapsto V_+$ giving rise to a corresponding symmetric bilinear form T with $T(e,f) = e[\Phi(f)] = T(f,e)$ and $T(e,e) \geqslant 0$ for all $e, f \in V^*$.

That is, T provides a semi-inner product on effects. In a similar way for strongly self-dual systems the inverse map Φ^{-1} leads to a semi-inner product on states.

PHYSICAL REVIEW A **87**, 052131 (2013)

Generalized probabilistic theories without the no-restriction hypothesis

Peter Janotta[1] and Raymond Lal[2]
[1]*Universität Würzburg, Am Hubland, Fakultät für Physik und Astronomie, 97074 Würzburg, Germany*
[2]*University of Oxford, Department of Computer Science, Quantum Group, Wolfson Building, Parks Road, Oxford OX1 3QD, United Kingdom*
(Received 21 February 2013; published 23 May 2013)

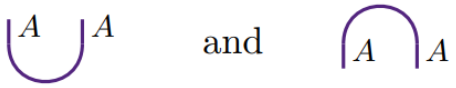
STRONGLY SELFDUAL GPTS HAVE CAPS AND CUPS?

Reconstructing quantum theory from diagrammatic postulates

John H. Selby[1], Carlo Maria Scandolo[2,3], and Bob Coecke[4]

[1]ICTQT, University of Gdańsk, Wita Stwosza 63, 80-308 Gdańsk, Poland
[2]Department of Mathematics & Statistics, University of Calgary, Canada
[3]Institute for Quantum Science and Technology, University of Calgary, Canada
[4]Cambridge Quantum Computing Ltd
2021-04-17

Definition 2.36 (Cups and caps)**.** A theory has *cups* and *caps* if for each system it has processes:



which satisfy:

(12)

Equivalently, this means that in diagrams inputs can be connected to inputs, outputs to outputs, and also that loops are allowed.

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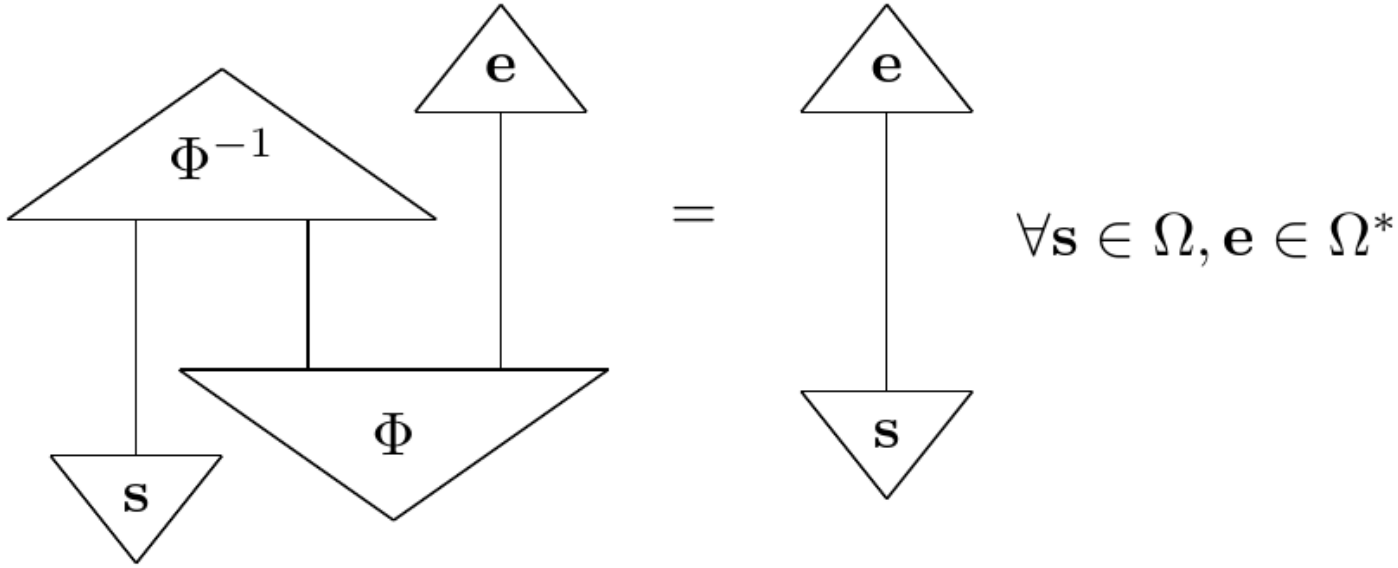

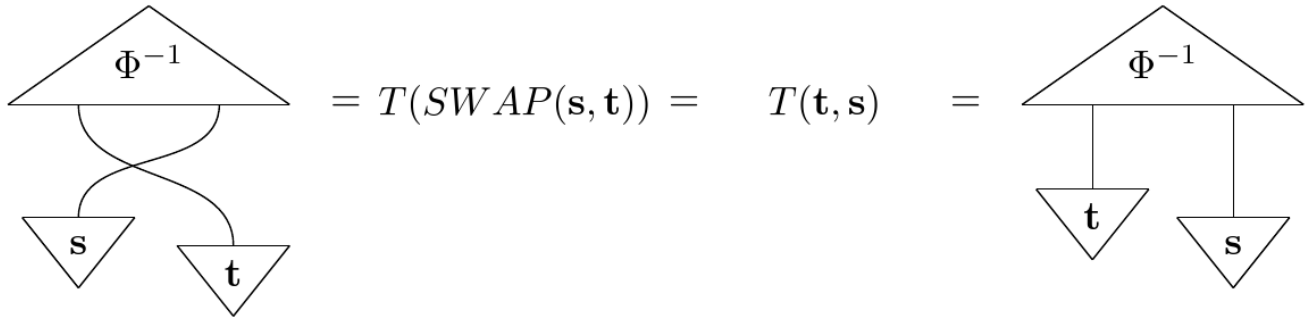

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