An introduction to contextuality and quantum advantage

Part 2

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INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

Quantum and Linear-Optical Computation (QLOC) group meeting
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- A hypothesis: this is related to **non-classical** features of quantum mechancics.
- In this talk, we focus on **non-local** and **contextual** behaviours as quantum resources.
- Contextuality is a feature of **empirical data** that is a key signature of non-classicality.

Recap on contextuality

‘*The sheaf-theoretic structure of non-locality and contextuality*’

Abramsky & Brandenburger, New Journal of Physics, 2011.

‘*Contextuality, cohomology, and paradox*’

Abramsky, B, Kishida, Lal, & Mansfield, CSL 2015.

(cf. Cabello–Severini–Winter, Acín–Fritz–Leverrier–Sainz)

The essence of contextuality

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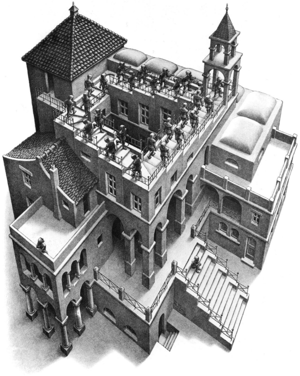

M. C. Escher, *Ascending and Descending*

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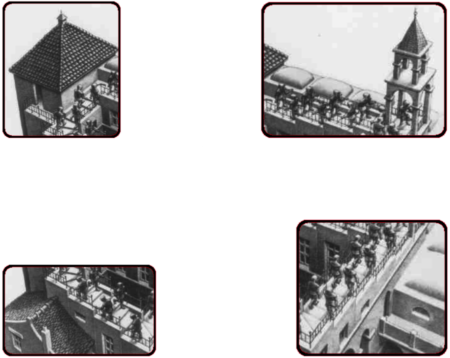

Local consistency

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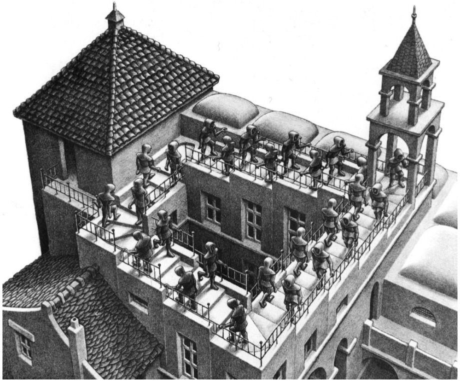

Local consistency *but* **Global inconsistency**

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E.g. $X = \{a_0, a_1, b_0, b_1\}$, $\quad O_x = \{0, 1\}$, $\quad \Sigma = \downarrow \{ \{a_0, b_0\}, \{a_0, b_1\}, \{a_1, b_0\}, \{a_1, b_1\} \}$.

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  - restricts to $[a, b \mapsto 0, 1]|_{\{a\}} = [a \mapsto 0] \in \mathbf{O}_{\{a\}}$.

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  - The requirement is that $e_\sigma|_\tau = e_\tau$.
  - So the statistics for a (joint) measurement τ are independent of what other measurements are also performed together with it.

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$$X = \{a_0, a_1, b_0, b_1\}, \qquad O_x = \{0,1\}, \qquad \Sigma = \downarrow \{\, \{a_0, b_0\}, \{a_0, b_1\}, \{a_1, b_0\}, \{a_1, b_1\} \,\}.$$

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An empirical model $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on a measurement scenario (X, Σ, O) is **non-contextual** if there is a probability distribution d on $\mathbf{O}_X = \prod_{x \in X} O_x$ such that:

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The import of Bell's and Bell–Kochen–Specker's theorems is that there are behaviours arising from quantum mechanics that are contextual.

Quantifying contextuality

Contextuality and advantages

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- Contextuality has been associated with quantum advantage in information-processing and computational tasks.
- Measure of contextuality $\rightsquigarrow$ **quantify such advantages**.

'*Contextuality fraction as a measure of contextuality*'
Abramsky, B, & Mansfield, Physical Review Letters, 2017.

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$$e = \lambda e^{NC} + (1 - \lambda) e'$$

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$$\mathrm{NCF}(e) = \lambda \qquad \mathrm{CF}(e) = 1 - \lambda$$

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- $\mathsf{CF}(e)$ is calculated via linear programming, the dual LP yields this inequality.

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- We have

$$\boxed{1 - \bar{p}_S \quad \geq \quad \mathrm{NCF}\, \frac{n-k}{n}}$$

Contextuality and advantage in quantum computation

- Measurement-based quantum computation (MBQC)

  '*Contextuality in measurement-based quantum computation*'
  Raussendorf, Physical Review A, 2013.

- Magic state distillation

  '*Contextuality supplies the 'magic' for quantum computation*'
  Howard, Wallman, Veitch, Emerson, Nature, 2014.

- Shallow circuits

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  - Contextuality analysis: Aasnæss, Forthcoming, 2020.

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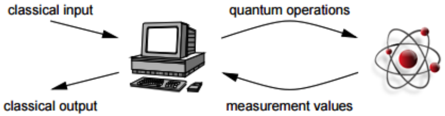

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- Probabilistic version: non-linear function computed with sufficently large probability of success implies contextuality.

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- Then,

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Further topics

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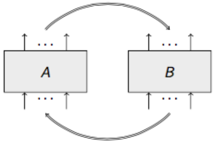

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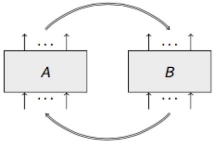

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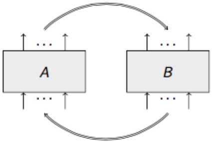

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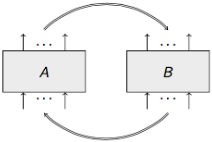

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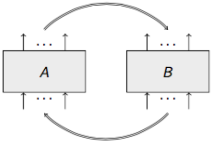

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Questions...

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