Quantum Walks, Algorithms and Implementations

Bruno Chagas

Irish Centre for High-End Computing (ICHEC)

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Overview

Table of Contents

First steps

Random Walks

- M. Dyer, A. Frieze, and R. Kannan (1991):
  A random polynomial-time algorithm for approximating the volume of convex bodies.
- R. Motwani and P. Raghavan (1995):
  Randomized Algorithms.
- U. Schöning (1999):
  A probabilistic algorithm for k-sat and constraint satisfaction problems.
- M. Jerrum, A. Sinclair, and E. Vigoda (2004):
  A polynomial-time approximation algorithm for permanent of a matrix with nonnegative entries.

Keep walking

Quantum Walks

- Y. Aharonov, L. Davidovich, and N. Zagury (1993):
  Quantum random walks.
- E. Farhi and S. Gutmann (1998):
  Quantum computation and decision trees.
- M. Szegedy (2004):
  Quantum speed-up of markov chain based algorithms
- A. Patel, K. Raghunathan, and P. Rungta (2005). Quantum random walks do not need a coin toss.
- A. Childs (2009):
  Universal computation by quantum walk.
- R. Portugal (2016):
  The staggered quantum walk model.

Table of Contents

CQW + Infinite Line Graph

The space of our quantum walk is composed by coin $\mathcal{H}_\mathcal{C}$ and walker spaces $\mathcal{H}_\mathcal{W}$, and we have $\mathcal{H} = \mathcal{H}_\mathcal{C} \otimes \mathcal{H}_\mathcal{W}$. The evolution operator consists of tossing a coin and performing a shift, and we say

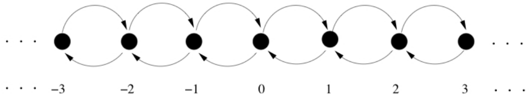
$$U = S(C \otimes I_W) \quad \longrightarrow \quad |\psi(t)\rangle = U^t |\psi(0)\rangle \tag{1}$$

We can describe the shift operator as

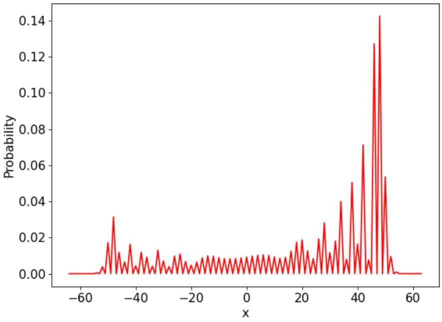
$$\begin{aligned} S|0\rangle|x\rangle &= |0\rangle|x+1\rangle \\ S|1\rangle|x\rangle &= |1\rangle|x-1\rangle \end{aligned} \tag{2}$$

and S in the computational basis has the format

$$S = |0\rangle\langle 0| \otimes \sum_x |x+1\rangle\langle x| + |1\rangle\langle 1| \otimes \sum_x |x-1\rangle\langle x| \tag{3}$$

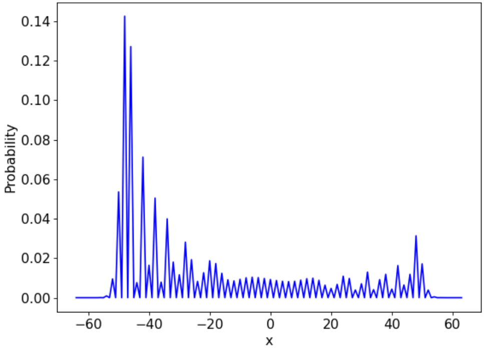


Simulation & Remarks



We consider $t = 70$, initial condition, and the Hadamard coin, as

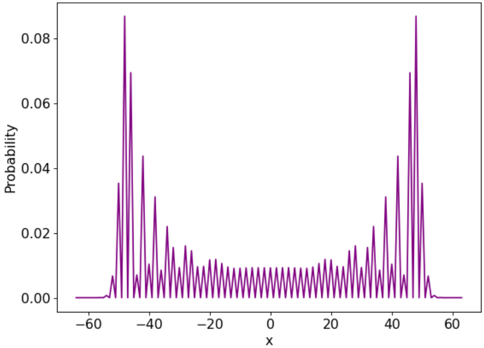
$$|\psi(0)\rangle = |0\rangle\,|0\rangle\,, C = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{4}$$



We consider $t = 70$, initial condition, and the Hadamard coin, as

$$|\psi(0)\rangle = |1\rangle\,|0\rangle\,, C = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{5}$$

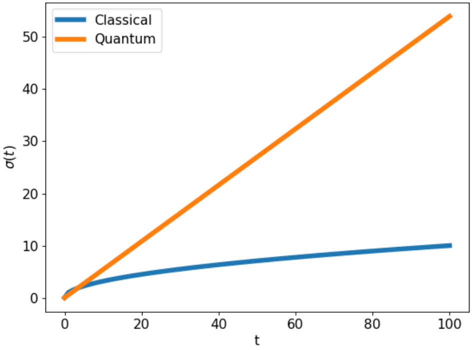
Simulation & Remarks



We consider $t = 70$, initial condition, and the Hadamard coin, as

$$|\psi(0)\rangle = \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}}\,|0\rangle\,, C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{6}$$

Simulation & Remarks



Classical: $\sigma(t) \sim \sqrt{t}$

Quantum: $\sigma(t) \sim t$

CTQW + Infinite Line Graph

Given a graph G defined by its Laplacian matrix $\mathcal{L}$, we can define a relation between a random walk and a quantum walk by

$$\frac{\partial p(x,t)}{\partial t} = \gamma \mathcal{L} p(x,t) \quad \longrightarrow \quad i \frac{\partial \left| \psi(x,t) \right\rangle}{\partial t} = H \left| \psi(x,t) \right\rangle \tag{7}$$

Considering γ a jumping-rate (amplitude per time) and $H = -\gamma \mathcal{L}$, we have the solution

$$U = e^{i\gamma \mathcal{L} t} \quad \longrightarrow \quad \left| \psi(t) \right\rangle = U \left| \psi(0) \right\rangle \tag{8}$$

We may define the adjacency matrix for an infinite line

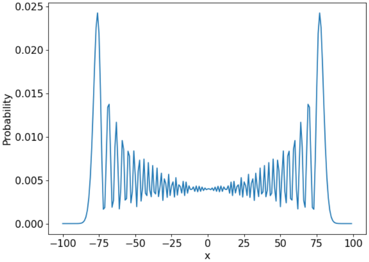
$$A = \sum_x \left| x+1 \right\rangle \left\langle x \right| + \left| x \right\rangle \left\langle x+1 \right| \tag{9}$$

and its Laplacian matrix as

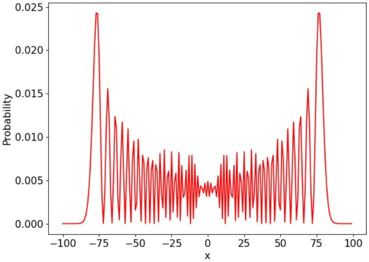
$$\mathcal{L} = A - D = \sum_x \left| x+1 \right\rangle \left\langle x \right| + \left| x \right\rangle \left\langle x+1 \right| - 2I \tag{10}$$

where D denotes the degree matrix of a graph.

Simulation & Remarks



(a) $|\psi_a\rangle$



(b) $|\psi_b\rangle$

We consider here $t = 40$, $\gamma = 1$ and initial conditions

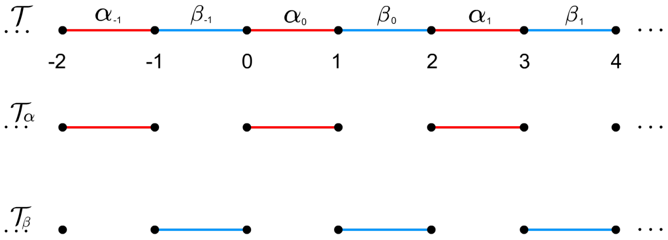
$$|\psi_a\rangle = |0\rangle \quad \text{and} \quad |\psi_b\rangle = |+\rangle \tag{11}$$

SQW + Tesselation & Infinite Line

We construct this walk based on a graph **tesselation**, that is a set of disjoint cliques over all vertices; and a graph **covering**, that is a family of tesselations, where every edge belongs to, at least, one tesselation.

Considering an infinite line, we can tesselate this graph as

$$\begin{aligned} \mathcal{T}_\alpha &= \{\{2x, 2x+1\} : x \in \mathbb{Z}\}, \\ \mathcal{T}_\beta &= \{\{2x+1, 2x+2\} : x \in \mathbb{Z}\}. \end{aligned}$$



SQW + Operators & Infinite Line

We may define states associated to each tesselation such as

$$\begin{aligned} |\alpha_x\rangle &= \frac{|2x\rangle + |2x+1\rangle}{\sqrt{2}}, \\ |\beta_x\rangle &= \frac{|2x+1\rangle + |2x+2\rangle}{\sqrt{2}}. \end{aligned}$$

and operators associated to each tesselation as

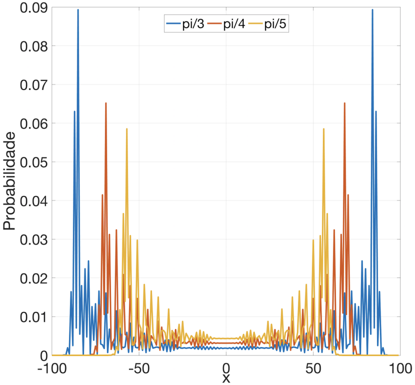
$$\begin{aligned} H_\alpha &= 2\sum_{x=-\infty}^{\infty} |\alpha_x\rangle\langle\alpha_x| - I, \\ H_\beta &= 2\sum_{x=-\infty}^{\infty} |\beta_x\rangle\langle\beta_x| - I. \end{aligned}$$

Finally, we can describe the evolution operator as

$$U = e^{i\theta_\beta H_\beta} e^{i\theta_\alpha H_\alpha}.$$

where $\theta_\alpha, \theta_\beta \in [0, \pi]$

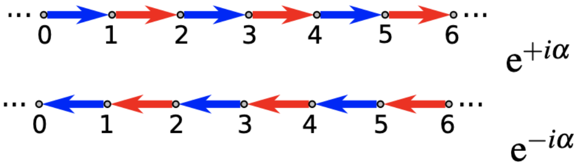
Simulation & Remarks



We consider $\theta_\alpha = \theta_\beta = \theta$, after $t = 50$, and initial condition

$$|\psi(0)\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \tag{12}$$

B. Chagas and R. Portugal (2020). Discrete-time quantum walks on oriented graphs.



We can modify the operators in order to have a sense of direction as

$$H_0 = \sum_x e^{-i\alpha} |2x-1\rangle \langle 2x| + e^{+i\alpha} |2x\rangle \langle 2x-1| ,$$
$$H_1 = \sum_x e^{-i\alpha} |2x\rangle \langle 2x+1| + e^{+i\alpha} |2x+1\rangle \langle 2x| .$$

and the evolution will be

$$U = e^{i\theta_\beta H_1} e^{i\theta_\alpha H_0} .$$

Simulation & Remarks

Defining the standard deviation as $\sigma(t) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$, we have the moments

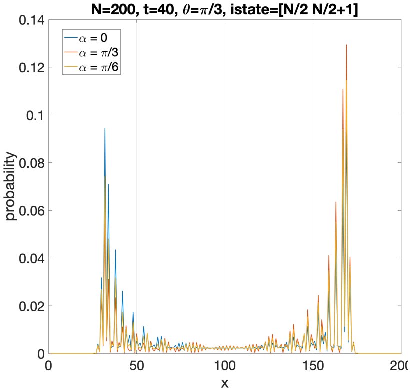
$$\begin{aligned}
\frac{\langle x \rangle}{t} &= 2(|a|^2 - |b|^2)(1 - \cos\theta) + \frac{i \sin 2\theta(\bar{a}be^{i\alpha} - a\bar{b}e^{-i\alpha})}{1 + |cos\theta|} + \mathcal{O}\left(\frac{1}{t}\right) \\
\frac{\langle x^2 \rangle}{t^2} &= 4(1 - |\cos\theta|) + \mathcal{O}\left(\frac{1}{t}\right)
\end{aligned}$$

considering the initial condition

$$|\psi(0)\rangle = a\,|0\rangle + b\,|b\rangle \tag{13}$$

where $|a|^2 + |b|^2 = 1$.

Simulation & Remarks



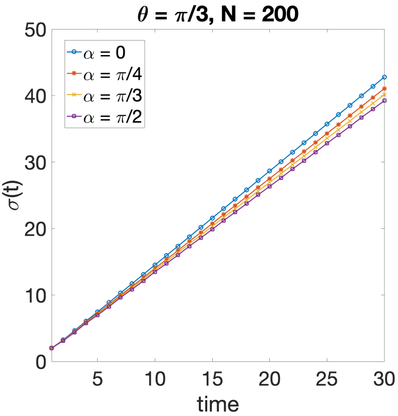


Table of Contents

Quantum Walk + Algorithms

- Triangle Finding
  - F. Magniez, M. Santha, and M. Szegedy (2003). Quantum algorithms for the triangle problem.
  - Problem: Given a graph G on n nodes, find a triangle, if there is any.
- Element Distinctness
  - A. Ambainis (2014). Quantum walk algorithm for element distinctness
  - Problem: determine whether the elements of a list are distinct.
- Matrix Product Verification
  - H. Buhrman, and R. Špalek (2005). Quantum verification of matrix products.
  - Problem: Let A, B, C be $n \times n$ matrices over any integral domain. A verification of a matrix product is deciding whether $AB = C$.
- Group Commutativity
  - F. Magniez, and A. Nayak (2005). Quantum complexity of testing group commutativity
  - Problem: Given a black-box group G with generators $g_1, ..., g_k$, decide if G is abelian

Searching Problems

Searching Algorithms: Given a list of elements, find a marked element, if there is any.

- L. Grover (1996):
  A fast quantum mechanical algorithm for database search.
- M. Boyer, et al (1996):
  Tight bounds on quantum searching.
- C. Zalka (1999):
  Grover's quantum searching algorithm is optimal.

Searching Algorithms Based on Quantum Walks

- A. Childs, and J. Goldstone (2004). Spatial search by quantum walk.
- R. Portugal (2018). Quantum walks and search algorithms.
- J. Janmark, D. Meyer, and T. Wong (2014). Global symmetry is unnecessary for fast quantum search.
- S. Chakraborty, L. Novo, A. Ambainis, and Y. Omar. Spatial search by quantum walk is optimal for almost all graphs.

CTQW + Searching Algorithm

Given a graph G defined by its Laplacian matrix $\mathcal{L}$, we can modify the Hamiltonian

$$H = -\gamma \mathcal{L} \tag{14}$$

by introducing the oracle Hamiltonian

$$H_w = - |w\rangle \langle w| \tag{15}$$

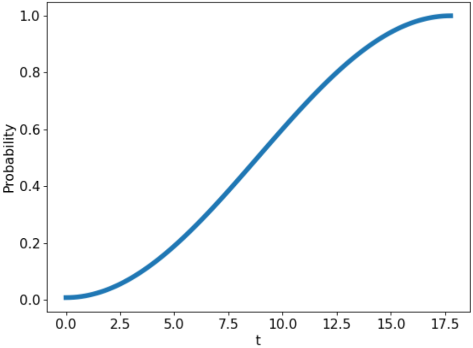
and now on we consider the time-independent Hamiltonian,

$$H = -\gamma L + H_w = -\gamma L - |w\rangle \langle w| . \tag{16}$$

Moreover, the evolution operator will be

$$|\psi(t)\rangle = e^{itH} |s\rangle \tag{17}$$

where $|s\rangle$ denotes a uniform superposition.



The optimal case, for one marked element over a clique graph, occurs when

$$t = \frac{\pi}{2}\sqrt{N} \quad \text{and} \quad \gamma = \frac{1}{N} \tag{18}$$

SQW + Searching Algorithm

Given the evolution operator for a general staggered quantum walk as

$$U = \prod_k e^{i\theta_k H_k} \tag{19}$$

we can add an oracle unitary operator, considering the marked element w, as

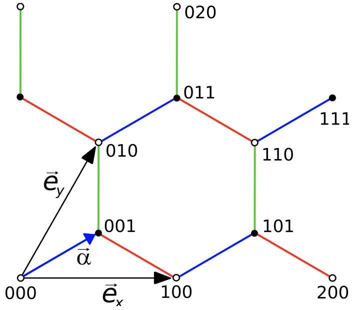
$$U_w = I - 2\left|w\right\rangle\left\langle w\right| \tag{20}$$

and the searching evolution consists of

$$\left|\psi(t)\right\rangle = (UU_w)^t \left|s\right\rangle, \tag{21}$$

where $\left|s\right\rangle$ is the uniform superposition state.

Hexagonal Lattice



$$|\alpha_{x,y}\rangle = \frac{1}{\sqrt{2}}\big(\,|x,y,1\rangle + |x+1,y,0\rangle\,\big)$$

$$|\beta_{x,y}\rangle = \frac{1}{\sqrt{2}}\big(\,|x,y,1\rangle + |x,y+1,0\rangle\,\big)$$

$$|\gamma_{x,y}\rangle = \frac{1}{\sqrt{2}}\big(\,|x,y,0\rangle + |x,y,1\rangle\,\big)$$

Evolution Operator

The evolution operator for this graph will be

$$U = e^{i\theta H_\gamma} e^{i\theta H_\beta} e^{i\theta H_\alpha} \tag{22}$$

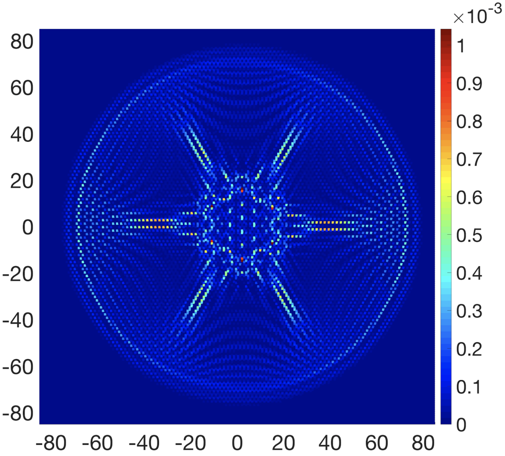
considering the operators

$$\begin{aligned}
H_\alpha &= 2 \sum_{x,y=0}^{n-1} |\alpha_{x,y}\rangle \langle \alpha_{x,y}| - I, \\
H_\beta &= 2 \sum_{x,y=0}^{n-1} |\beta_{x,y}\rangle \langle \beta_{x,y}| - I, \\
H_\gamma &= 2 \sum_{x,y=0}^{n-1} |\gamma_{x,y}\rangle \langle \gamma_{x,y}| - I,
\end{aligned}$$

where

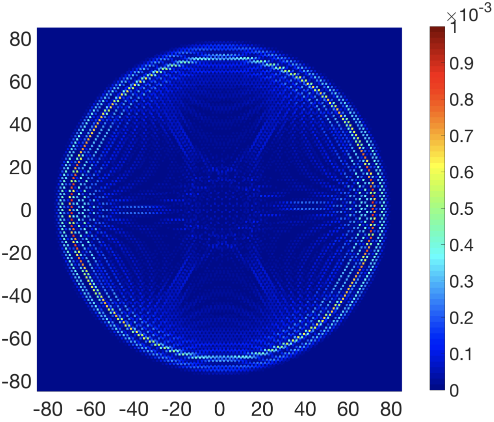
$$|\psi_t\rangle = U^t |\psi_0\rangle$$

n=121, t=58, $\theta = \pi/3$, $\left|\psi_0^b\right\rangle$



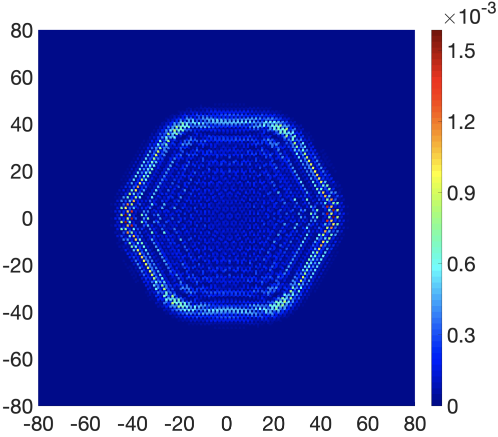
$$\left|\psi_0^b\right\rangle = \frac{1}{\sqrt{2}}(|1,1,0\rangle + |1,0,1\rangle)$$

n=121, t=58, $\theta = \pi/3$, $|\psi_0^c\rangle$



$$|\psi_0^c\rangle = \frac{1}{\sqrt{6}}(|1,1,0\rangle + |1,0,1\rangle + |1,0,0\rangle + |0,0,1\rangle + |0,1,0\rangle + |0,1,1\rangle)$$

n=121, t=58, $\theta = \pi/6$, $|\psi_0^c\rangle$



$$|\psi_0^c\rangle = \frac{1}{\sqrt{6}}(|1,1,0\rangle + |1,0,1\rangle + |1,0,0\rangle + |0,0,1\rangle + |0,1,0\rangle + |0,1,1\rangle)$$

Searching Problem

B. Chagas, R. Portugal, S. Boettcher, and E. Segawa (2018). Staggered Quantum Walk on Hexagonal Lattices.

We considered the evolution operator

$$|\psi(t)\rangle = (UU_w)^t |s\rangle, \quad (23)$$

where U is the staggered quantum walk operator for the hexagonal graph, and U_w the oracle operator. We've got the following time execution

$$t = \Theta(\sqrt{N \ell n N})$$

Table of Contents

Some questions

What about plannar graphs?

- B. Chagas, R. Portugal, S. Boettcher, and E. Segawa (2018). Staggered Quantum Walk on Hexagonal Lattices.
- R. Portugal, and T. Fernandes (2017). Quantum search on the two-dimensional lattice using the staggered model with Hamiltonians

Some questions

What about physical realizations of continuous-time quantum walks?

- R. Balu, D. Castillo, and G. Siopsis (2018). Physical realization of topological quantum walks on IBM-Q and beyond Staggered Quantum Walk on Hexagonal Lattices.
- F. Acasiete, F. Agostini, J. Moqadam, and R. Portugal (2020). Experimental Implementation of Quantum Walks on IBM Quantum Computers

Some questions

What's the relation between angles and tesselation in staggered quantum walks?

- A. Abreu, L. Cunha, T. Fernandes, C. de Figueiredo, L. Kowada, F. Marquezino, D. Posner, R. Portugal (2017). The tessellation problem of quantum walks
- R. Santos (2018). The role of tessellation intersection in staggered quantum walks.

Questions?

Obrigado!