Feynman path integrals in linear optics

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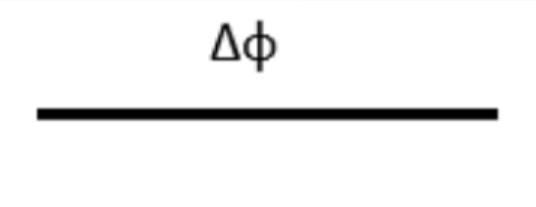
Linear optics

- An interferometer consists of a network of beam splitters and phase shifters connected by waveguides



- Photons can exist in different spatial modes one for each of the possible waveguide tracks

$$\hat{a}_1^\dagger |0\rangle = |1\rangle \quad \hat{a}_1^\dagger \hat{a}_2^\dagger |1,1\rangle \quad \frac{1}{\sqrt{2!}}(\hat{a}_1^\dagger)^2 \hat{a}_3^\dagger = |2,0,1\rangle$$

Linear optics

- Beam splitters and phase shifters are described by unitary matrices acting on the modes

$$\begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1$$

$$\hat{a}^\dagger \mapsto \alpha \hat{a}^\dagger + \beta \hat{b}^\dagger$$

$$\hat{b}^\dagger \mapsto -\beta^* \hat{a}^\dagger + \alpha^* \hat{b}^\dagger$$

$\Delta\phi$

$$\hat{a}^\dagger \mapsto e^{i\phi} \hat{a}^\dagger$$

Linear optics

- An interferometer is described by an mxm unitary matrix

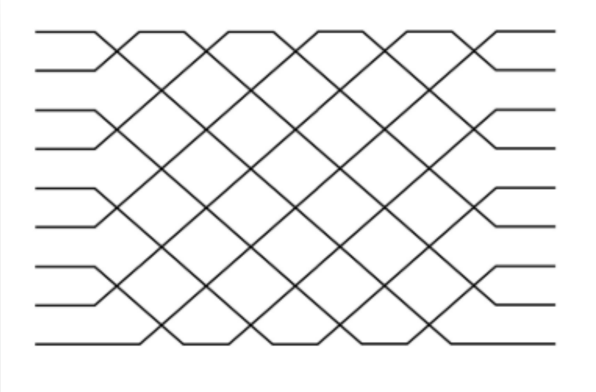
$$U$$

- The unitary acts of the set of creation operators

$$U\left|n_1, n_2, \ldots, n_M\right\rangle = \prod_i^M \frac{1}{\sqrt{n_i!}}(U\hat{a}_i^\dagger)^{n_i}\left|0\right\rangle = \prod_i^M \frac{1}{\sqrt{n_i!}}(\sum_{j=1}^M U_{ij}\hat{a}_j^\dagger)^{n_i}\left|0\right\rangle$$

Linear optics comparison to quantum circuits

- Quantum circuit -> Interferometer



- Qubit -> Single mode Fock state

$$\frac{1}{\sqrt{n!}}\left(\hat{a}^{\dagger}\right)^{n}|0\rangle = |n\rangle$$

- Qubits -> Multi mode Fock state

$$\frac{1}{\sqrt{n_1!n_2!\ldots n_m!}}\left(\hat{a_1}^{\dagger}\right)^{n_1}\left(\hat{a_2}^{\dagger}\right)^{n_2}\ldots\left(\hat{a_m}^{\dagger}\right)^{n_m}|0\rangle = |n_1, n_2\ldots, n_m\rangle$$

Permanents

- To simulate the behaviour of the optical circuit we need to calculate transition amplitudes

$$\langle k_1, k_2, ..., k_M | U | n_1, n_2, ..., n_M \rangle$$

- These can be related to the permanent of some matrix constructed from matrix elements in the unitary

$$\mathrm{perm}(A) = \sum_{\sigma \in S_N} \prod_{i}^{N} A_{i\sigma(i)}$$

$$\langle k_1, k_2, ..., k_M | U | n_1, n_2, ..., n_M \rangle = \left(\prod n_i!\right)^{-\frac{1}{2}} \left(\prod k_j!\right)^{-\frac{1}{2}} \mathrm{perm}(U[\psi' | \psi])$$

S. Scheel,"Permanents in linear optical networks,"2004. [Online]. Available:http://arxiv.org/abs/quant-ph/0406127

Permanents

- The constructed matrix depends on the input and output state

$$U[(1^1, 2^2, 3^1)|(1^2, 2^2, 3^0)] = \begin{bmatrix} U_{11} & U_{11} & U_{12} & U_{12} \\ U_{21} & U_{21} & U_{22} & U_{22} \\ U_{21} & U_{21} & U_{22} & U_{22} \\ U_{31} & U_{31} & U_{32} & U_{32} \end{bmatrix}$$

- A consequence is that the permanent of U is related to the transition amplitude

$$\mathrm{perm}(U) = \langle 1_1, 1_2, .., 1_M | \, U \, | 1_1, 1_2, ..., 1_M \rangle$$

Permanents

- The complexity of simulating linear optics is then the complexity of calculating permanents

- The Best known algorithm is given by Ryser

$$\mathrm{perm}(A) = (-1)^N \sum_{S \subseteq \{1,\dots,N\}} (-1)^{|S|} \prod_{i=1}^{N} \sum_{j \in S} A_{ij}$$

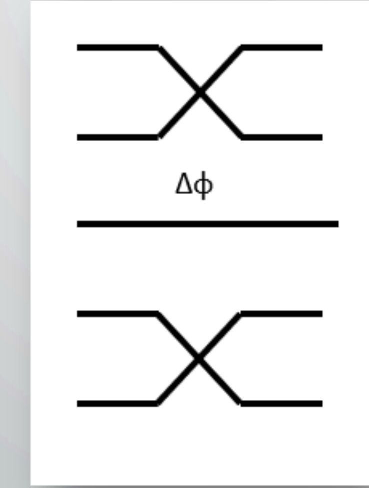
$$\text{Run time} = O(N 2^{N-1})$$

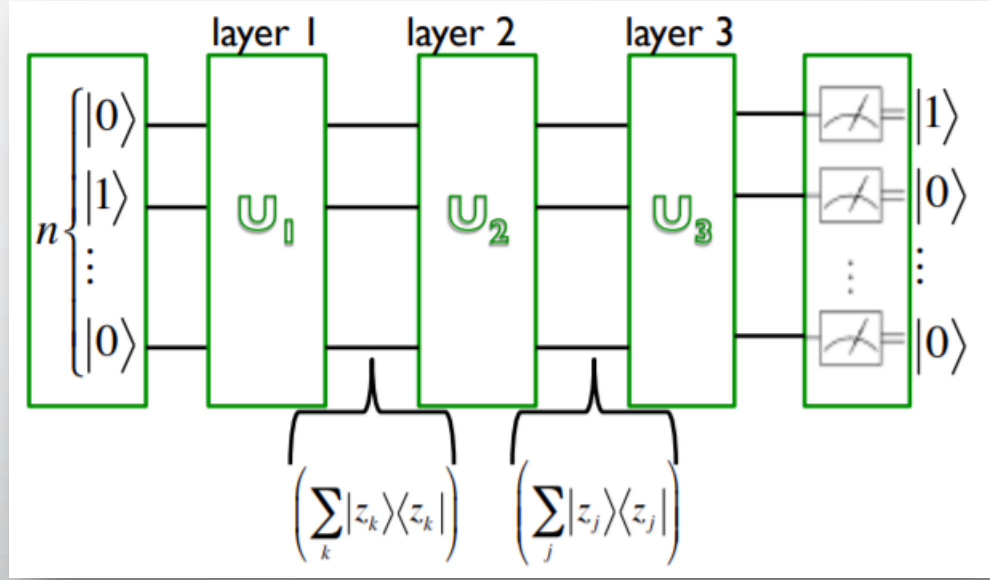
$$\text{Memory} = O\left(N^2\right)$$

A. Nijenhuis and W. Herbert,Combinatorial Algorithms, 1978

Feynman path integral approach

- Similar to the qubit case we break down our optical circuit into layers





Feynman path integral approach

- For three layers the transition amplitude is expressed as the sum over paths

$$\begin{aligned}\langle y|U|x\rangle = \langle y|U_3U_2U_1|x\rangle &= \langle y|U_3\left(\sum_i |z_i\rangle\langle z_i|\right)U_2\left(\sum_j |z_j\rangle\langle z_j|\right)U_1|x\rangle \\ &= \sum_{i,j}\langle y|U_3|z_i\rangle\langle z_i|U_2|z_j\rangle\langle z_j|U_1|x\rangle\,.\end{aligned}$$

Feynman path integral approach

- For a general circuit the transition amplitude would be given by

$$\langle y|\, U \,|x\rangle = \sum_{z \in B^{D+1}} \prod_{i=0}^{D-1} \langle z_i|\, U_i \,|z_{i+1}\rangle$$

- Where $z_0 = x$ and $z_D = y$

Feynman path integral approach

- The photon number N, must be conserved at each stage in the optical circuit
- The number of paths in the sum is then given by the number of N photon states across M modes to the power of D-1

$$\frac{(N+M-1)^{N(D-1)}}{N^N} \leq \binom{N+M-1}{N}^{D-1} \leq \frac{(N+M-1)^{N(D-1)}}{N!}$$

Feynman path integral approach

- Each path in the sum contributes a product of transition amplitudes

$$\langle y| U_3 |z_i\rangle \langle z_i| U_2 |z_j\rangle \langle z_j| U_1 |x\rangle$$

- So we need a method to calculate the transition amplitude for a single layer

- This would be given by some permanent

Permanents of layers

- We could use Rysers formula, although this would not be efficient

- Since the transition amplitudes are for single layers we can exploit their structure to calculate their permanents more efficiently

Permanents of layers

- For a layer in the interferometer in the worst case we would have around m/2 beam splitters

- Following a result from V. S. Shchesnovich using a modified Glyns formula the time complexity for calculating the permanent for a beam splitter with n input photons, m_1 and m_2 photons in the output modes

$$O\left(n\frac{(m_1+1)(m_2+1)}{\min(m_l+1)}\right)$$

- Maximising this complexity for the worst case gives

$$O(n^2)$$

V. Shchesnovich, "On the classical complexity of sampling from quantum interference of in-distinguishable bosons," pp. 1–15, 2019. [Online]. Available: http://arxiv.org/abs/1904.02013

Permanents of layers

- For m/2 beam splitters the complexity would become

$$O(n_1^2 + ... + n_{\frac{m}{2}}^2)$$

- Which if we maximise for the worst case we get

$$O\left(\frac{N^2}{M}\right)$$

Feynman path integral complexity

- In total we have to perform D of these permanent calculations for each path in the sum

- The time complexity for calculating the path integral is then given by

$$O\left(D\frac{N^2}{M} \times \#paths\right)$$

Feynman path integral complexity

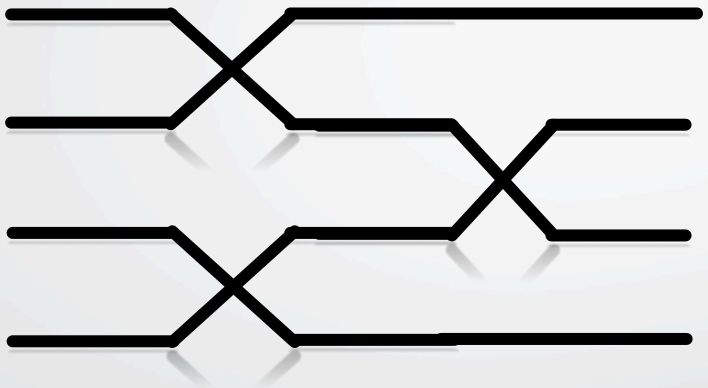
- For the memory complexity we consider what needs to be stored at any point in the algorithm
  1. A path
  2. Term phase
  3. Total phase
  4. Working phase

- A path consists of D+1 lists of M numbers whose size is a most N

$$\text{Memory} = O\left(\log\left(N\right)M\left(D+1\right)\right)$$

Special cases

- Consider the case for a depth two optical circuit in the Clements scheme



- The unitary for a layer will take the from of a block diagonal matrix with block sizes of at most two

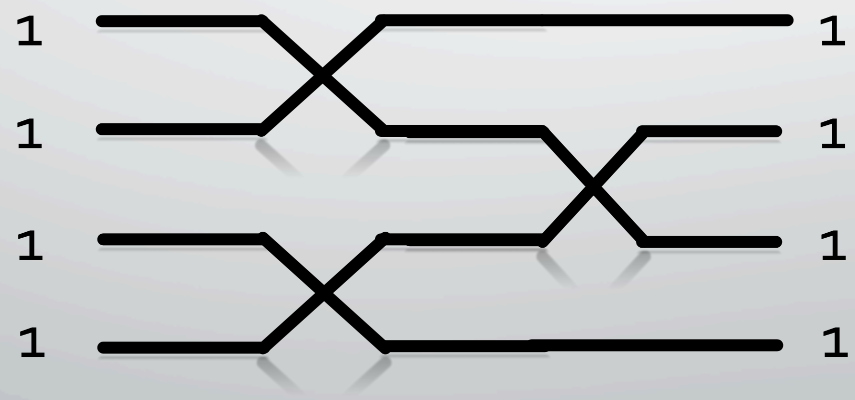
- The full unitary will be band diagonal with bandwidth at most four

W. R. Clements, P. C. Humphreys, B. J. Metcalf, W. S. Kolthammer, and I. A. Walsmley,“Optimal design for universal multiport interferometers,”Optica, vol. 3, no. 12, p. 1460, 2016.

Special cases

- If we wanted to use FPI to calculate the permanent of such a unitary we would need to know which paths to consider in the sum

- For depth two we only need to consider one state in the path



Special cases

- So for depth two there is only one path that can give a contribution

- Since we have a photon in each mode the complexity comes out as

$$O\left(D\frac{N^2}{M}\right) = O\left(2N\right) = O\left(N\right)$$

- Which is linear in the number of photons

Special cases

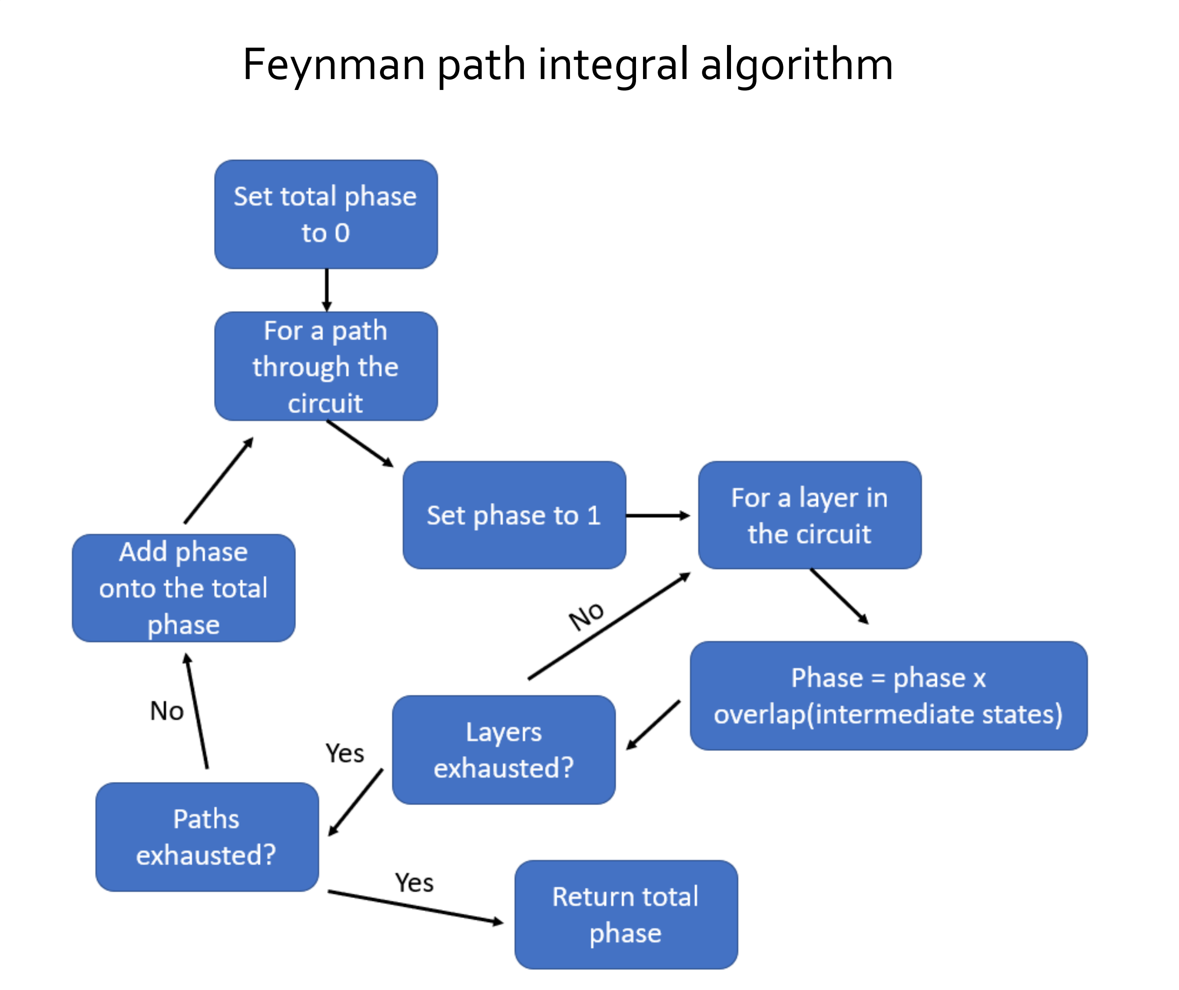
- We can perform a similar calculation for depth three circuits in the Clements scheme

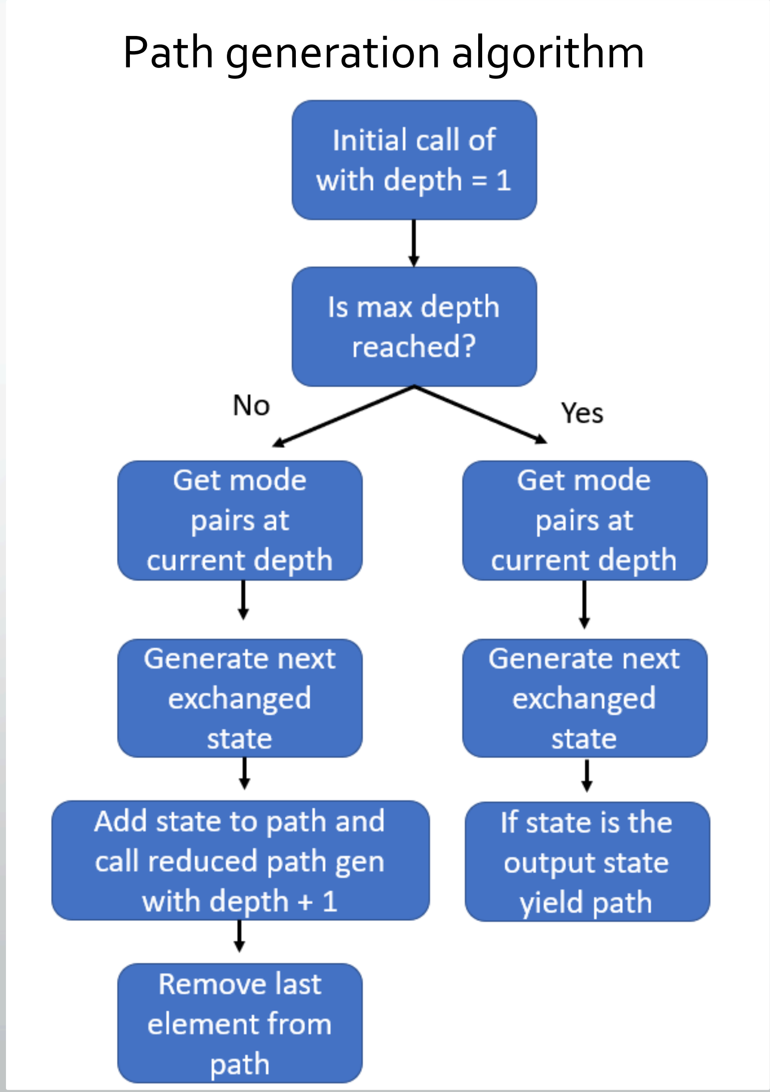
- This time however the number of paths is not constant and it can be shown that it depends on N

$$\#paths(D = 3) = 3^{\frac{N}{2}}$$

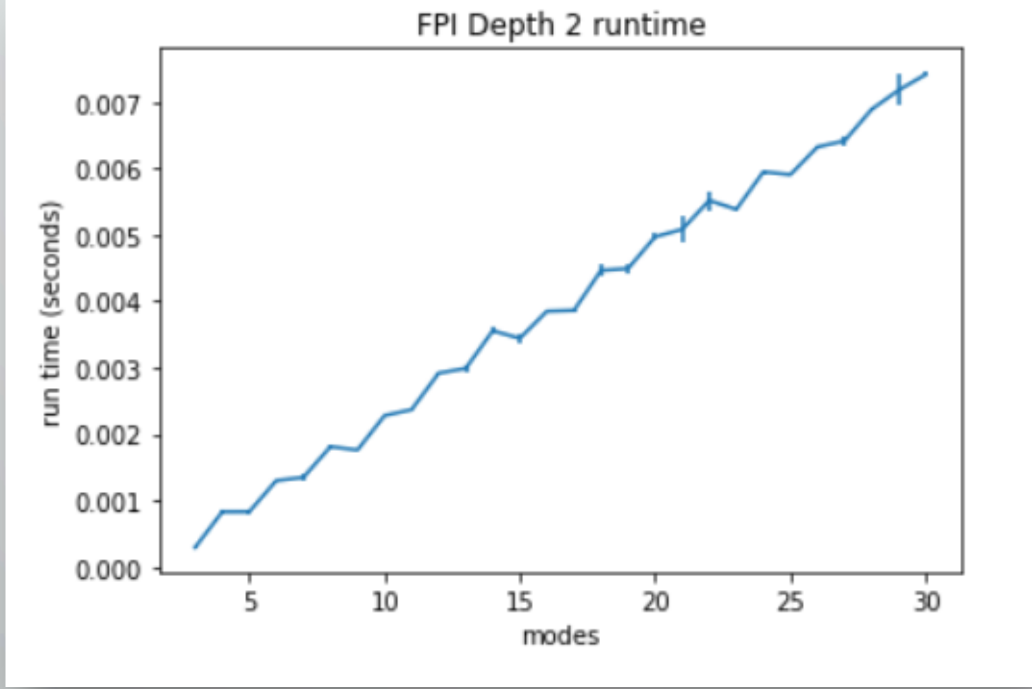
$$\text{Run time} = O\left(N3^{\frac{N}{2}}\right)$$

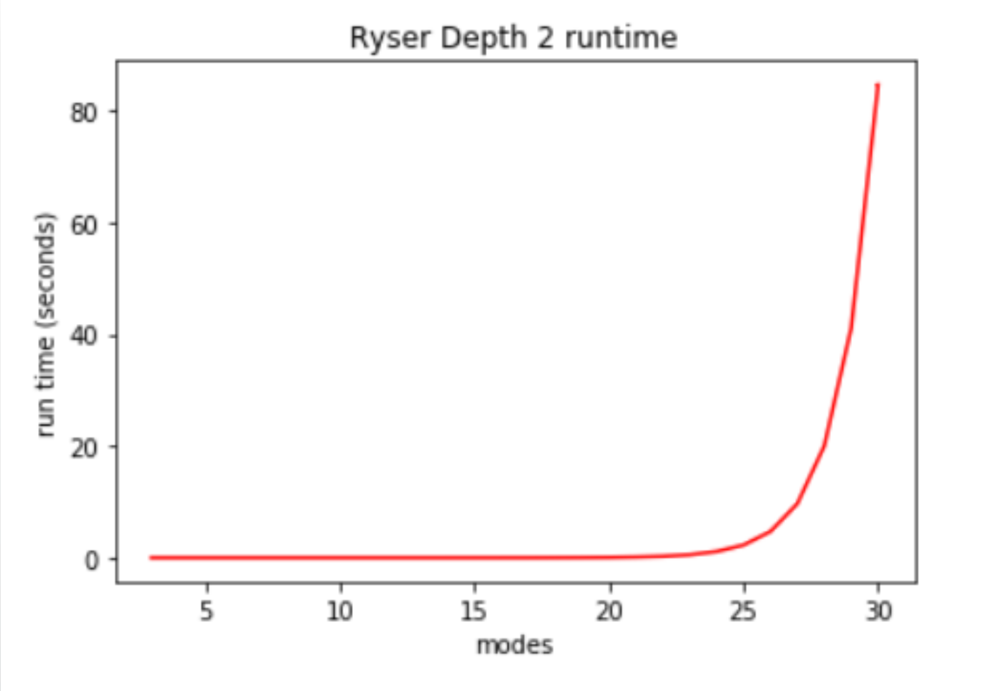
Python implementation





Comparison with Ryser





$$\text{Run time} = O(N)$$

$$\text{Run time} = O(N2^{N-1})$$

Future work

- Extending the method to calculate permanents of arbitrary matrices
  - Beam splitter elements would be replaced with arbitrary 2x2 matrices
  - Beam splitters elements can act on non adjacent modes

- There is no specification in the FPI protocol or the method form V. S. Shchesnovich that requires the matrices to be unitary

- Non local beam splitter layers can be viewed as local layers together with some permutation

$$\text{Run time} = O\left(N3^{\frac{N}{2}}\right)$$

Future work

- There exist efficient algorithms for calculating permanents of block factorizable and band diagonal matrices

$$A = F_1 F_2 \cdots F_L$$

$$\text{Block run time} = O\left(N 2^{3L^2}\right)$$

$$\text{Block memory} = O\left(N 2^{2L^2}\right)$$

$$\text{Banded run time} = O\left(N 2^{\omega}\right)$$

D. Cifuentes and P. A. Parrilo, “An efficient tree decomposition method for permanents and mixed discriminants,” pp. 1–32, 2018.

K. Temme and P. Wocjan, “Efficient Computation of the Permanent of Block Factorizable Matrices,” pp. 1–16, 2012. [Online]. Available: http://arxiv.org/abs/1208.6589

Summary

- Demonstrated how Feynman path integrals give a physically motivated method to simulating linear optics
- Feynman path integrals can be used to calculate permanents of certain sparse matrices that outperform Rysers' algorithm
- These results are comparable to other methods for calculating permanents of spare matrices that correspond to optical circuits of short depth