Rediscovering a family of hybrid algorithms with quantum singular value transforms

Miguel Murça[1,3,(2)], Duarte Magano[1,2]

18 October 2022

1 2 3

[Submitted on 29 Jul 2022]

Simplifying a classical-quantum algorithm interpolation with quantum singular value transformations

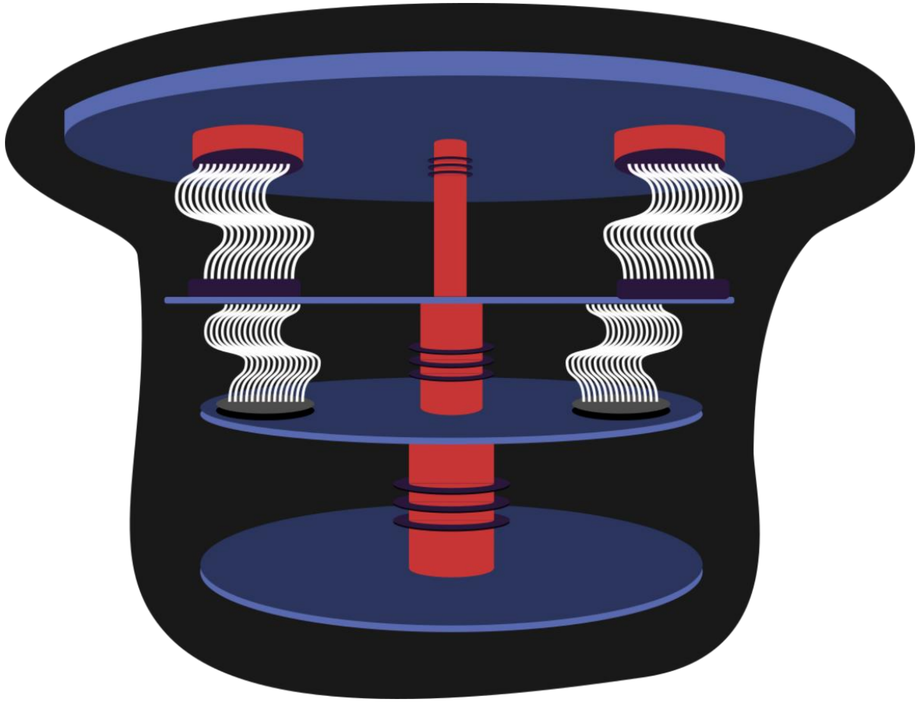
Duarte Magano, Miguel Murça

https://arxiv.org/abs/2207.14810

Outline

- Introduction
- Quantum Singular Value Transformations
- Quantum Phase Estimations
- "α-Eigenvalue Estimation"
- Conclusion

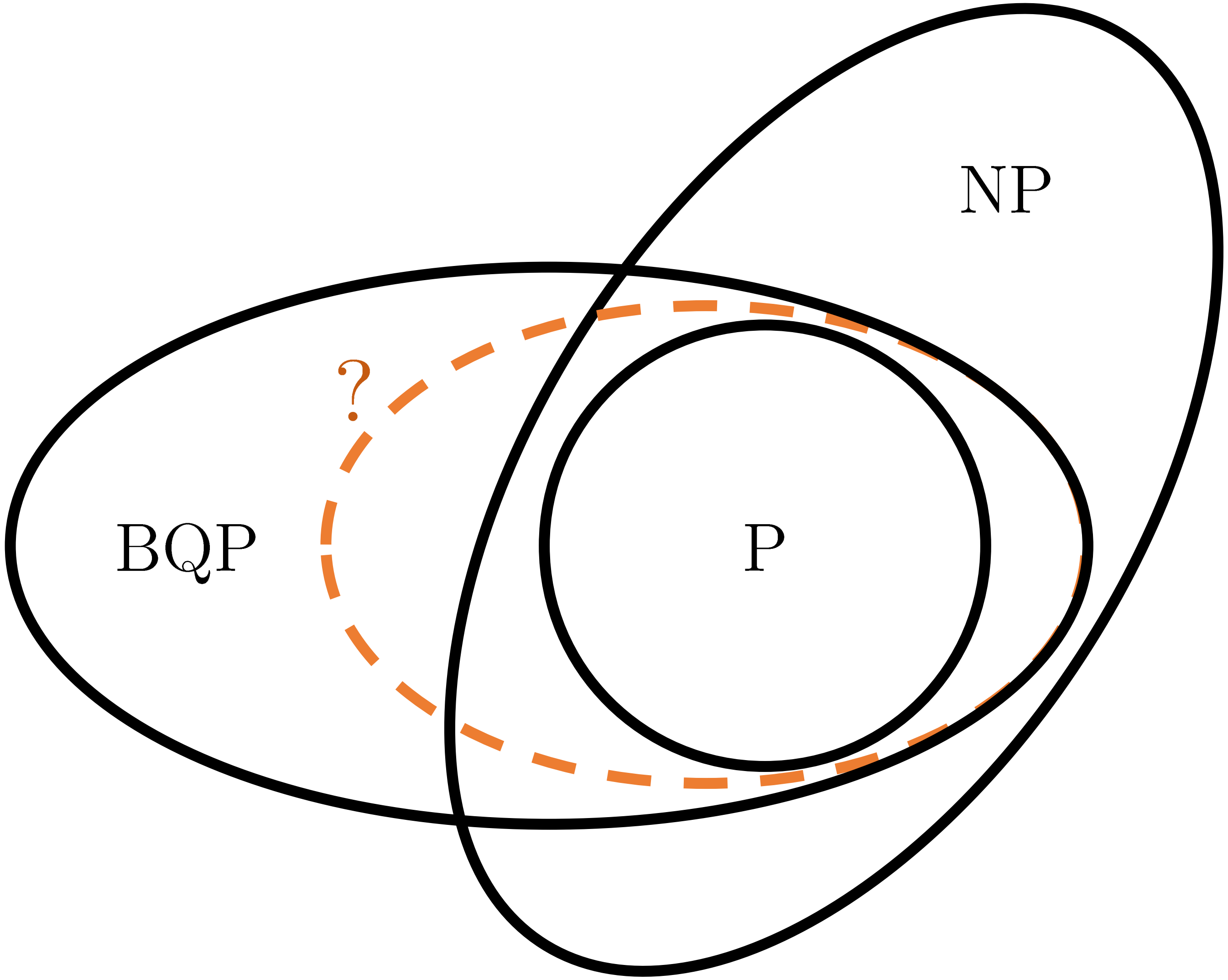
Introduction



Cannot fit in your theory team's room
Cannot run your brand new algorithm
Hard to get funding to buy one

Can fit in your theory team's room
Can simulate your proof-of-concept on 5 qubits
Your research project may fund one

Introduction



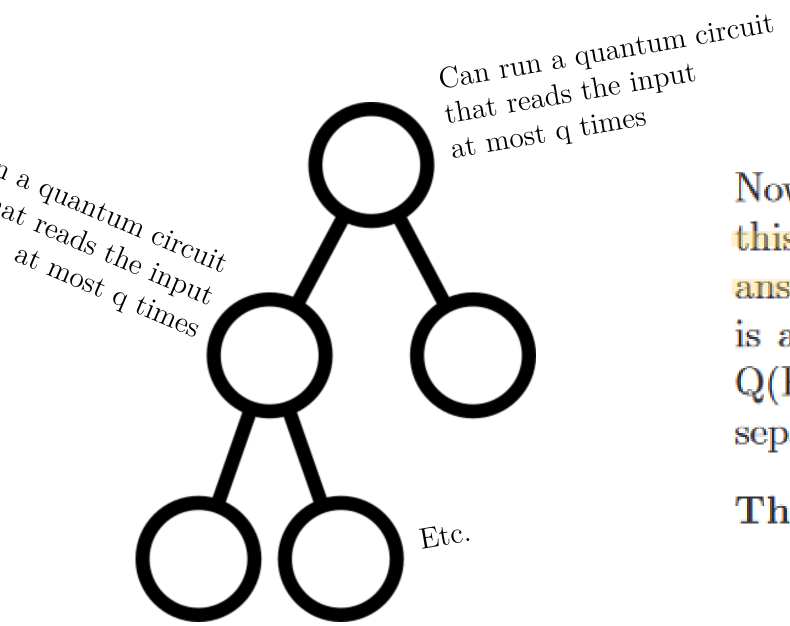
Introduction

[Submitted on 29 Nov 2019]

Hybrid Decision Trees: Longer Quantum Time is Strictly More Powerful

Xiaoming Sun, Yufan Zheng

https://arxiv.org/abs/1911.13091



Now that we have defined the hybrid decision tree model, one may ask: is the computing power of this model *strictly* stronger than the classical one, or weaker than the quantum one? The positive answer to the first question comes immediately, because of the Forrelation function FOR, which is a partial function satisfying $\mathrm{Q}(\textsc{For}) = 1$ and $\mathrm{R}(\textsc{For}) = \Omega(\sqrt{n}/\log n)$ [AA18], which implies $\mathrm{Q}(\textsc{For}; 1) = 1$. However, what if we require the function to be total? Theorem 1.3 says that the separation still exists.

Theorem 1.3. *There exists a total Boolean function f such that* $\mathrm{Q}(f; 1) = \widetilde{O}(\mathrm{R}(f)^{4/5})$.

Quantum Singular Value Transformations

Quantum Singular Value Transformations

[Submitted on 5 Jun 2018]

Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics

András Gilyén, Yuan Su, Guang Hao Low, Nathan Wiebe

[Submitted on 6 May 2021 (v1), last revised 10 Dec 2021 (this version, v5)]

A Grand Unification of Quantum Algorithms

John M. Martyn, Zane M. Rossi, Andrew K. Tan, Isaac L. Chuang

[Submitted on 20 Oct 2016 (v1), last revised 11 Jul 2019 (this version, v3)]

Hamiltonian Simulation by Qubitization

(Quantum Signal Processing)

Guang Hao Low, Isaac L. Chuang

Quantum Singular Value Transformations

- Quantum Signal Processing

$$W(a) = \begin{bmatrix} a & i\sqrt{1-a^2} \\ i\sqrt{1-a^2} & a \end{bmatrix}$$

"Signal Rotation Operator"

$$S(\phi) = e^{i\phi Z}$$

"Signal Processing Rotation Operator"

$$U_\phi = e^{i\phi_0 Z} \prod_{k=1}^{d} W(a) e^{i\phi_k Z}$$

"Quantum Signal Processing Sequence"

Quantum Singular Value Transformations

- Quantum Signal Processing

QSP Theorem

$$e^{i\phi_0 Z}\prod_{k=1}^{d} W(a)e^{i\phi_k Z} = \begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix}$$

(in the computational basis)

$$P \in \mathbb{R}_d[x]$$

$$Q \in \mathbb{R}_{d-1}[x]$$

Parity P,Q ≡ Parity $d,d\text{-}1$

$$|P|^2 + (1-a^2)|Q|^2 = 1$$

$$\forall\, a \in [-1,1]$$

Quantum Singular Value Transformations

- Quantum Eigenvalue Transformations

$$\mathcal{H} = \sum_{\lambda} \lambda \, |\lambda\rangle\langle\lambda|$$

$$U = Z \otimes \mathcal{H} + X \otimes \sqrt{I - \mathcal{H}^2}$$

$$U = \begin{bmatrix} \mathcal{H} & \sqrt{I - \mathcal{H}^2} \\ \sqrt{I - \mathcal{H}^2} & -\mathcal{H} \end{bmatrix} = \begin{bmatrix} \mathcal{H} & \cdot \\ \cdot & \cdot \end{bmatrix}$$

(1,1)–Block-encoding

Quantum Singular Value Transformations

- Quantum Eigenvalue Transformations

$$\Pi_\phi = e^{i\phi Z} \otimes I$$

$$U_\phi = \prod_{k=1}^{d/2} \Pi_{\phi_{2k-1}} U^\dagger \Pi_{\phi_{2k}} U$$

Quantum Singular Value Transformations

- Quantum Eigenvalue Transformations

$$U = \bigoplus_{\lambda} \underbrace{\begin{bmatrix} \lambda & \sqrt{1-\lambda^2} \\ \sqrt{1-\lambda^2} & -\lambda \end{bmatrix}}_{R(\lambda)}$$

$$U_\phi = \bigoplus_{\lambda} [\prod_{k=1}^{d/2} e^{i\phi_{2k-1}Z} R(\lambda) e^{i\phi_{2k}Z} R(\lambda)]$$

$$W(a) = \begin{bmatrix} a & i\sqrt{1-a^2} \\ i\sqrt{1-a^2} & a \end{bmatrix}$$

Note that $R(a) = -ie^{i\frac{\pi}{4}Z} W(a) e^{i\frac{\pi}{4}Z}$

$$\implies U_\phi = \bigoplus_{\lambda} e^{i\phi'_0 Z} \prod_{k=1}^{d} W(\lambda) e^{i\phi'_k Z}$$

And now use the Quantum Signal Processing Theorem

Quantum Singular Value Transformations

$$\implies U_\phi = \bigoplus_\lambda \begin{pmatrix} P(\lambda) & \cdot \\ \cdot & \cdot \end{pmatrix}$$

Quantum Eigenvalue Transformation Theorem

$$U_\phi = \prod_{k=1}^{d/2} \Pi_{\phi_{2k-1}} U^\dagger \Pi_{\phi_{2k}} U = \begin{bmatrix} P(\mathcal{H}) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} P(\lambda_1) & & & \\ & \ddots & & \cdot \\ & & P(\lambda_n) & \\ & \cdot & & \cdot \end{bmatrix}$$

Quantum Phase Estimation

Quantum Phase Estimation

Inputs:

U $N \times N$ unitary operator

U_ψ to prepare $|\psi\rangle$, i.e., $U_\psi |0\rangle = |\psi\rangle$ (Formally: access to controlled U_ψ, $U_\psi^\dagger$)

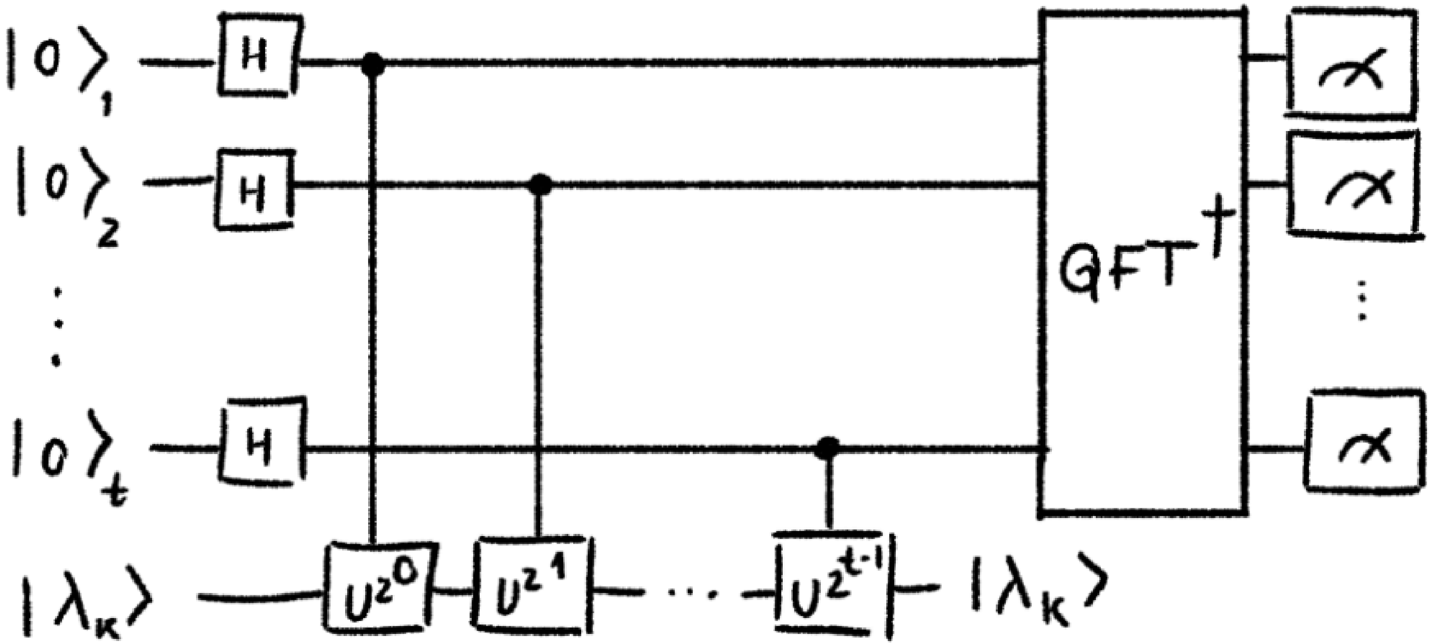
$|\psi\rangle$ $\in \mathbb{C}^N$ such that $U|\psi\rangle = e^{i\phi}|\psi\rangle$ $\phi \in [0, 2\pi)$

ϵ > 0 precision parameter

Outputs: ϕ up to precision ϵ with bounded error probability

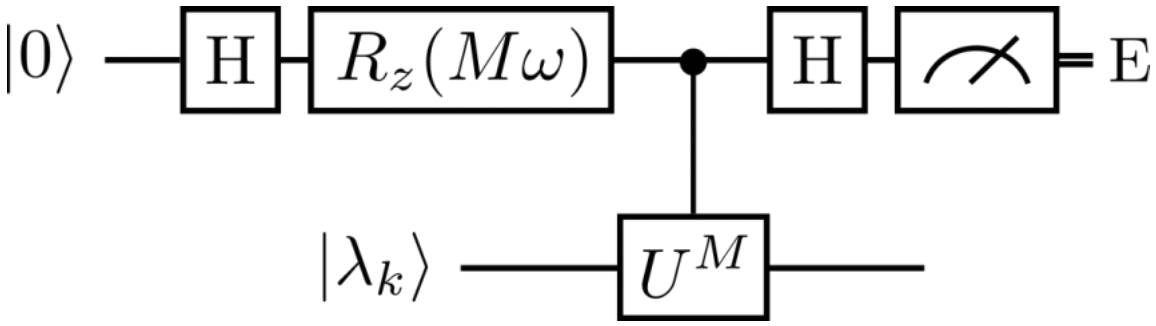
Quantum Phase Estimation

- Textbook QPE Algorithm



Quantum Phase Estimation

- Kitaev's/Iterative Phase Estimation Algorithm



$$\phi = 0.\phi_1\phi_2 \dots \phi_m \dots$$

$$\downarrow$$

$$\phi = 0.\phi_m \dots$$

$$\downarrow$$

$$\phi = 0.\phi_{m-1}\phi_m \dots - 0.0\phi_m$$

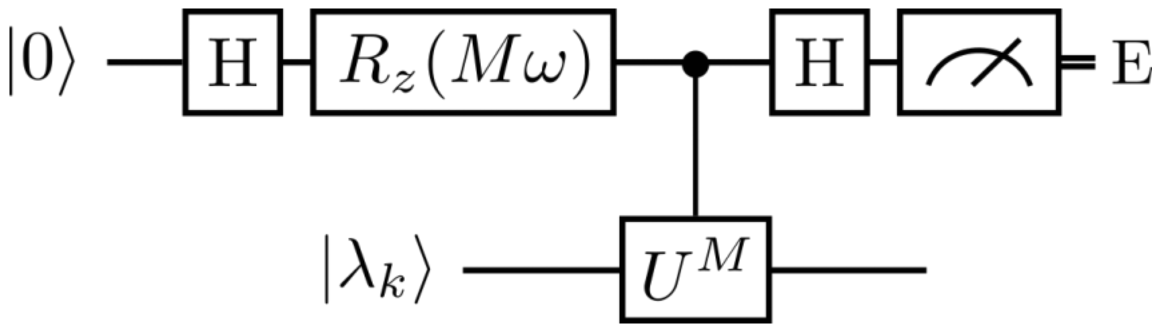
$$\downarrow$$

$$\dots$$

$$(M,\theta)_1 = (2^{m-1}, 0)$$
$$(M,\theta)_2 = (2^{m-2}, -\pi \cdot 2^{-m} \cdot \phi_m)$$
$$(M,\theta)_3 = (2^{m-3}, -\pi \cdot (2^{-m} \cdot \phi_m + 2^{-m+1} \cdot \phi_{m-1}))$$
$$\vdots$$
$$(M,\theta)_m = (1, -\pi \cdot (2^{-m} \cdot \phi_m + \cdots + 2^{-2} \cdot \phi_2))$$

Quantum Phase Estimation

- Faster Phase Estimation



$$P(0|\phi;\theta,M) = \frac{1+\cos(M[\phi+\theta])}{2},$$
$$P(1|\phi;\theta,M) = \frac{1-\cos(M[\phi+\theta])}{2}.$$

[Submitted on 2 Apr 2013]

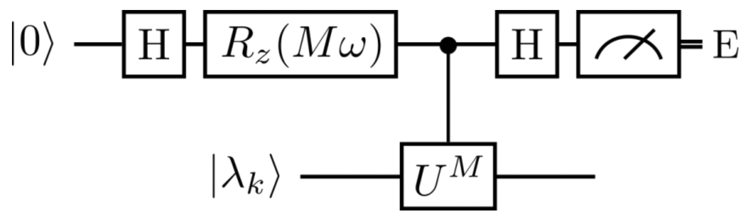
Faster Phase Estimation

Krysta M. Svore, Matthew B. Hastings, Michael Freedman

Idea: new "schedules" for M and ω

Plus: Informational perspective

Quantum Phase Estimation



- α-Quantum Phase Estimation

Efficient Bayesian Phase Estimation

Nathan Wiebe and Chris Granade
Phys. Rev. Lett. 117, 010503 – Published 30 June 2016

$$M \sim 1/\sigma, \theta = \mu$$

Accelerated Variational Quantum Eigensolver

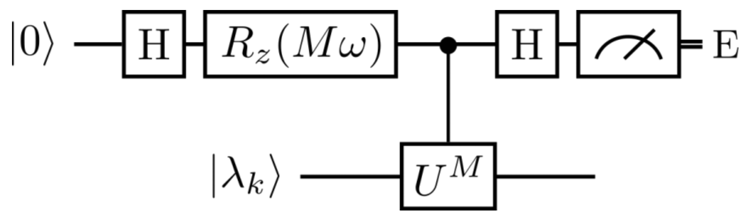
Daochen Wang, Oscar Higgott, and Stephen Brierley
Phys. Rev. Lett. 122, 140504 – Published 12 April 2019

$$(M, \theta) = \left(\frac{1}{\sigma^\alpha}, \mu - \sigma\right)$$

$$N(\alpha) = \begin{cases} \frac{2}{1-\alpha}\left(\frac{1}{\epsilon^{2(1-\alpha)}} - 1\right) & \text{if } \alpha \in [0,1) \\ 4\log(1/\epsilon) & \text{if } \alpha = 1 \end{cases}$$

$$D(\alpha) = \mathcal{O}(1/\epsilon^\alpha)$$

Quantum Phase Estimation



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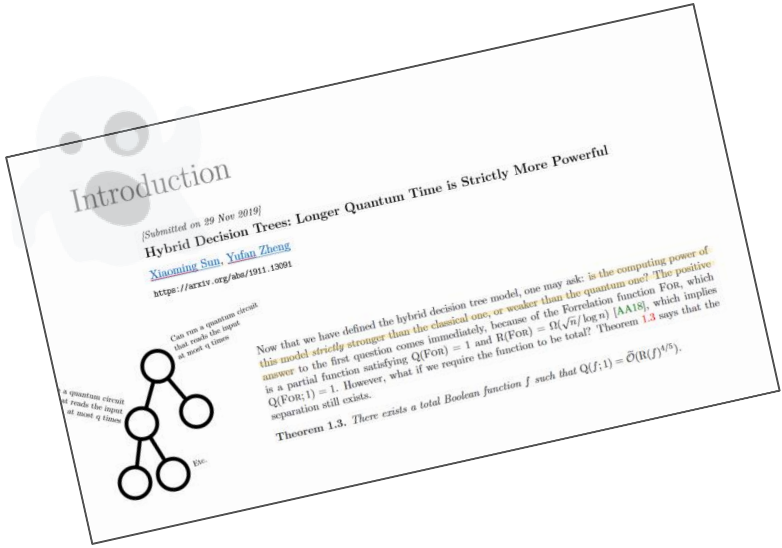
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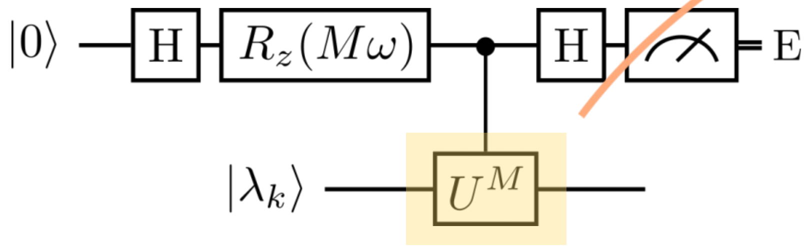
Theorem 1.3. *There exists a total Boolean function f such that* $Q(f;1) = \widetilde{O}(R(f)^{4/5})$.

Quantum Phase Estimation

- Query Perspective

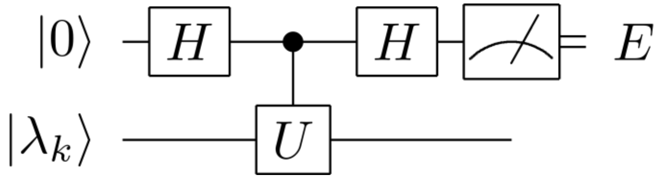
Let this be our query (how we measure our depth)



$$P(0|\phi;\theta,M) = \frac{1+\cos(M[\phi+\theta])}{2},$$
$$P(1|\phi;\theta,M) = \frac{1-\cos(M[\phi+\theta])}{2}.$$

$$P(x\,|\,\phi;\theta,M) = \frac{1+(-1)^x\cos(M[\phi+\theta])}{2}$$

Classical Sampling

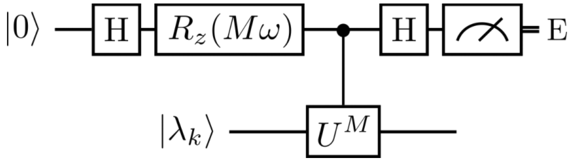


$D = \mathcal{O}(1)$ trivially

$N = \mathcal{O}(1/\epsilon^2)$ use Cramér-Rao bound

$T = N \times D = \sum D = \mathcal{O}(1/\epsilon^2)$

Fully Coherent



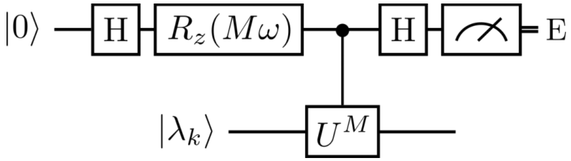
Plus Kitaev schedule

$D = \mathcal{O}(1/\epsilon)$

$T = \tilde{\mathcal{O}}(1/\epsilon)$

By inspection of the schedule

Hybrid (α-QPE)

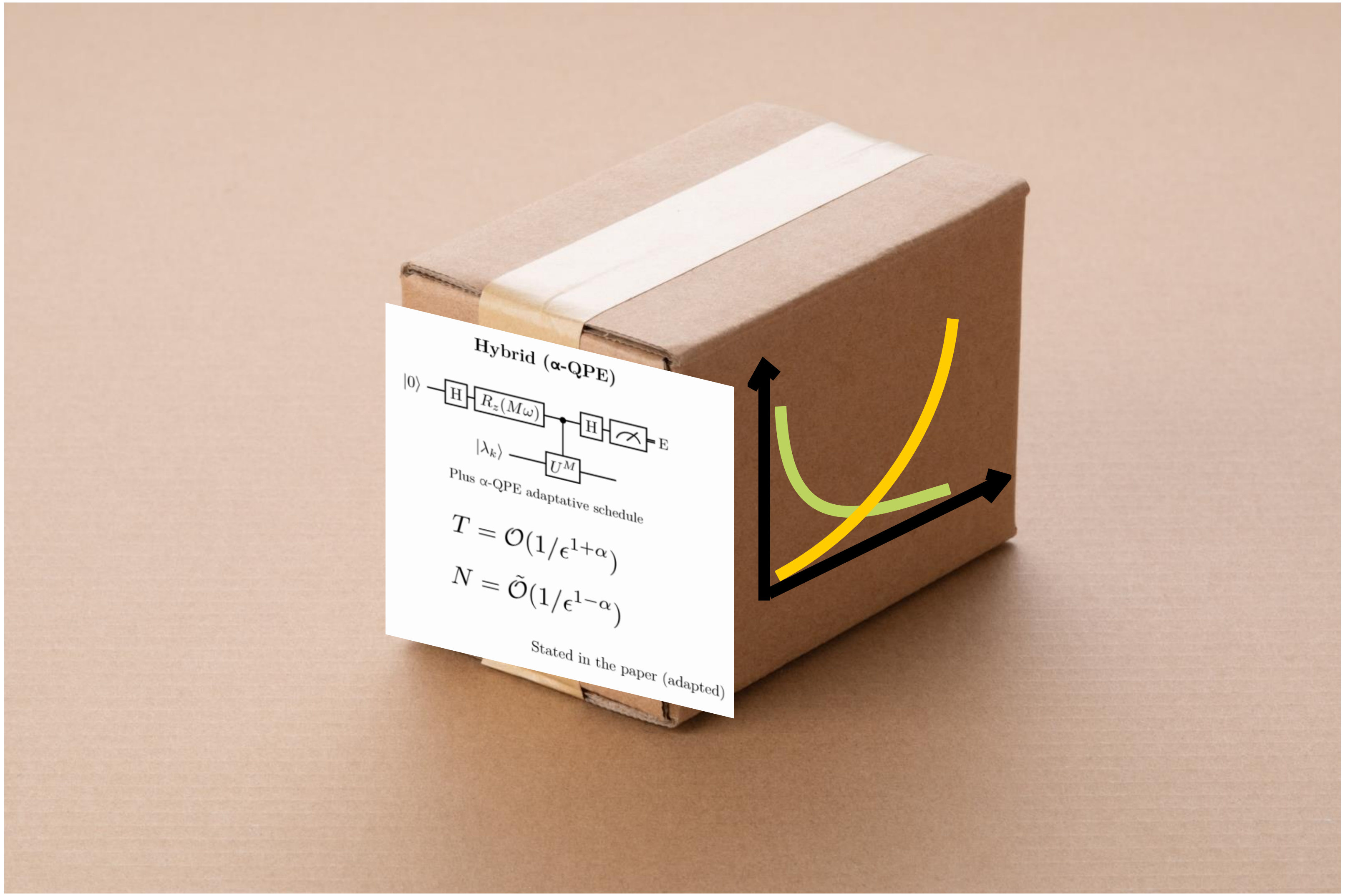


Plus α-QPE adaptative schedule

$D = \mathcal{O}(1/\epsilon^{1-\alpha})$

$T = \tilde{\mathcal{O}}(1/\epsilon^{1+\alpha})$

Stated in the paper (adapted)


Hybrid (α-QPE)
|0⟩
H
$R_z(M\omega)$
H
E
$|\lambda_k\rangle$
U^M
Plus α-QPE adaptative schedule
$T = \mathcal{O}(1/\epsilon^{1+\alpha})$
$N = \tilde{\mathcal{O}}(1/\epsilon^{1-\alpha})$
Stated in the paper (adapted)

α-Eigenvalue Estimation

α-Eigenvalue Estimation

- "Phase Estimation is a weaker form of Eigenvalue Estimation"

Plan: $\text{PE} \preceq \text{AE} \preceq \text{EE}$

Define:

Amplitude Estimation

$$A\,|0^m\rangle = \sqrt{p}\,|\text{good}\rangle + \sqrt{1-p^2}\,|\text{bad}\rangle$$

$$O_A\,|\text{good/bad}\rangle = \pm\,|\text{good/bad}\rangle$$

Input: A, $A^\dagger$, O_A, $\epsilon > 0$

Output: $|p|$ up to ϵ with bounded error probability

Eigenvalue Estimation

$\mathcal{H} \in \mathbb{H}_N \qquad \mathcal{H}\,|\psi\rangle = E\,|\psi\rangle$

U_H is a (γ, m)–BE of $\mathcal{H}$

$U_\psi\,|0^m\rangle = |\psi\rangle$

Input: U_ψ, U_H, $U_H^\dagger$, γ, $\epsilon > 0$

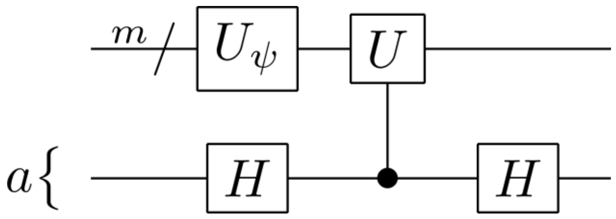
Output: E up to ϵ with bounded error probability

α-Eigenvalue Estimation

- "Phase Estimation is a weaker form of Eigenvalue Estimation"

$$\mathrm{PE} \preceq \mathrm{AE}$$

$A =$



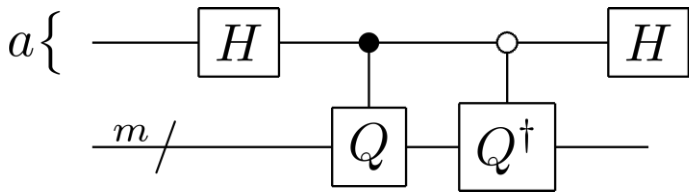
$$A|0^m\rangle|0\rangle = \cos(\phi/2)|\psi\rangle|0\rangle - i\sin(\phi/2)|\psi\rangle|1\rangle$$

$$\downarrow$$

$$\cos(\phi/2) \rightsquigarrow |\phi|$$

$$\mathrm{AE} \preceq \mathrm{EE}$$

$U_H =$



$$Q = A(2|0^m\rangle\langle 0^m| - I)A^\dagger O_A$$

$$\langle 0|_a U_H |0\rangle_a U_\psi |0^m\rangle = (1-2p)U_\psi |0^m\rangle$$

$$\downarrow$$

$$\mu = (1-2p) \rightsquigarrow p$$

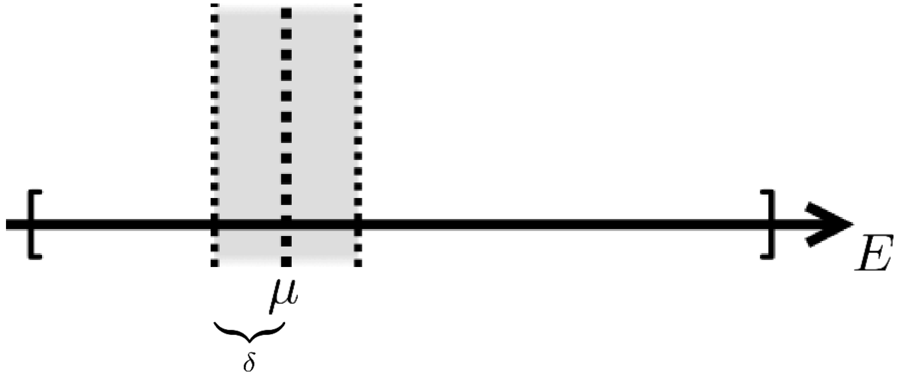
α-Eigenvalue Estimation

- QSVT

⇒ If we focus on EE (and find an interpolation for EE) we solve everything else

Now note:

If you can solve the decision problem



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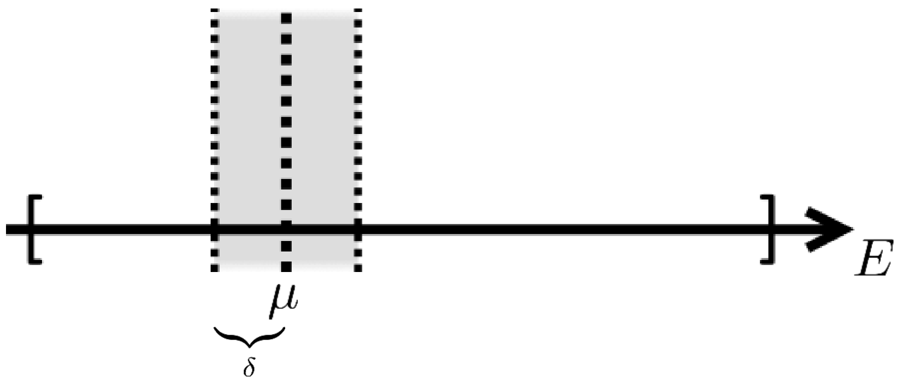
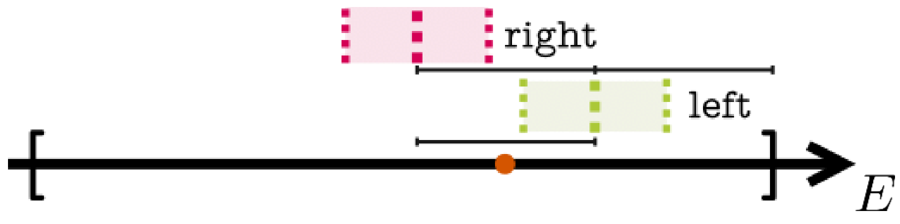


Then you can solve EE with a binary search



α-Eigenvalue Estimation

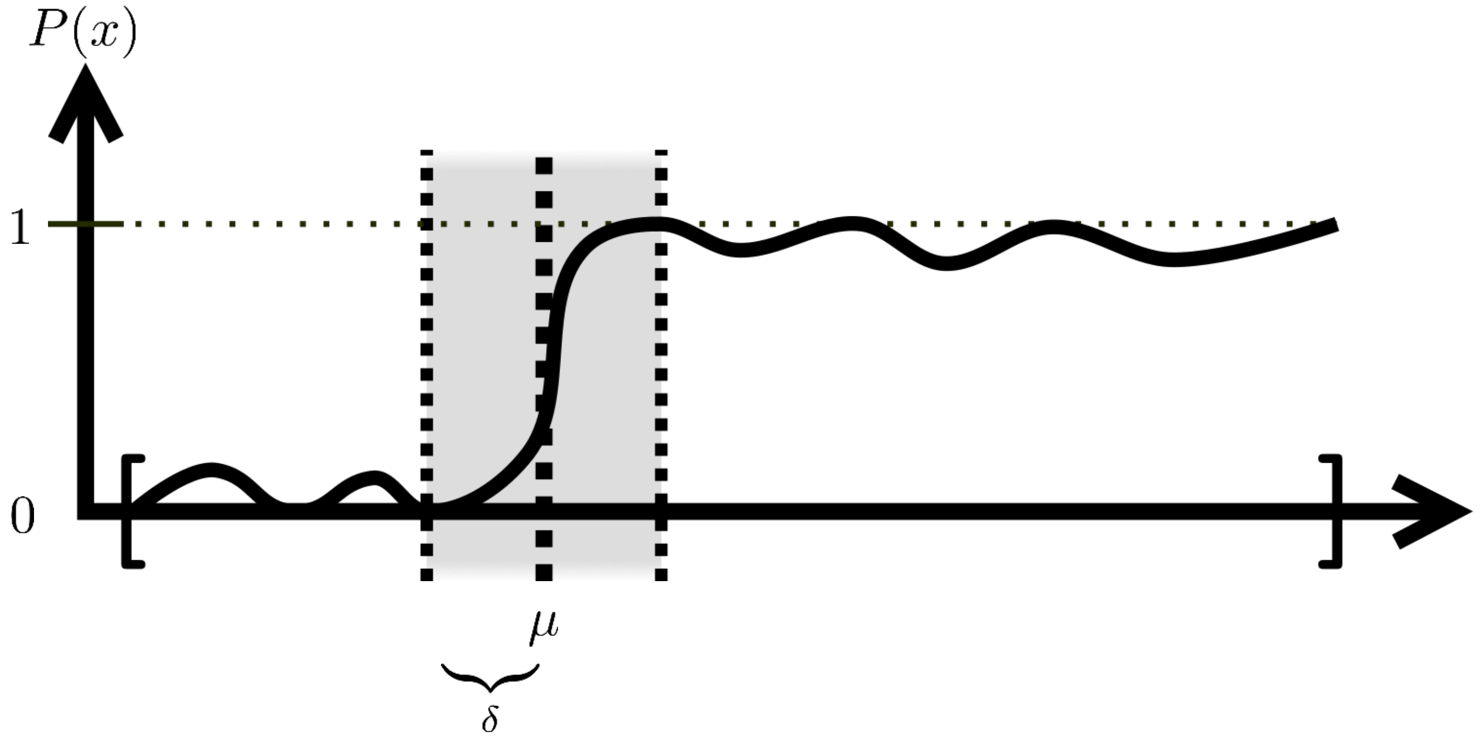
bang

- QSVT

$$\begin{pmatrix} P(\mathcal{H}) & \cdot \\ \cdot & \cdot \end{pmatrix} |0\rangle \otimes |\lambda_k\rangle \rightsquigarrow P(\lambda_k)\,|0\rangle\,|\lambda_k\rangle + |1\rangle \otimes (\cdots)$$

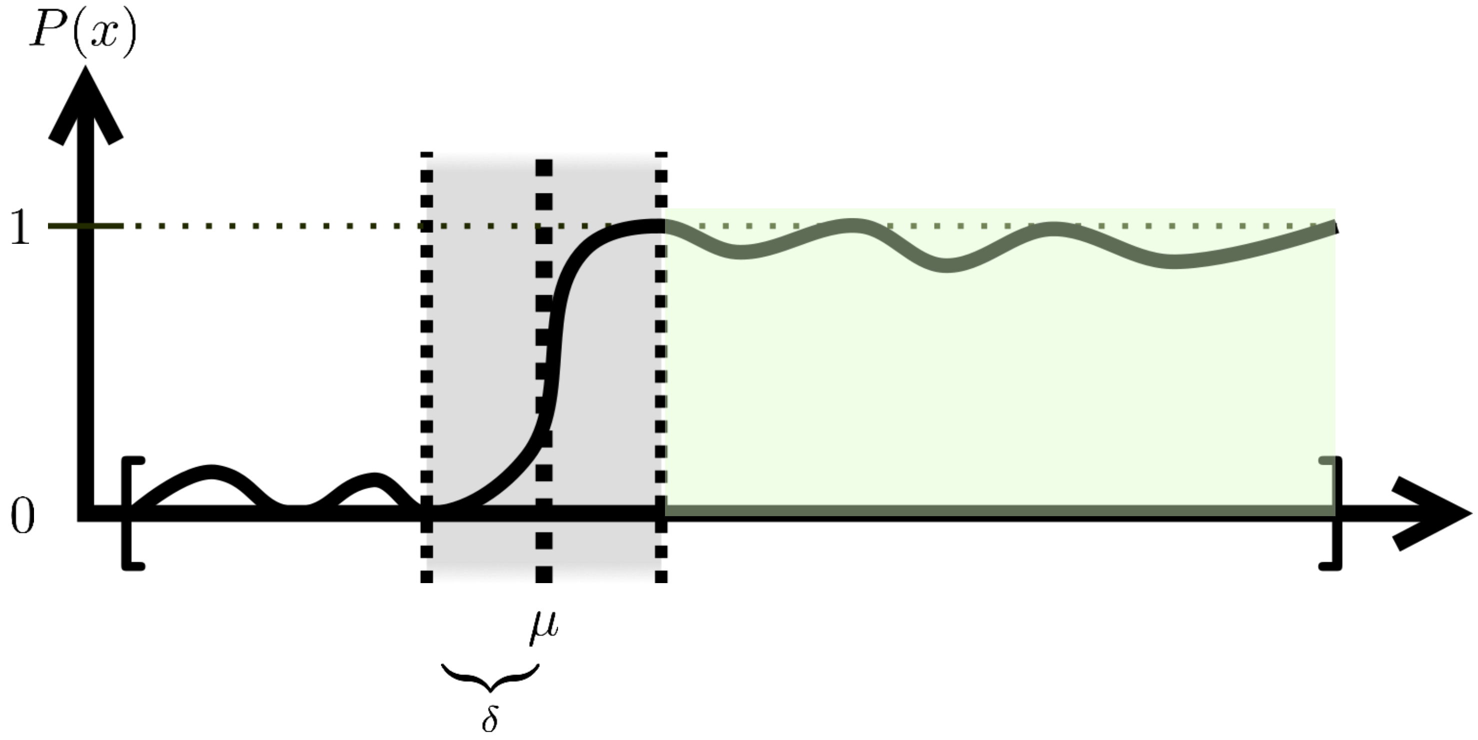
α-Eigenvalue Estimation

- QSVT



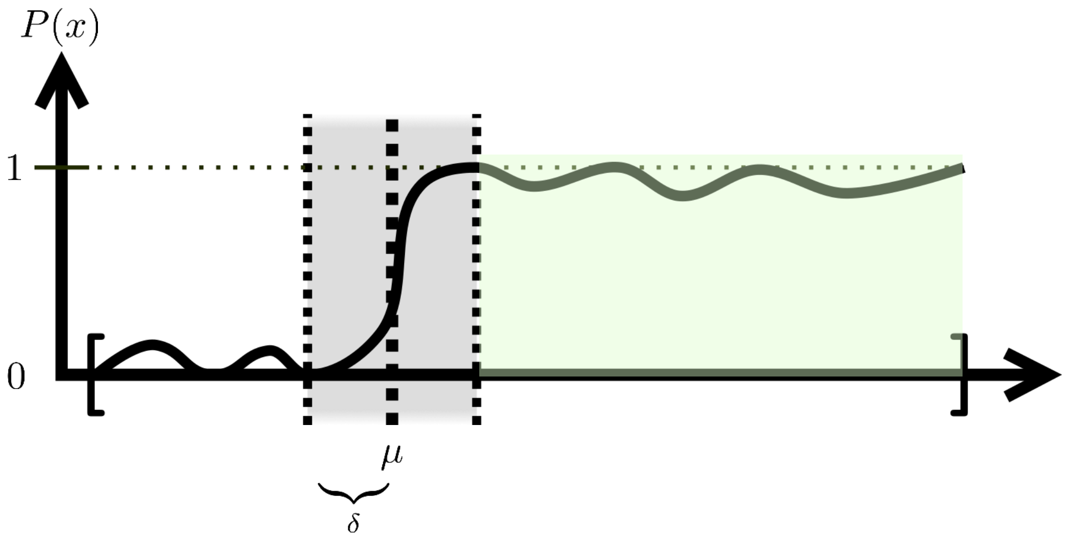
α-Eigenvalue Estimation

- QSVT



α-Eigenvalue Estimation

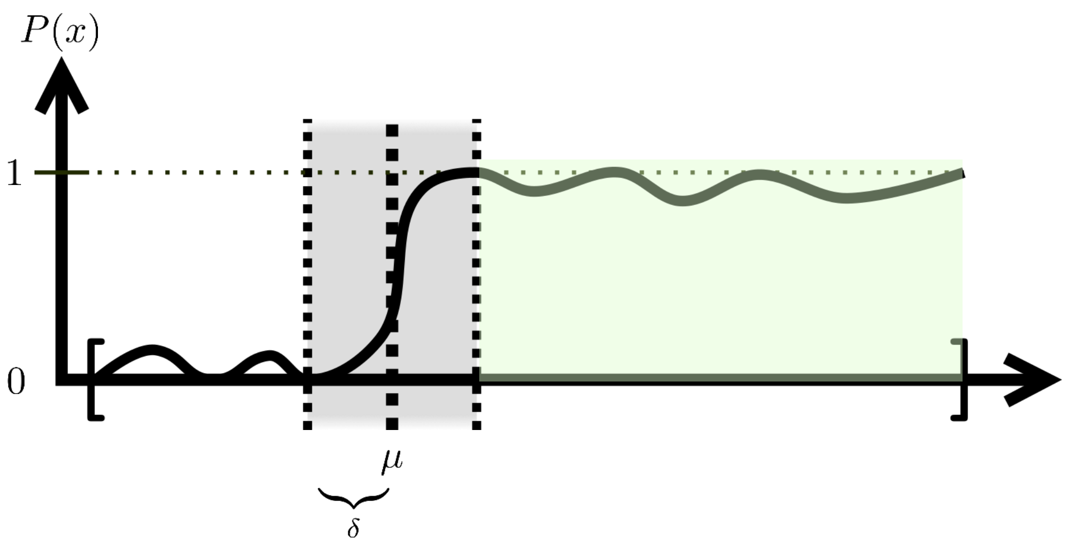
- QSVT



$$P(\lambda_k)\,|0\rangle\,|\lambda_k\rangle \approx |0\rangle\,|\lambda\rangle$$

α-Eigenvalue Estimation

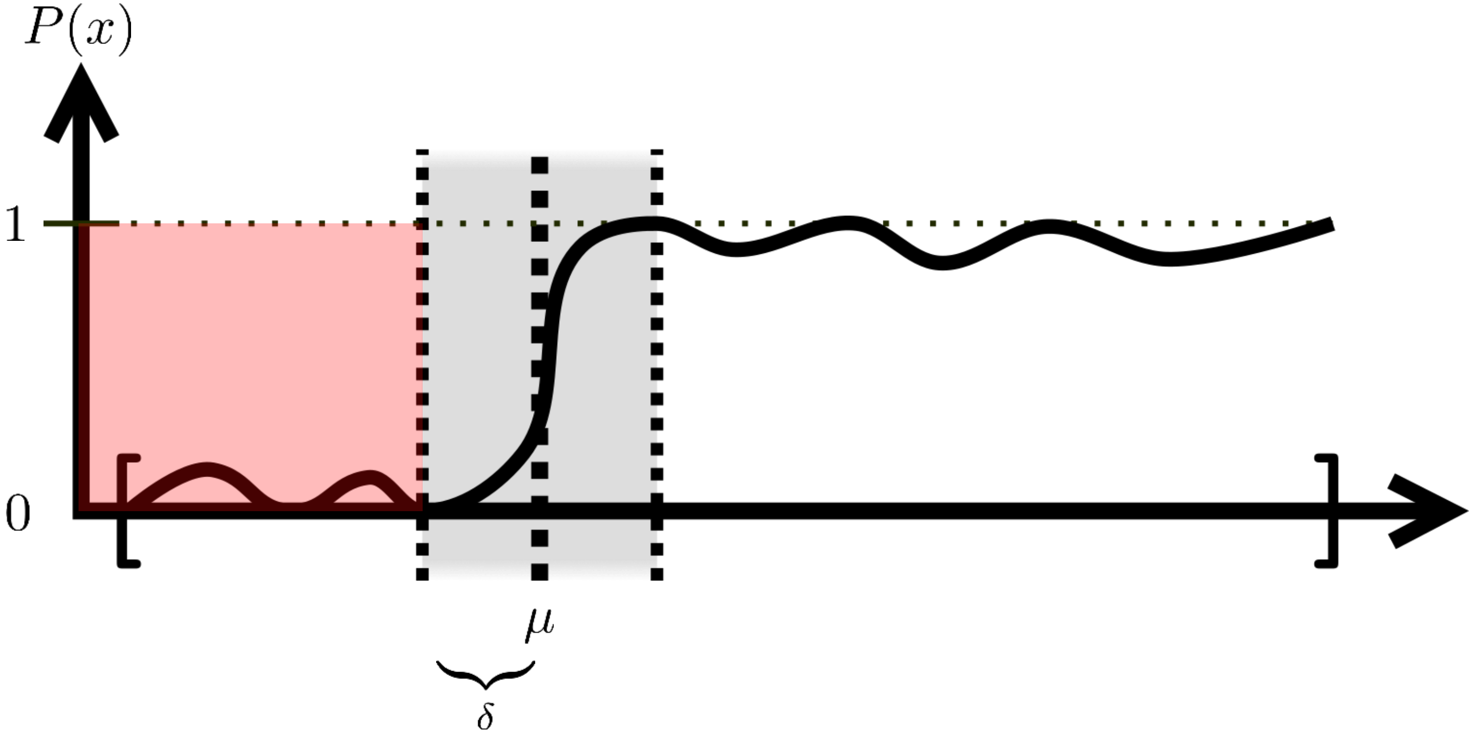
- QSVT



$$P(\lambda_k)\,|0\rangle\,|\lambda_k\rangle \approx |0\rangle\,|\lambda\rangle$$

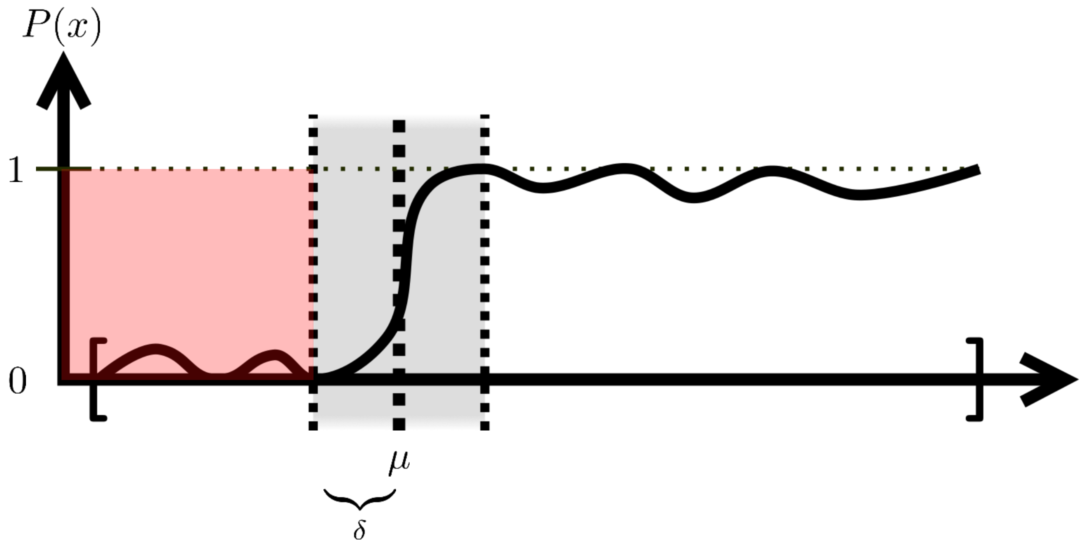
α-Eigenvalue Estimation

- QSVT



α-Eigenvalue Estimation

- QSVT

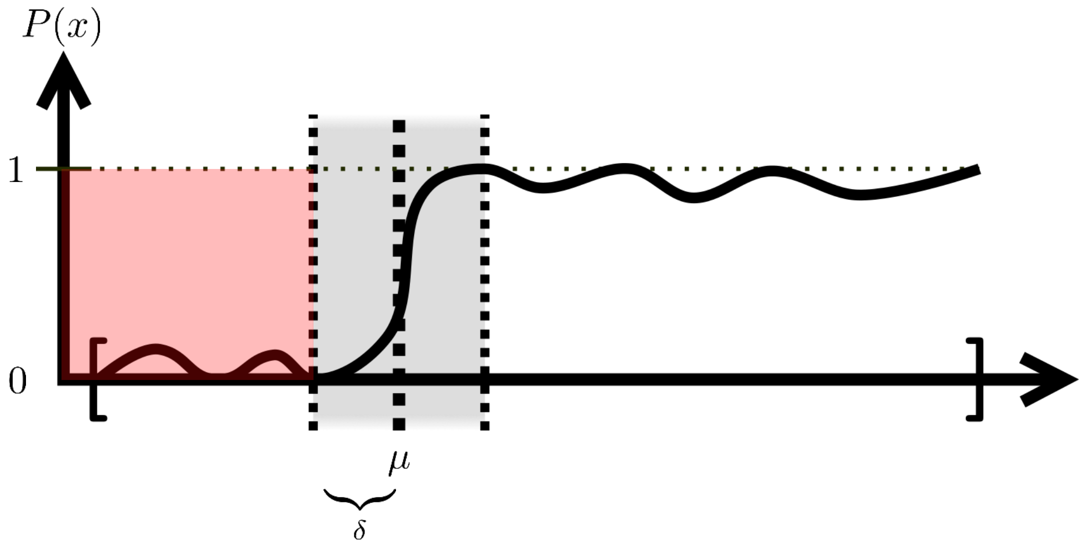


$$\begin{pmatrix} P(\mathcal{H}) & \cdot \\ \cdot & \cdot \end{pmatrix} = U_H \left|0\right\rangle \left|\lambda_k\right\rangle = P(\lambda_k) \left|0\right\rangle \left|\lambda_k\right\rangle + \left|1\right\rangle \left|u_j\right\rangle = \left|1\right\rangle \left|u_j\right\rangle$$

Must be unitary!

α-Eigenvalue Estimation

- QSVT

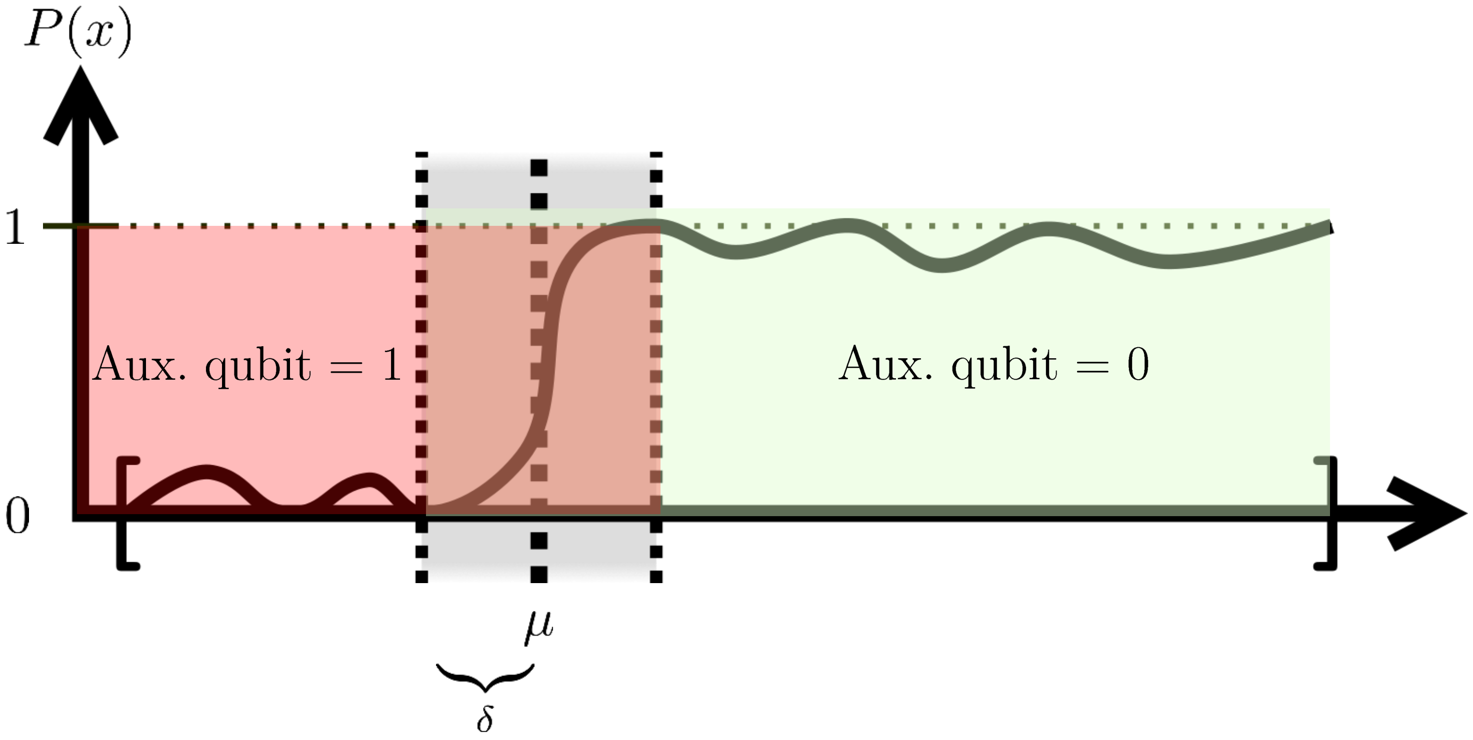


$$\begin{pmatrix} P(\mathcal{H}) & \cdot \\ \cdot & \cdot \end{pmatrix} = U_H |0\rangle |\lambda_k\rangle = P(\lambda_k) |0\rangle |\lambda_k\rangle + |1\rangle |u_j\rangle = |1\rangle |u_j\rangle$$

Must be unitary!

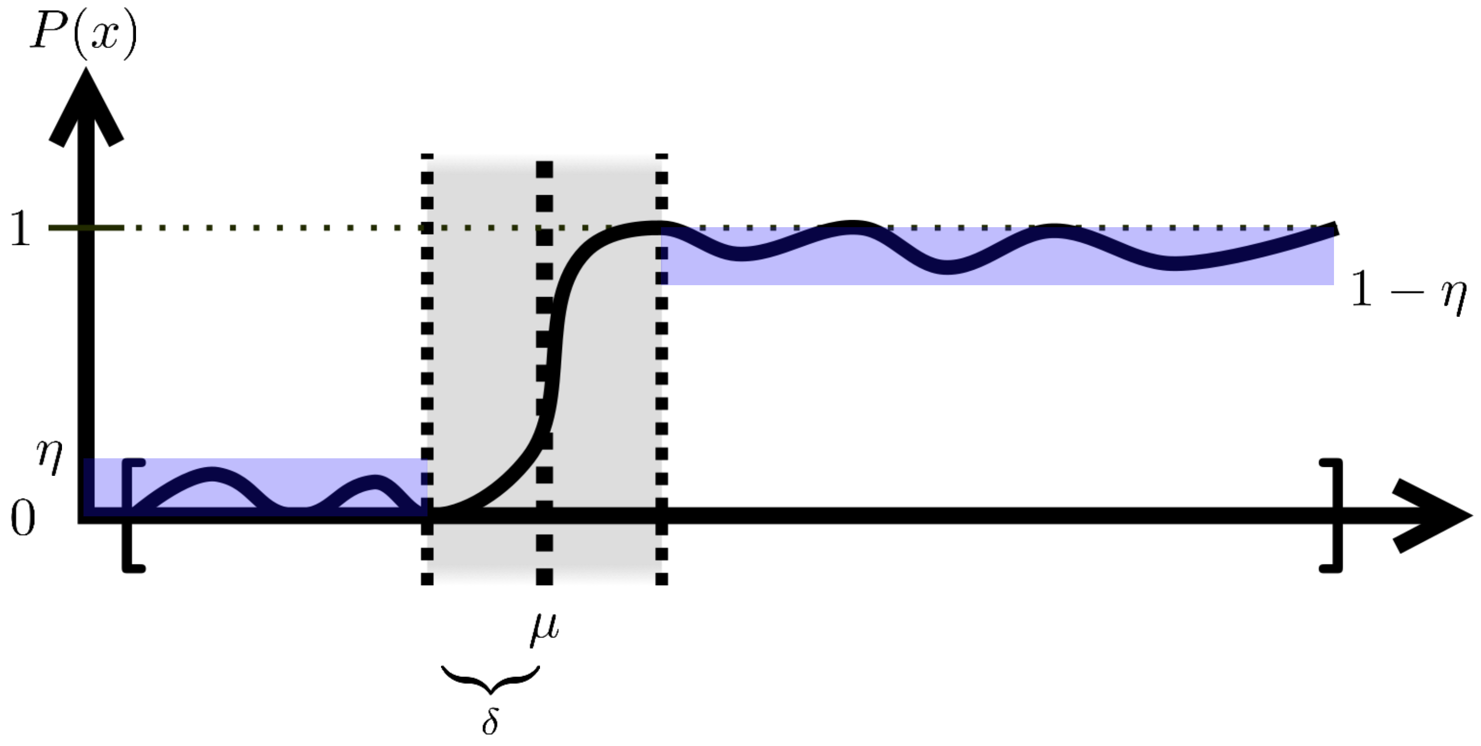
α-Eigenvalue Estimation

- QSVT



α-Eigenvalue Estimation

- QSVT



α-Eigenvalue Estimation

- Decision problem

⇒ If you can implement a step function, you can solve EE.

α-Eigenvalue Estimation

- Decision problem

⇒ If you can implement a step function, you can solve EE.

… Can you implement a step function?

α-Eigenvalue Estimation

- Decision problem

$\Rightarrow$ If you can implement a step function, you can solve EE.

… Can you implement a step function?

Near-optimal ground state preparation

Lin Lin[1,2] and Yu Tong[1]

[1]Department of Mathematics, University of California, Berkeley, CA 94720, USA
[2]Computational Research Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

Published: 2020-12-14, volume 4, page 372
Citation: Quantum 4, 372 (2020).

Yep! (Sort of)

α-Eigenvalue Estimation

- Decision problem

Near-optimal ground state preparation

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Published: 2020-12-14, volume 4, page 372
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Let U_H be a (γ, m)-block-encoding of a Hermitian matrix H and $\mu_0 \in [0, \gamma]$. Then, there is a $(1, m+3)$–block-encoding of $P\left(\frac{H-\mu_0 I}{\gamma+\mu_0}; \delta, \eta\right)$, where P satisfies

$$\forall x \in [-1, -\delta], 0 \leq P(x; \delta, \eta) \leq \eta/2 \quad (1)$$

$$\text{and } \forall x \in [\delta, 1], 1 - \eta/2 \leq P(x; \delta, \eta) \leq 1, \quad (2)$$

using $\mathcal{O}\left(\frac{1}{\delta}\log\left(\frac{1}{\eta}\right)\right)$ queries of U_H and $U_H^\dagger$.

(Adapted from lemma 5)

α-Eigenvalue Estimation

- Decision problem

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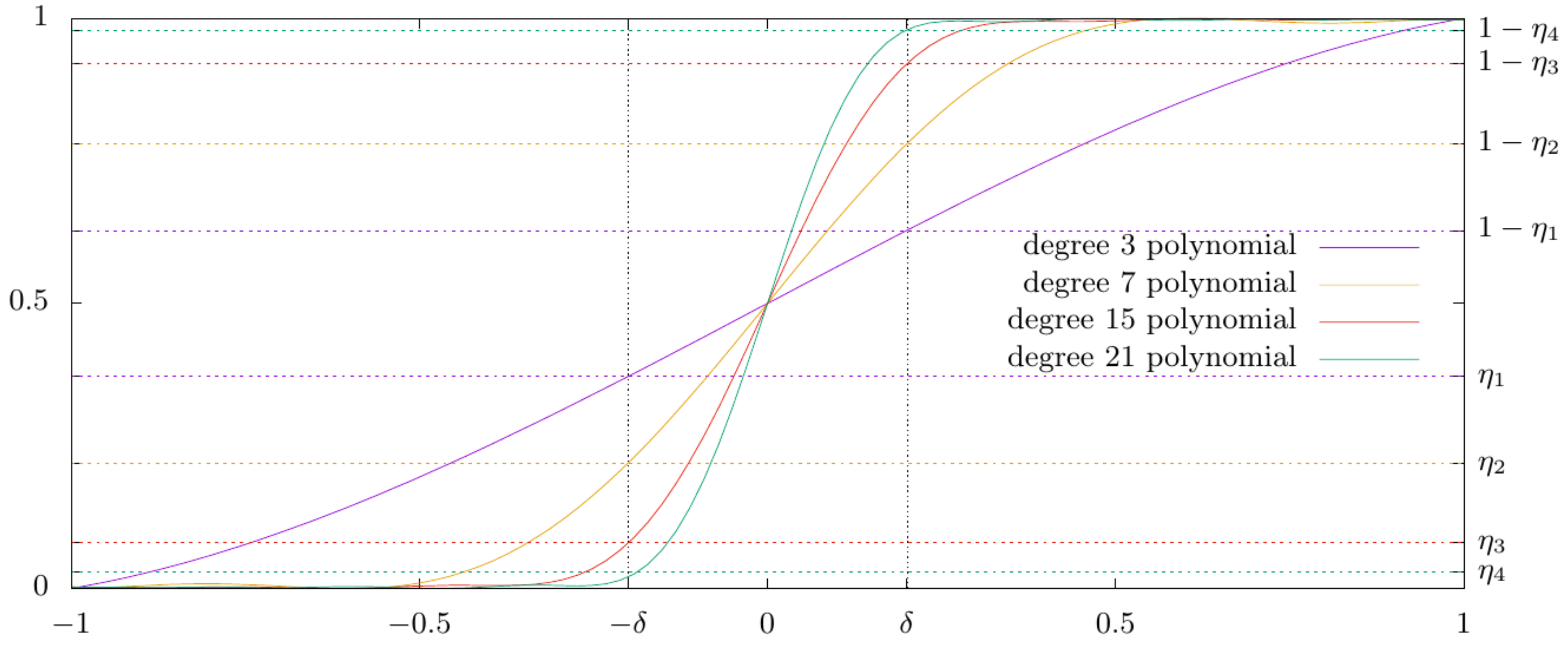
using $\mathcal{O}\left(\frac{1}{\delta}\log\left(\frac{1}{\eta}\right)\right)$ queries of U_H and $U_H^\dagger$.



α-Eigenvalue Estimation

- Bringing it all together

What scalings do we get as a function of ε?

α-Eigenvalue Estimation

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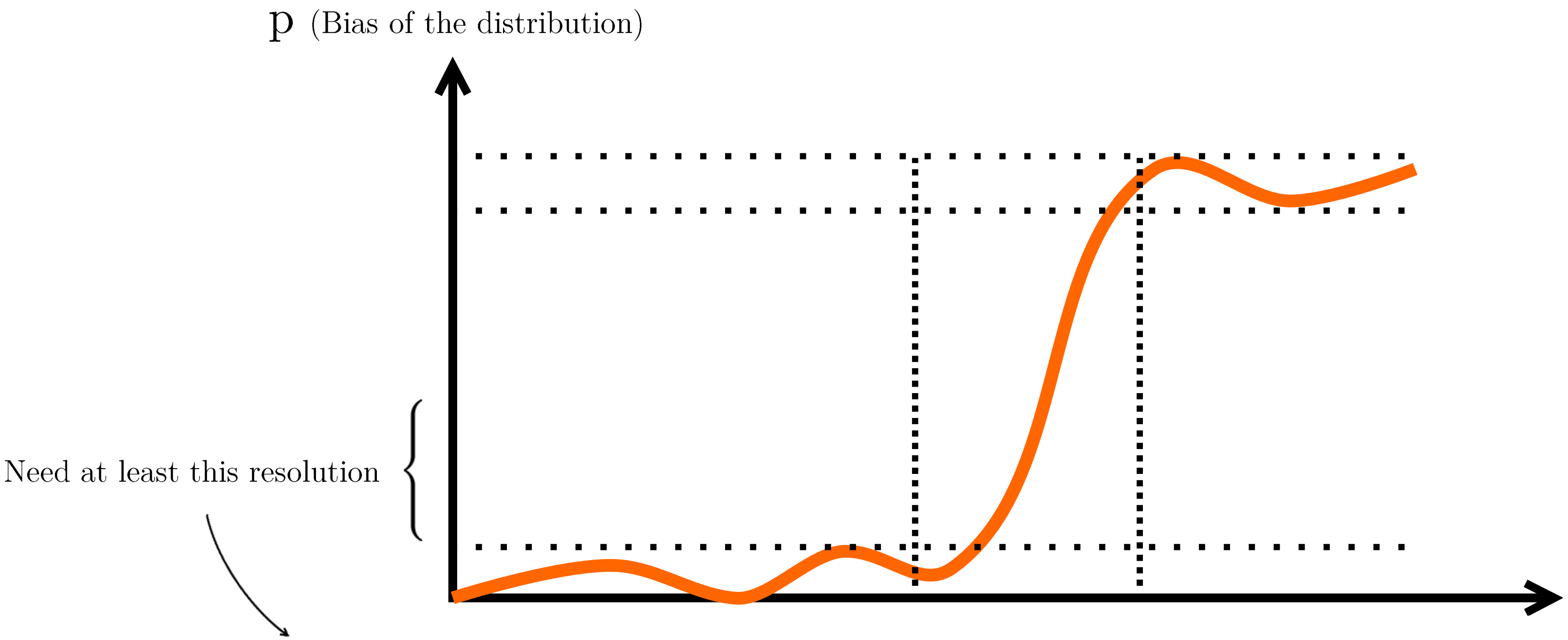

$$\frac{1-2\eta}{2}$$ by Chebyshev's inequality $\Longrightarrow$ $N = \mathcal{O}\left(1/(1-2\eta)^2\right)$

α-Eigenvalue Estimation

- Decision problem

$$\eta = \frac{1}{2} - \frac{1}{4}\left(\frac{\epsilon}{4}\right)^{\alpha}$$

$$\delta = \epsilon/4$$

$$\Longrightarrow \quad \begin{cases} D(\alpha) = \mathcal{O}\left(\frac{1}{\epsilon}\log\left(\frac{1}{1-\epsilon^{\alpha}}\right)\right) = \mathcal{O}\left(\left(\frac{1}{\epsilon}\right)^{1-\alpha}\right) \\ \\ N(\alpha) = \mathcal{O}\left(\scriptstyle 1/(1-2\eta)^2\right) = \mathcal{O}\left(D(\alpha)\left(\frac{1}{\epsilon}\right)^{2\alpha}\right) = \mathcal{O}\left(\left(\frac{1}{\epsilon}\right)^{1+\alpha}\right) \end{cases}$$

α-Eigenvalue Estimation

- α-Eigenvalue Estimation

$$\Longrightarrow \quad \begin{cases} D(\alpha) = \mathcal{O}\left(\left(\frac{1}{\epsilon}\right)^{1-\alpha}\right) \\ T(\alpha) = \mathcal{O}\left(\left(\frac{1}{\epsilon}\right)^{1+\alpha}\log^2\left(\frac{1}{\epsilon}\right)\right) = \tilde{\mathcal{O}}\left(\left(\frac{1}{\epsilon}\right)^{1+\alpha}\right) \end{cases}$$

...which is the α-QPE scaling.

In Conclusion

- α-Quantum Phase Estimation can be "upgraded" to α-Eigenvalue Estimation
- You can think of the problem as "how well can I approximate a step function"
- Quantum Singular Value Transformations might be a good tool for finding hybrid algorithms
  - (Think: what do I need to do if I only have a poor approximation of my target function — and what *is* my target function)

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Rediscovering a family of hybrid algorithms with quantum singular value transforms

Miguel Murça[1,3,(2)], Duarte Magano[1,2]

18 October 2022

1 2 3

With acknowledgement and thanks to the support from FCT – Fundação para a Ciência e a Tecnologia (Portugal), namely through projects UIDB/50008/2020 and UIDB/04540/2020, as well as projects QuantHEP and HQCC supported by the EU H2020 QuantERA ERA-NET Cofund in Quantum Technologies and by FCT (QuantERA/0001/2019 and QuantERA/004/2021, respectively). DM and MM acknowledge the support from FCT through scholarships 2020.04677.BD and 2021.05528.BD, respectively.