KIRKWOOD-DIRAC QUASIPROBABILITY DISTRIBUTIONS BEYOND QUANTUM STATES

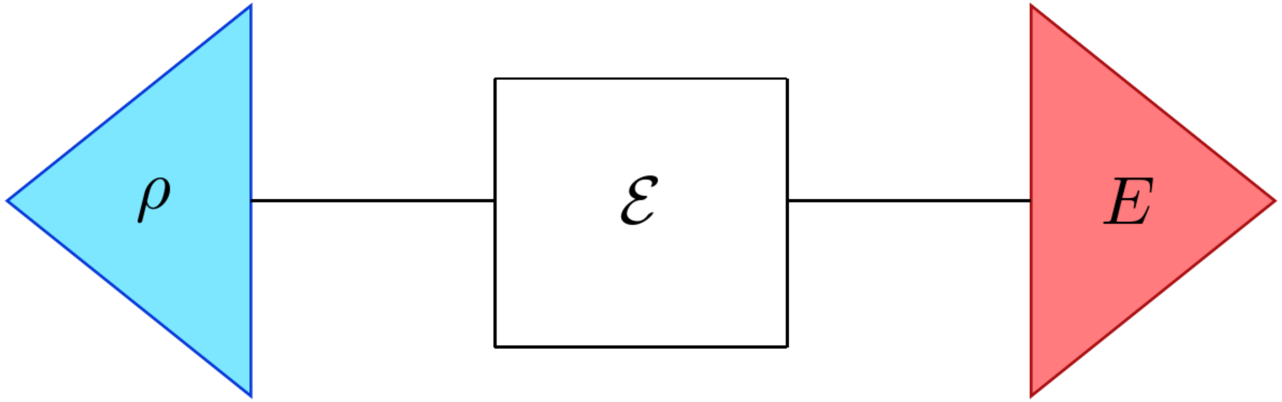
Rafael Wagner

International Iberian Nanotechnology Laboratory (INL)

University of Minho

University of Pisa

QUANTUM THEORY IN PHASE SPACE



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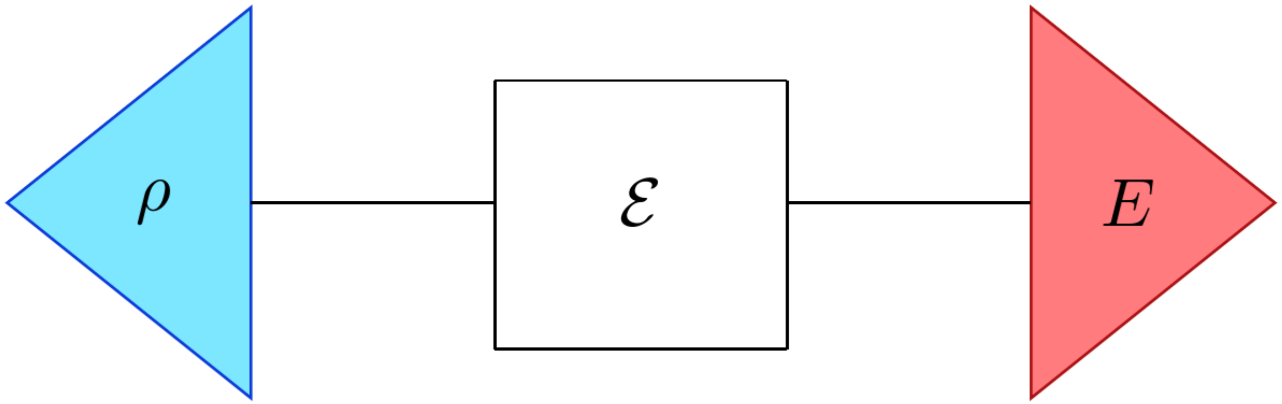


Quantum state

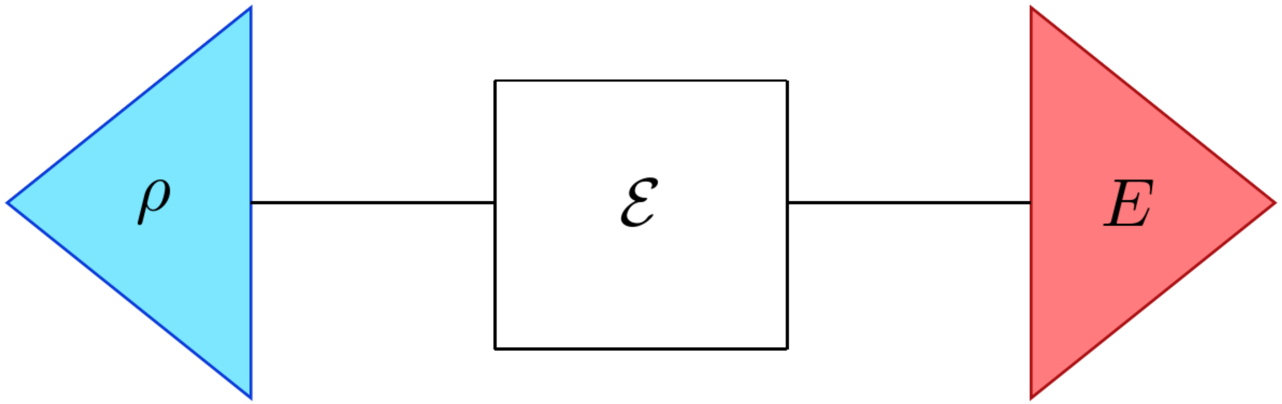
Quantum channel

POVM element

QUANTUM THEORY IN PHASE SPACE

$$\Lambda$$

Phase space



Quantum state	Quantum channel	POVM element
$\mu(\lambda\vert\rho)$	$\Gamma(\lambda'\vert\lambda,\mathcal{E})$	$\xi(E\vert\lambda)$

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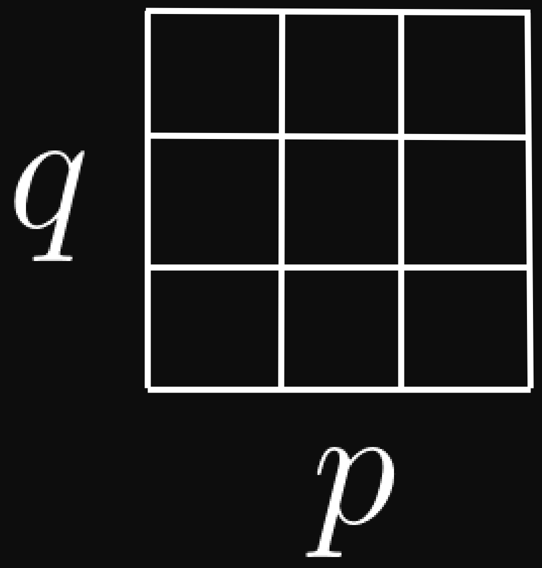

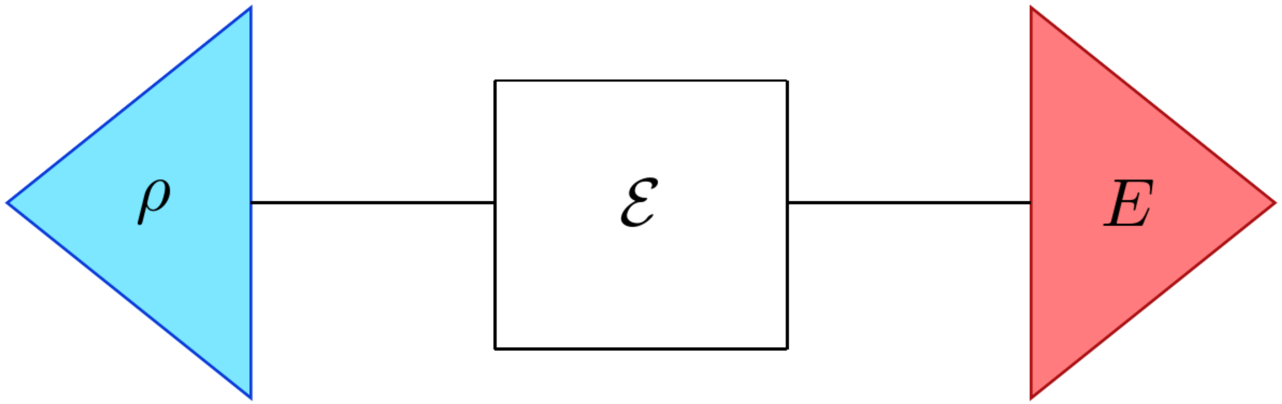

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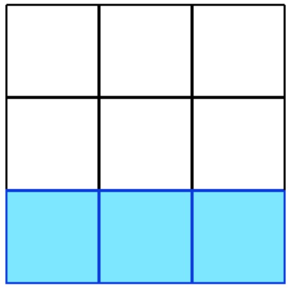

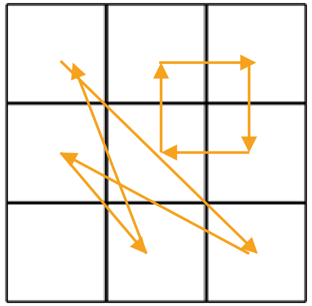

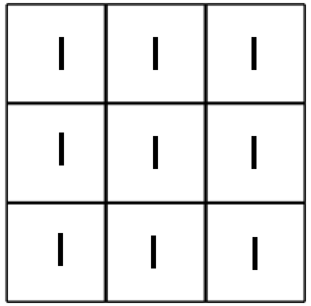

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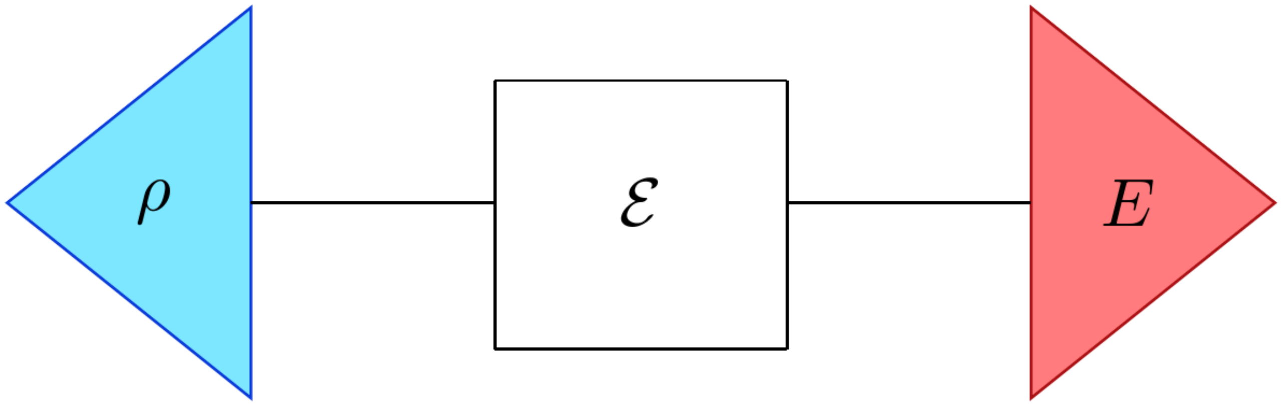

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$$\mu(\lambda|\rho) \qquad \Gamma(\lambda'|\lambda,\mathcal{E}) \qquad \xi(E|\lambda)$$

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John G. Kirkwood, Phys. Rev. 44, 31 (1933)

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Rabei, Arvind, Mukunda, and Simon, PRA **60**, 3397 (1999)
V. Bargmann, J. Math. Phys. 5, 862 (1964)

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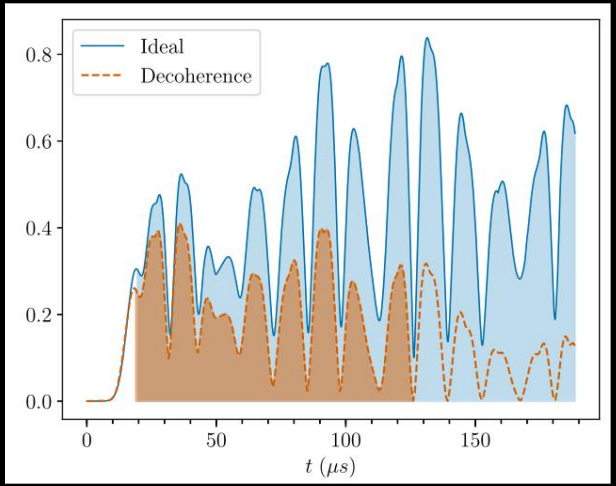
$\mathrm{Re}[\mu]$

H Margenau, RN Hill, *Prog. of Theor. Phys.*, 26, 5 (1961)
Y. P. Terletsky, Zh. Eksp. Teor. Fiz. 7, 1290 (1937)

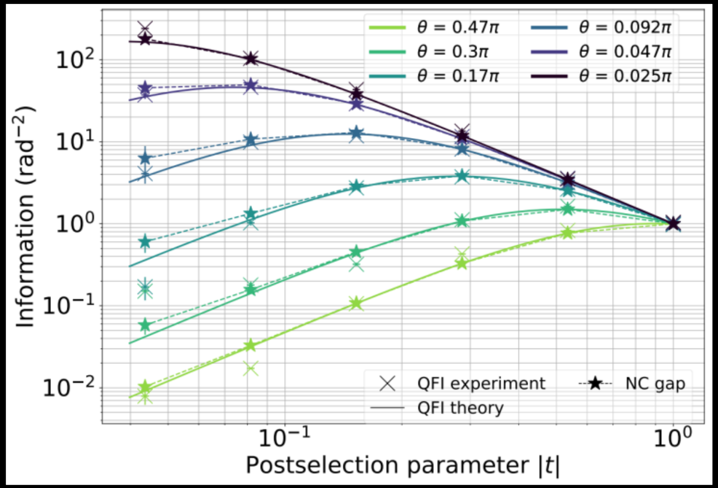


Scrambling

PRL **122**, 040404 (2019)

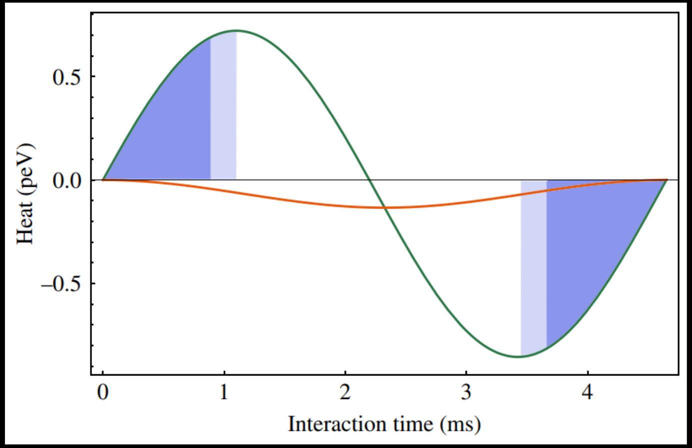
PRA **97**, 042105 (2018)

PRL **128**, 220504 (2022)
PRA **106**, 042404 (2022)



Quantum Metrology

$$\mu(i, i'|\rho) = \langle a_i|\rho|a'_{i'}\rangle\langle a'_{i'}|a_i\rangle$$



Quantum Thermodynamics

PRL **120**, 040602 (2018)
Quantum 7, 1128 (2023).
PRX Quantum 1, 010309 (2020)

WHAT IS IT GOOD FOR?

BASIC PROPERTIES

$$\boxed{\mu(i, i'|\rho) = \langle a_i|\rho|a'_{i'}\rangle\langle a'_{i'}|a_i\rangle}$$

$$\sum_{i,i'} \mu(i, i'|\rho) = 1$$

BASIC PROPERTIES

$$\boxed{\mu(i, i'|\rho) = \langle a_i|\rho|a'_{i'}\rangle\langle a'_{i'}|a_i\rangle}$$

$$\sum_i \mu(i, i'|\rho) = \langle a'_{i'}|\rho|a'_{i'}\rangle$$

$$\sum_{i'} \mu(i, i'|\rho) = \langle a_i|\rho|a_i\rangle$$

BASIC PROPERTIES

$$\boxed{\mu(i, i'|\rho) = \langle a_i|\rho|a'_{i'}\rangle\langle a'_{i'}|a_i\rangle}$$

$$\mu(i, i'|\rho) \in \mathbb{C}$$

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$$\mu(i, i'|\rho) \in \mathbb{C}$$

$$\mu(0, +|\rho) = \underbrace{\langle 0|+_i\rangle}_{\frac{1}{\sqrt{2}}}\underbrace{\langle +_i|+\rangle}_{\frac{1-i}{2}}\underbrace{\langle +|0\rangle}_{\frac{1}{\sqrt{2}}}$$

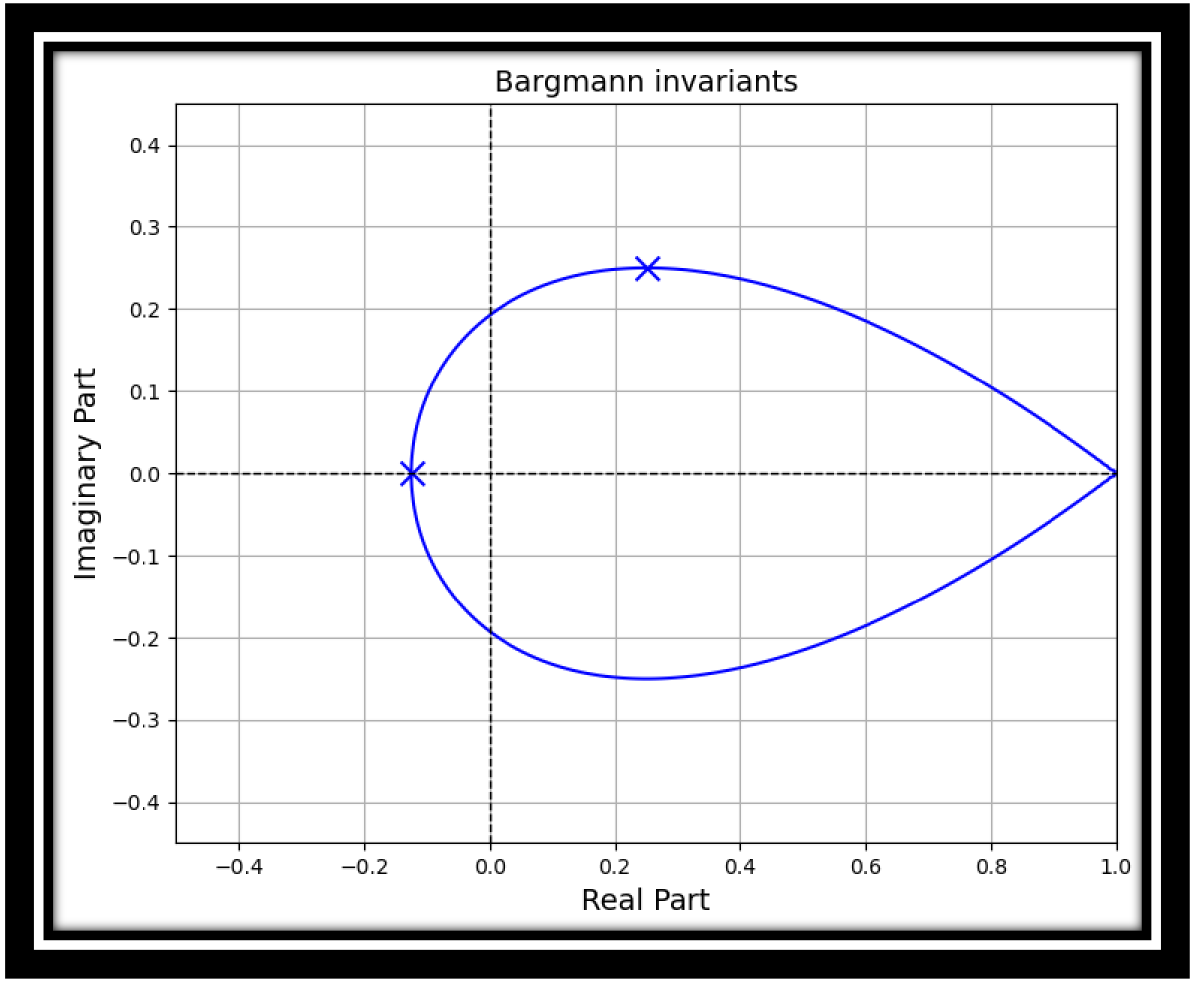
BASIC PROPERTIES

$$\{|0\rangle, |1\rangle\}$$

$$\{|+\rangle, |-\rangle\}$$

$$\rho = |+_i\rangle\langle +_i|$$

$$\mu(i, i'|\rho) = \langle a_i|\rho|a'_{i'}\rangle\langle a'_{i'}|a_i\rangle$$



BASIC PROPERTIES

$$\Delta = |\Delta|e^{i\phi}$$

$$1 - 3|\Delta|^{\frac{2}{3}} + 2|\Delta|\cos(\phi) \geq 0$$

arXiv: 2403.15066

$$\boxed{\mu(i, i'|\rho) = \langle a_i|\rho|a'_{i'}\rangle\langle a'_{i'}|a_i\rangle}$$

BASIC PROPERTIES

$$[\rho, A] \neq 0$$

$$[\rho, A'] \neq 0$$

$$[A, A'] \neq 0$$

$$\mu(i, i'|\rho) = \langle a_i|\rho|a'_{i'}\rangle\langle a'_{i'}|a_i\rangle$$

BASIC PROPERTIES

ρ	$\|0\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	$\|1\rangle\|0\rangle$	$\|1\rangle\|1\rangle$
$\|0\rangle\|+\rangle$	0	0	0	0
$\|0\rangle\|-\rangle$	0	0	0	0
$\|1\rangle\|0\rangle$	0	0	$\frac{1}{2}$	0
$\|1\rangle\|1\rangle$	0	0	0	$\frac{1}{2}$

$$\rho = |1, +\rangle\langle 1, +|$$

$$[\rho, A] \neq 0$$

$$[\rho, A'] \neq 0$$

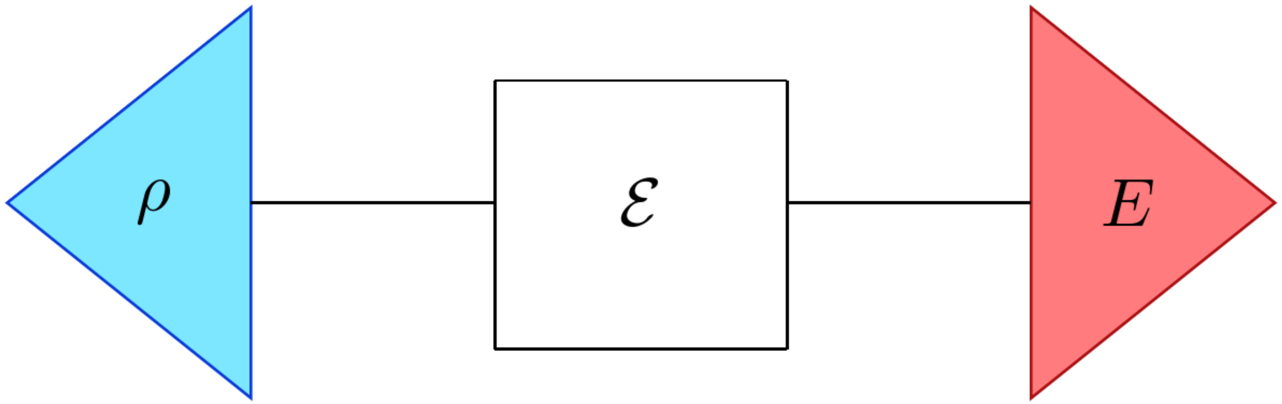
$$[A, A'] \neq 0$$

D.R.M. Arvidsson-Schukur, et. al., *J. Phys. A: Math. Theor.* **54** 284001(2021)

PHASE SPACE REPRESENTATION

$$\Lambda$$

Phase space



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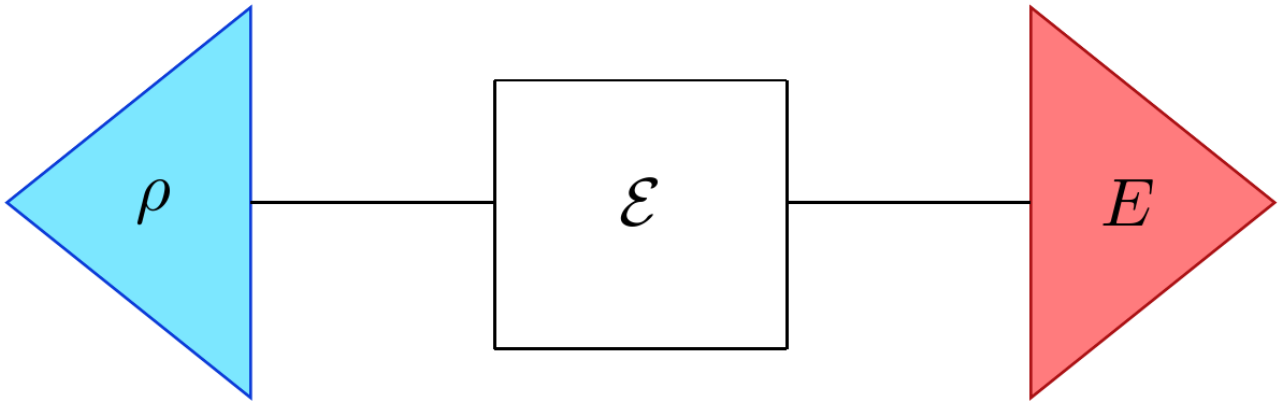
LACKING FOR KIRKWOOD AND DIRAC

$$\Lambda$$

Phase space



Quantum state | Quantum channel | POVM element

$$\mu(i, i'|\rho)$$ | ? | ?

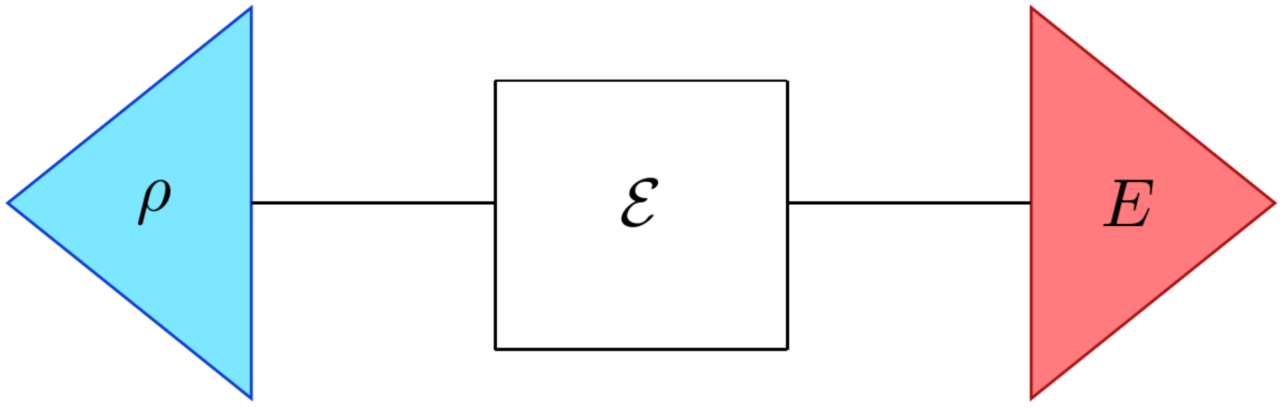
LACKING FOR KIRKWOOD AND DIRAC

Kirkwood-Dirac representations beyond quantum states (and their relation to noncontextuality)

David Schmid,[1,*] Roberto D. Baldijão,[1] Yìlè Yīng,[2,3] Rafael Wagner,[4,5,†] and John H. Selby[1]

[1] *International Centre for Theory of Quantum Technologies, University of Gdańsk, 80-308 Gdańsk, Poland*
[2] *Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario Canada N2L 2Y5*
[3] *Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1*
[4] *INL – International Iberian Nanotechnology Laboratory, Av. Mestre José Veiga s/n, 4715-330 Braga, Portugal*
[5] *Centro de Física, Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal*

arXiv: 2405.04573



Quantum state	Quantum channel	POVM element
↓	↓	↓
$\mu(\lambda\|\rho)$	$\Gamma(\lambda'\|\lambda,\mathcal{E})$	$\xi(E\|\lambda)$

BASIS AND CANONICAL DUAL BASIS

$$A = \{|a_i\rangle\}_i, A' = \{|a_i'\rangle\}_i$$

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$$F_{i,i'} := \langle a'_{i'}|a_i\rangle|a'_{i'}\rangle\langle a_i|$$

This is a basis for the Hilbert space $\mathcal{B}(\mathcal{H})$

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$$A = \{|a_i\rangle\}_i, A' = \{|a'_i\rangle\}_i$$

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$$D_{i,i'} := \frac{|a_i\rangle\langle a'_{i'}|}{\langle a'_{i'}|a_i\rangle}$$

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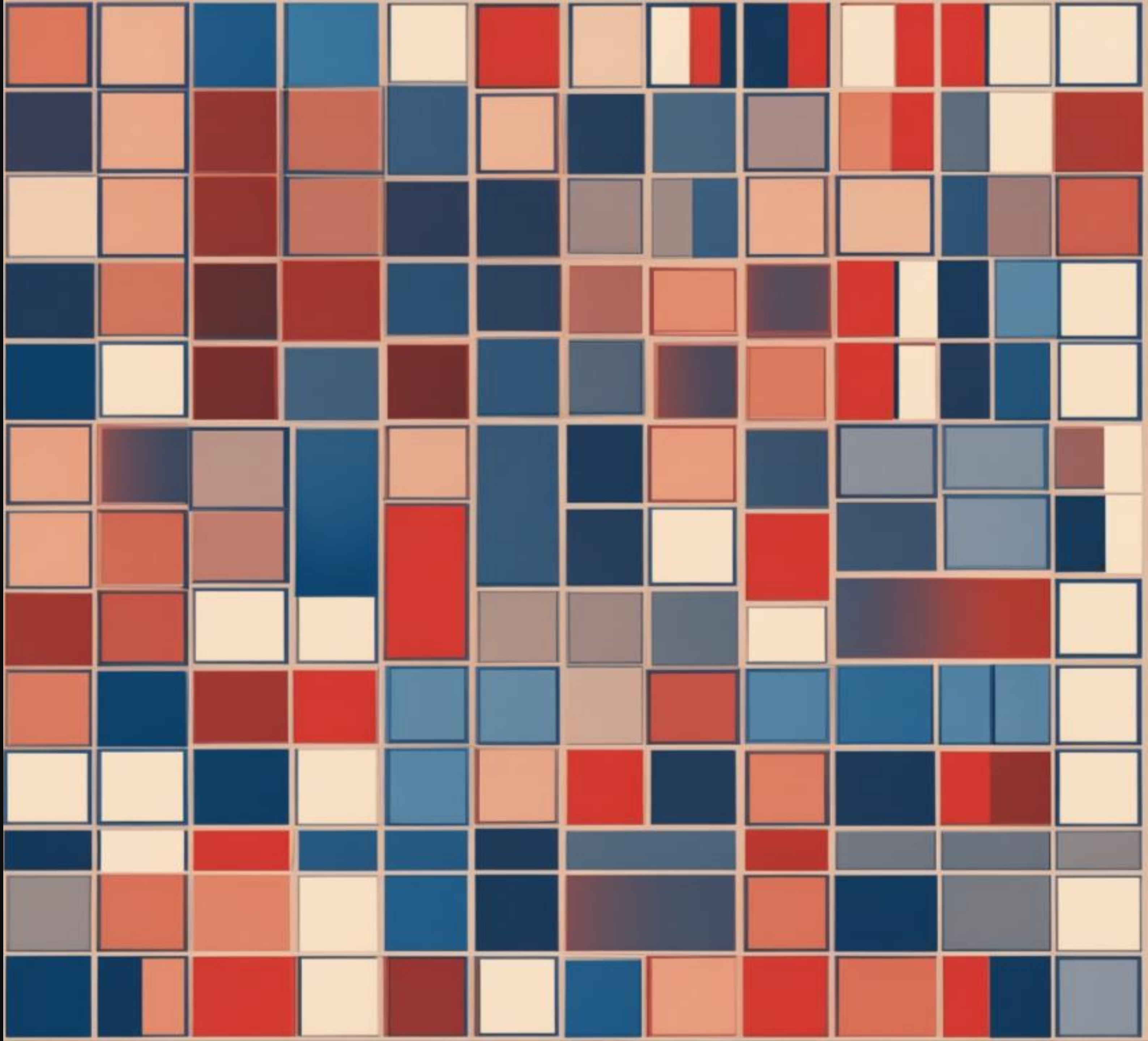
BASIS AND CANONICAL DUAL BASIS

$$A = \{|a_i\rangle\}_i,\ A' = \{|a'_i\rangle\}_i$$

$$\langle a_i|a'_{i'}\rangle \neq 0\ \forall i, i'$$

$$\boxed{\mathrm{Tr}(F_{i,i'}D_{j,j'}) = \delta_{i,j}\delta_{i',j'}}$$

KIRKWOOD-DIRAC REPRESENTATION



STATES

$$F_{i,i'} := \langle a'_{i'}|a_i\rangle|a'_{i'}\rangle\langle a_i|$$

$$\mu(i, i'|\rho) = \mathrm{Tr}(F_{i,i'}\rho)$$

STATES

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EFFECTS

$$D_{i,i'} := \frac{|a_i\rangle\langle a'_{i'}|}{\langle a'_{i'}|a_i\rangle}$$

$$\xi(E|i,i') = \mathrm{Tr}(ED_{i,i'})$$

EFFECTS

$$D_{i,i'} := \frac{|a_i\rangle\langle a'_{i'}|}{\langle a'_{i'}|a_i\rangle}$$

$$\xi(E|i,i') = \mathrm{Tr}(ED_{i,i'})$$

$$\boxed{\xi(E|i,i') = \frac{\langle a'_{i'}|E|a_i\rangle}{\langle a'_{i'}|a_i\rangle}}$$

Weak value

QUANTUM CHANNELS

$$F_{i,i'} := \langle a'_{i'}|a_i\rangle|a'_{i'}\rangle\langle a_i|$$

$$D_{i,i'} := \frac{|a_i\rangle\langle a'_{i'}|}{\langle a'_{i'}|a_i\rangle}$$

$$\Gamma(j, j'|i, i', \mathcal{E}) := \mathrm{Tr}(F_{j,j'}\mathcal{E}(D_{i,i'}))$$

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Summing over the 'rows' gives us one: 'Quasi' stochastic matrix

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Summing over the 'rows' gives us one: 'Quasi' stochastic matrix

$$\Gamma(j,j'|i,i',\mathrm{id}) = \delta_{i,j}\delta_{i',j'}$$

REPRESENTATION OF THE IDENTITY PROOF

$$\Gamma(\bar{i}, \bar{i}' | i, i', \mathcal{I}) = \mathrm{Tr}\left[F_{\bar{i},\bar{i}'} \mathcal{I}(D_{i,i'})\right]$$

$$= \mathrm{Tr}\left[F_{\bar{i},\bar{i}'} D_{i,i'}\right] = \delta_{\bar{i},i} \delta_{\bar{i}',i'},$$

EMPIRICAL ADEQUACY

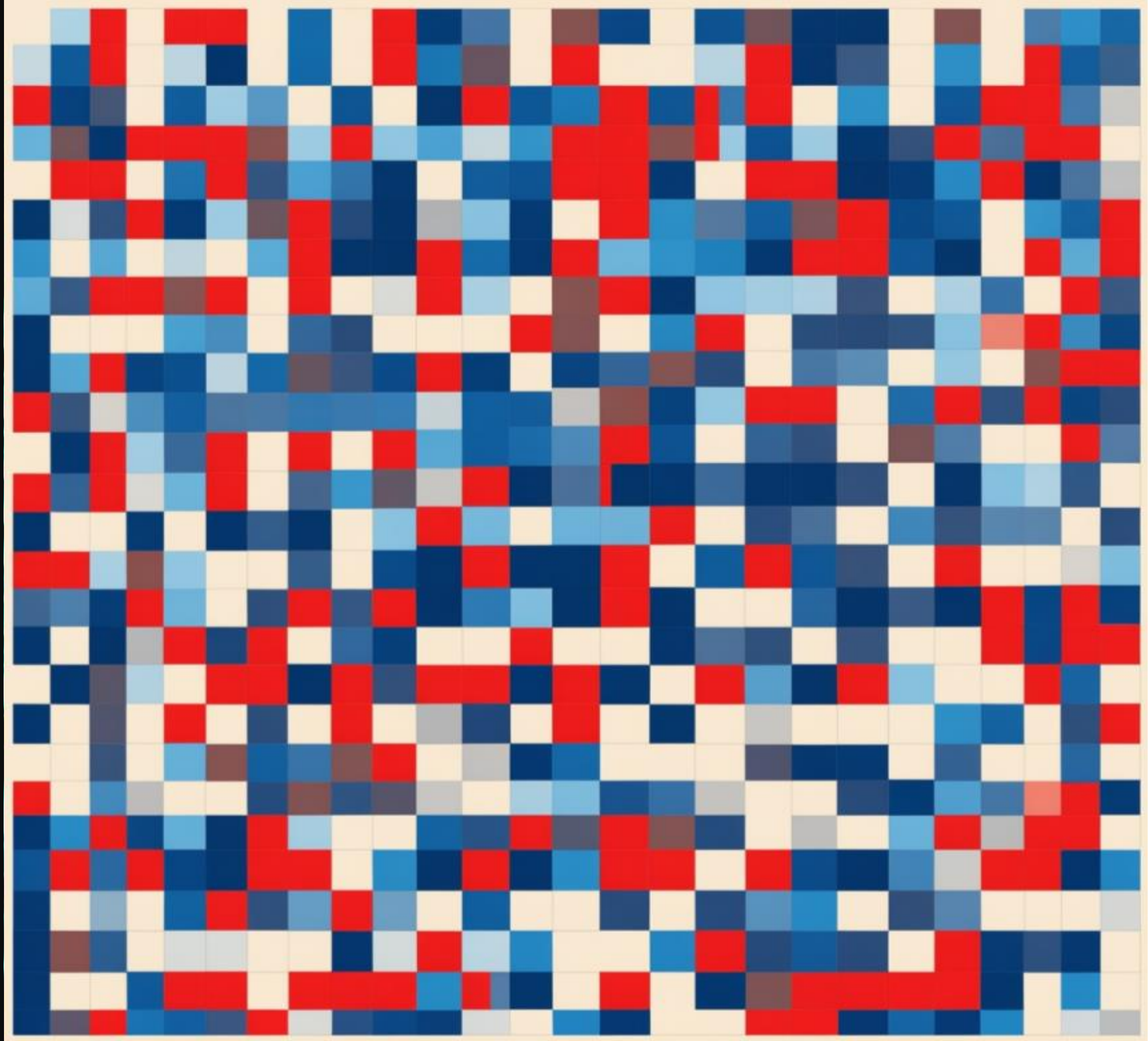
$$\mathrm{Tr}(E\mathcal{E}(\rho)) = \sum_{\lambda,\lambda'\in\Lambda} \xi(E|\lambda')\Gamma(\lambda'|\lambda,\mathcal{E})\mu(\lambda|\rho)$$

$$\boxed{\mathrm{Tr}(E\mathcal{E}(\rho)) = \sum_{j,j',i,i'} \xi(E|i,i')\Gamma(j,j'|i,i',\mathcal{E})\mu(i,i'|\rho)}$$

EMPIRICAL ADEQUACY PROOF

$$
\begin{aligned}
&\sum_{j,j',i,i'} \xi(E|j,j')\Gamma(j,j'|i,i',\mathcal{E})\mu(i,i'|\rho) \\
&= \sum_{j,j',i,i'} \xi(E|j,j')\,\mathrm{Tr}[F_{j,j'}\mathcal{E}(D_{i,i'})]\mu(i,i'|\rho) \\
&= \mathrm{Tr}\left[\left(\sum_{j,j'}\xi(E|j,j')F_{j,j'}\right)\mathcal{E}\left(\sum_{i,i'}\mu(i,i'|\rho)D_{i,i'}\right)\right] \\
&= \mathrm{Tr}(E\mathcal{E}(\rho)),
\end{aligned}
$$

APPLICATIONS



TOMOGRAPHY

$$\rho = \sum_{i,i'} \mu(i, i'|\rho) D_{i,i'}$$

$$E = \sum_{i,i'} \xi(E|i, i') F_{i,i'}$$

$$\mathcal{E}(\cdot) = \sum_{j,j',i,i'} \Gamma(j, j'|i, i', \mathcal{E}) \operatorname{Tr}[F_{i,i'} \ \cdot \ ] D_{j,j'}$$

TOMOGRAPHY

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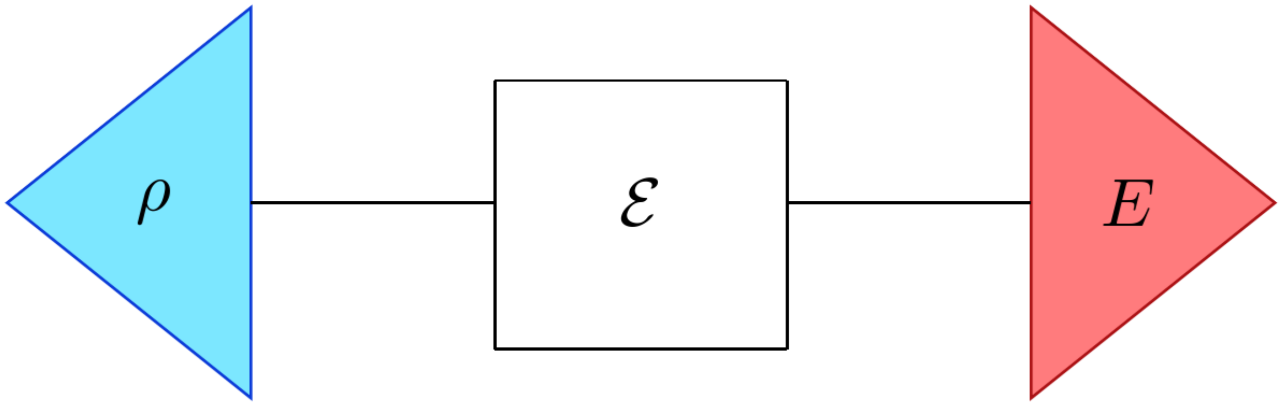
RW *et al*, *Quantum Sci. Technol.* **9** 015030 (2024)
Michał Oszmaniec *et al*, *New J. Phys.* **26** 013053 (2024)
Yihui Quek et al, Quantum 8, 1220 (2024).
…

(GENERALISED) CONTEXTUALITY

$$\Lambda$$

Phase space



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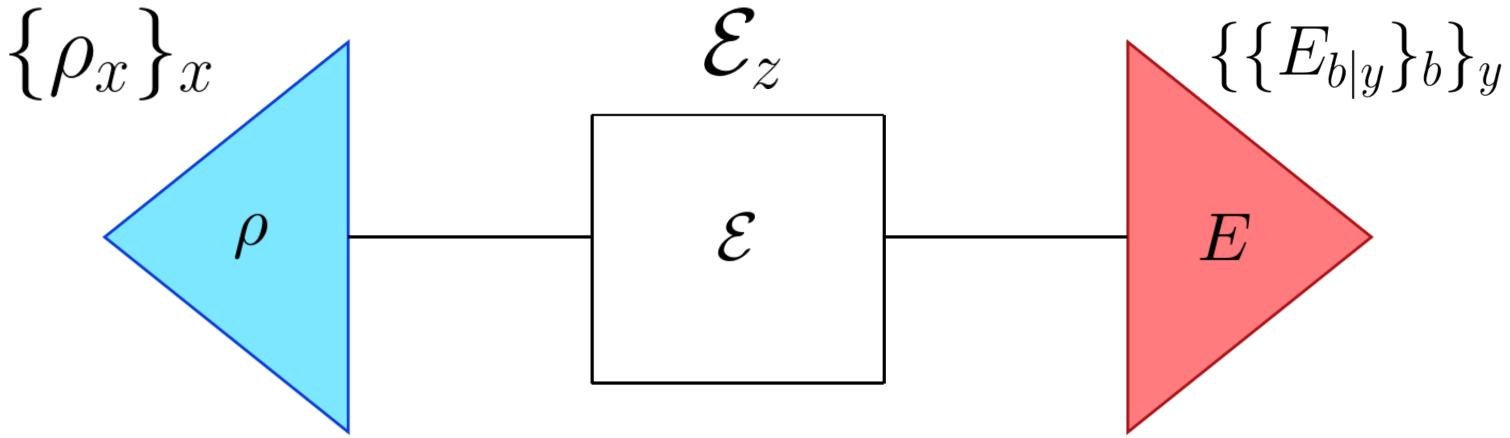

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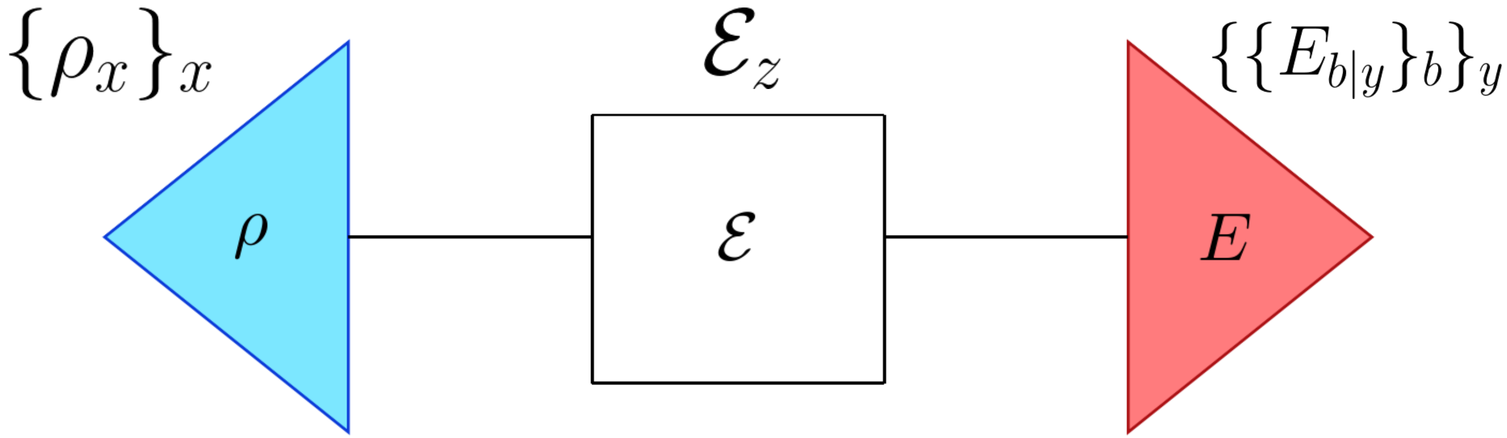
$$\Lambda$$

Phase space

This subtheory/fragment is noncontextual

$\updownarrow$

There exists some real nonnegative (probabilistic) representation



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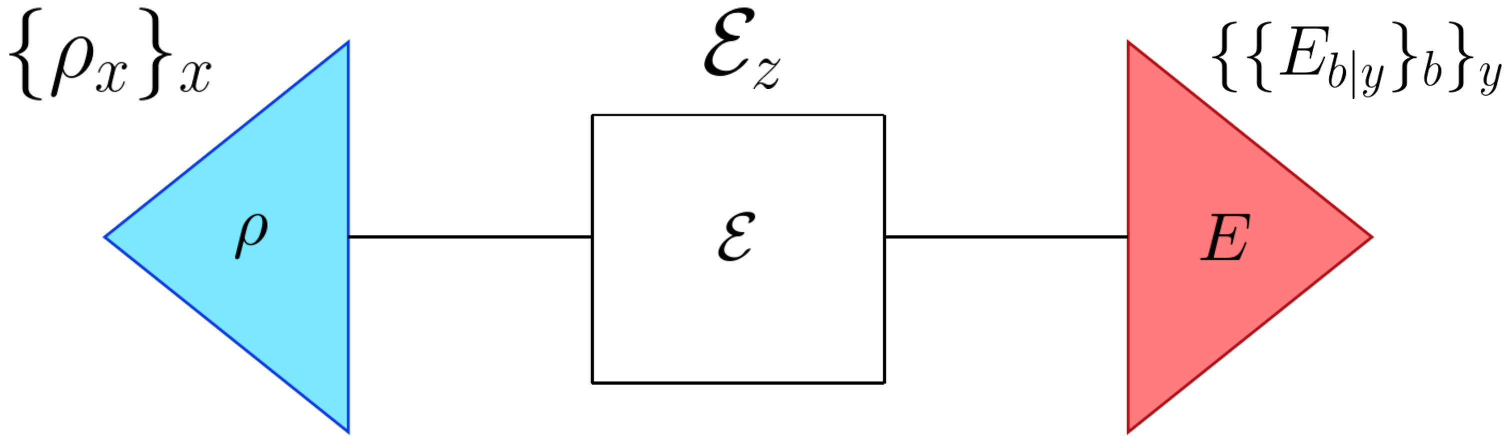

Quantum state

Quantum channel

POVM element

Real nonnegative KD representation $\Rightarrow$ There is a noncontextual model for that subtheory

(GENERALISED) CONTEXTUALITY

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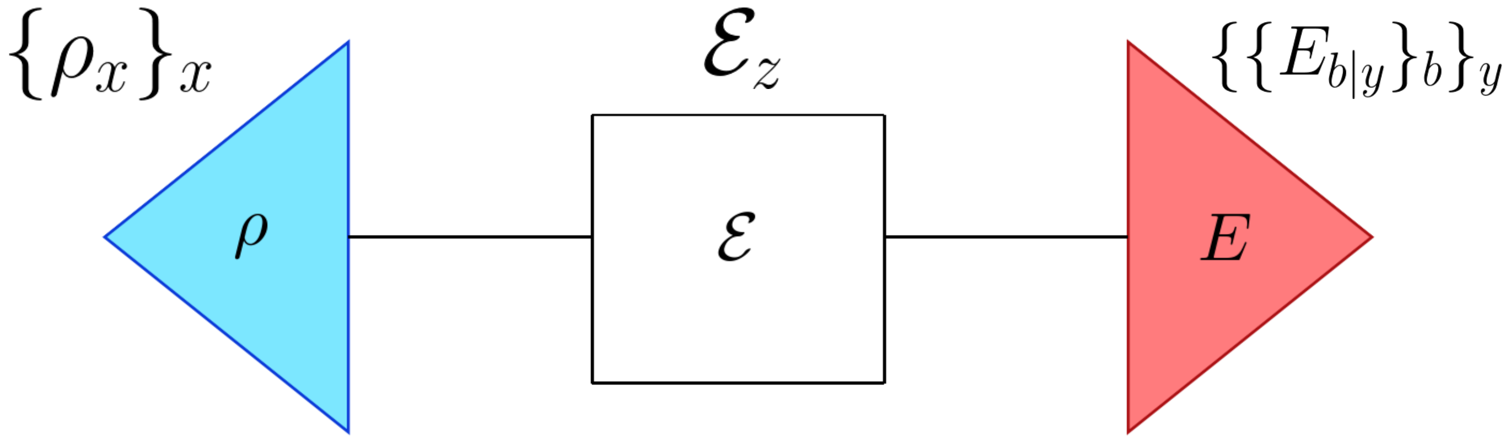
This subtheory/fragment is noncontextual

$\updownarrow$

There exists some real nonnegative (probabilistic) representation



Quantum state

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Real nonnegative KD representation → There is a noncontextual model for that subtheory

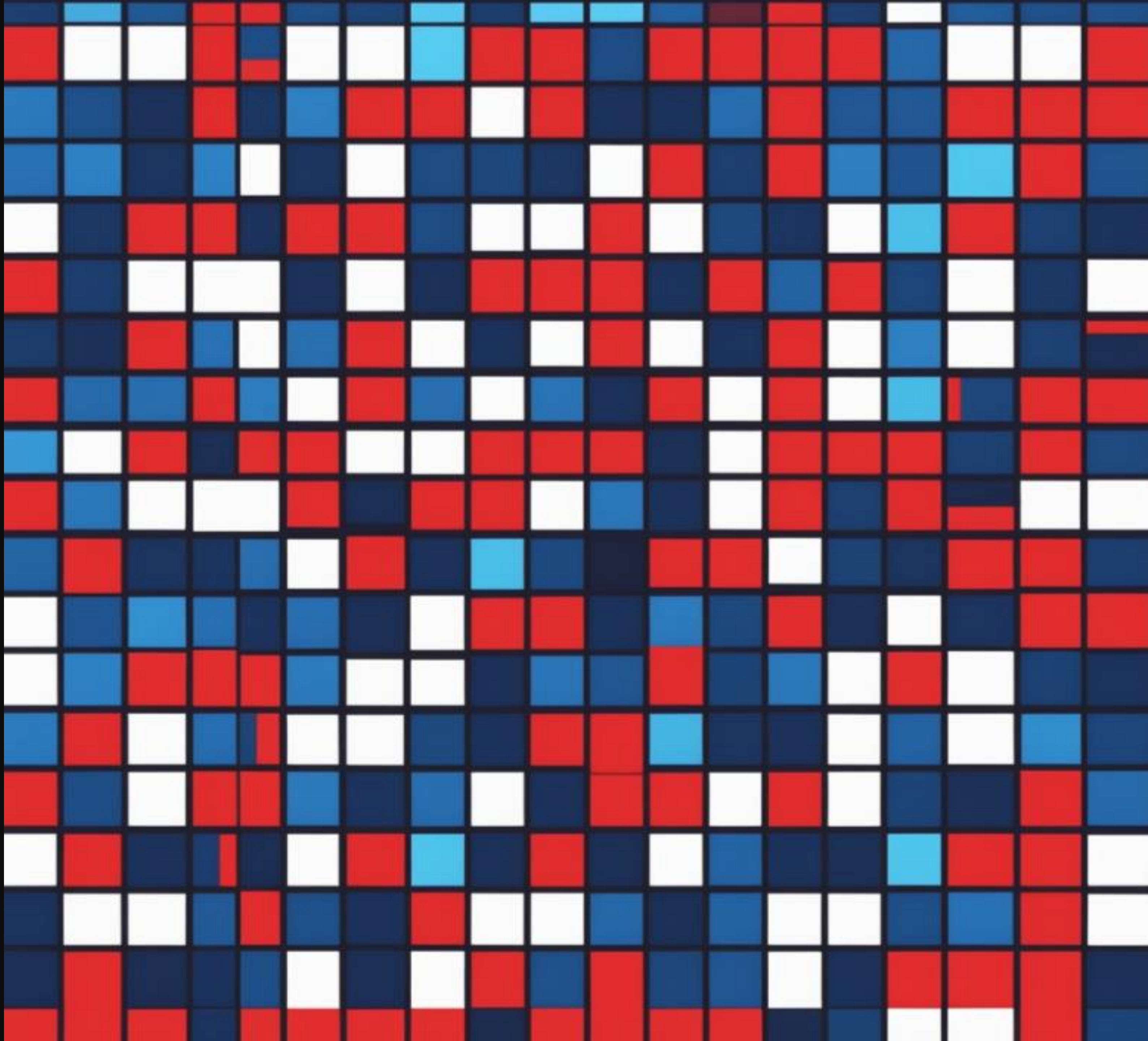
Negative/imaginary KD representations → Inconclusive

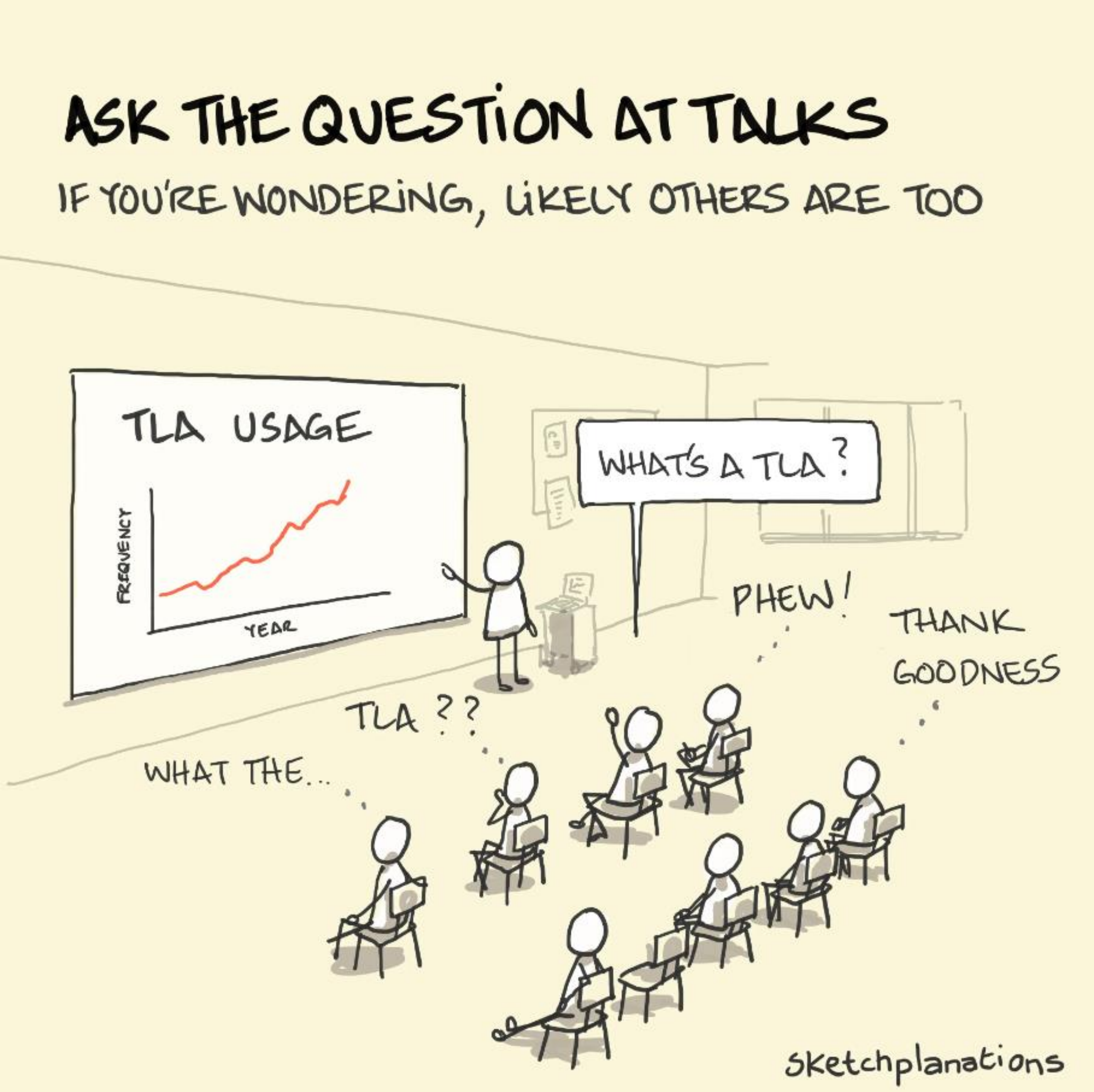
SUMMARY

KD distributions are useful

New KD representations beyond states

Negativity/imaginarity of a KD is not equivalent to contextuality





THANK YOU

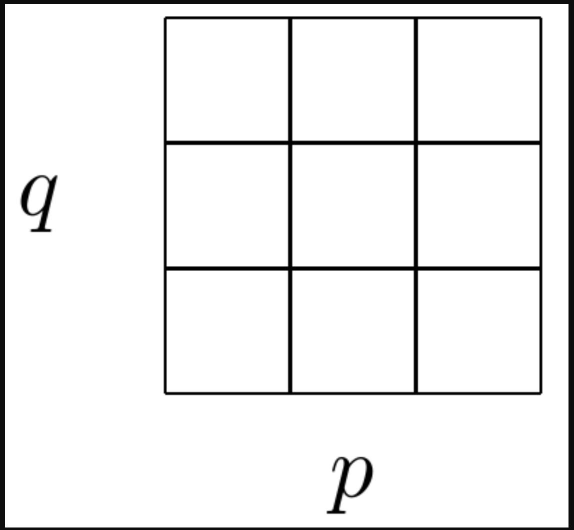
- rafael.wagner@inl.int
- @QuantumRW

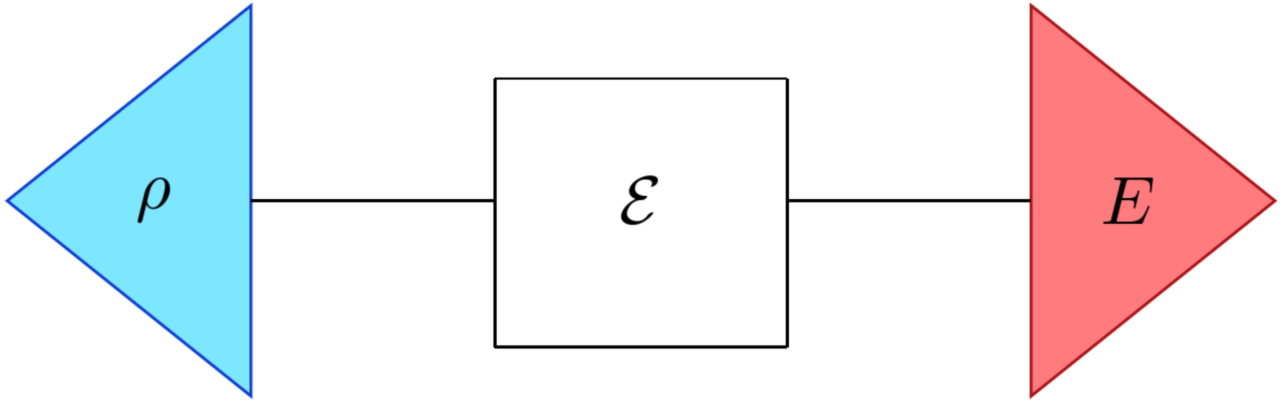
MORE INFORMATION

QUANTUM THEORY IN PHASE SPACE

$$\Lambda$$

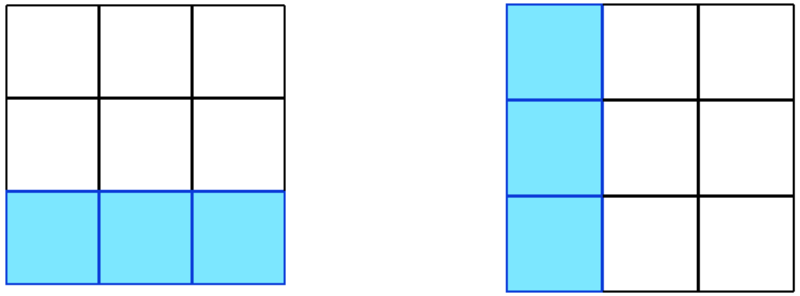
Phase space





Quantum state | Quantum channel | POVM element

$$\mu(\lambda|\rho) \qquad \Gamma(\lambda'|\lambda,\mathcal{E}) \qquad \xi(E|\lambda)$$



FRAME THEORY

FRAMES

$$\{|f_k\rangle\}_k$$

Countable family

$$\exists a, b > 0$$

$$a\|f\| \leq \sum_k |\langle f, f_k\rangle|^2 \leq b\|f\|$$

$$\forall |f\rangle \in \mathcal{H}$$

Ole Christensen, *An introduction to frames and Riesz bases*, (2003)

FRAMES

A family of elements $\{|f_k\rangle\}_k$ is a frame iff

$$\mathrm{span}(\{|f_k\rangle\}) = \mathcal{H}$$

- Frames can be *overcomplete*
- Every basis is an example of a frame

Ole Christensen, *An introduction to frames and Riesz bases*, (2003)

FRAMES

$$\{|f_k\rangle\}_k$$ Orthonormal basis

$$\{|f_k\rangle\}_k \overset{U}{\mapsto} \{|e_k\rangle\}_k$$

$$U^\dagger = U^{-1}$$

$$U \in \mathcal{B}(\mathcal{H})$$

Ole Christensen, *An introduction to frames and Riesz bases*, (2003)

FRAMES

$$\{|f_k\rangle\}_k$$ Frame

$$\{|f_k\rangle\}_k \overset{U}{\mapsto} \{|e_k\rangle\}_k$$

Surjective operator

$$U \in \mathcal{B}(\mathcal{H})$$

Ole Christensen, *An introduction to frames and Riesz bases*, (2003)

FRAMES

$$\{|f_k\rangle\}_k$$

Frame

$$S(f) = \sum_k \langle f, f_k \rangle f_k$$

Frame operator

- Frame operators are selfadjoint and invertible
- The inverse of the frame operator defines a way of finding the canonical dual frame

Ole Christensen, *An introduction to frames and Riesz bases*, (2003)

FRAMES

$$\{|f_k\rangle\}_k$$ Frame

There is a unique set $\{g_k\}_k$ such that

$$f = \sum_k \langle f, g_k \rangle f_k$$

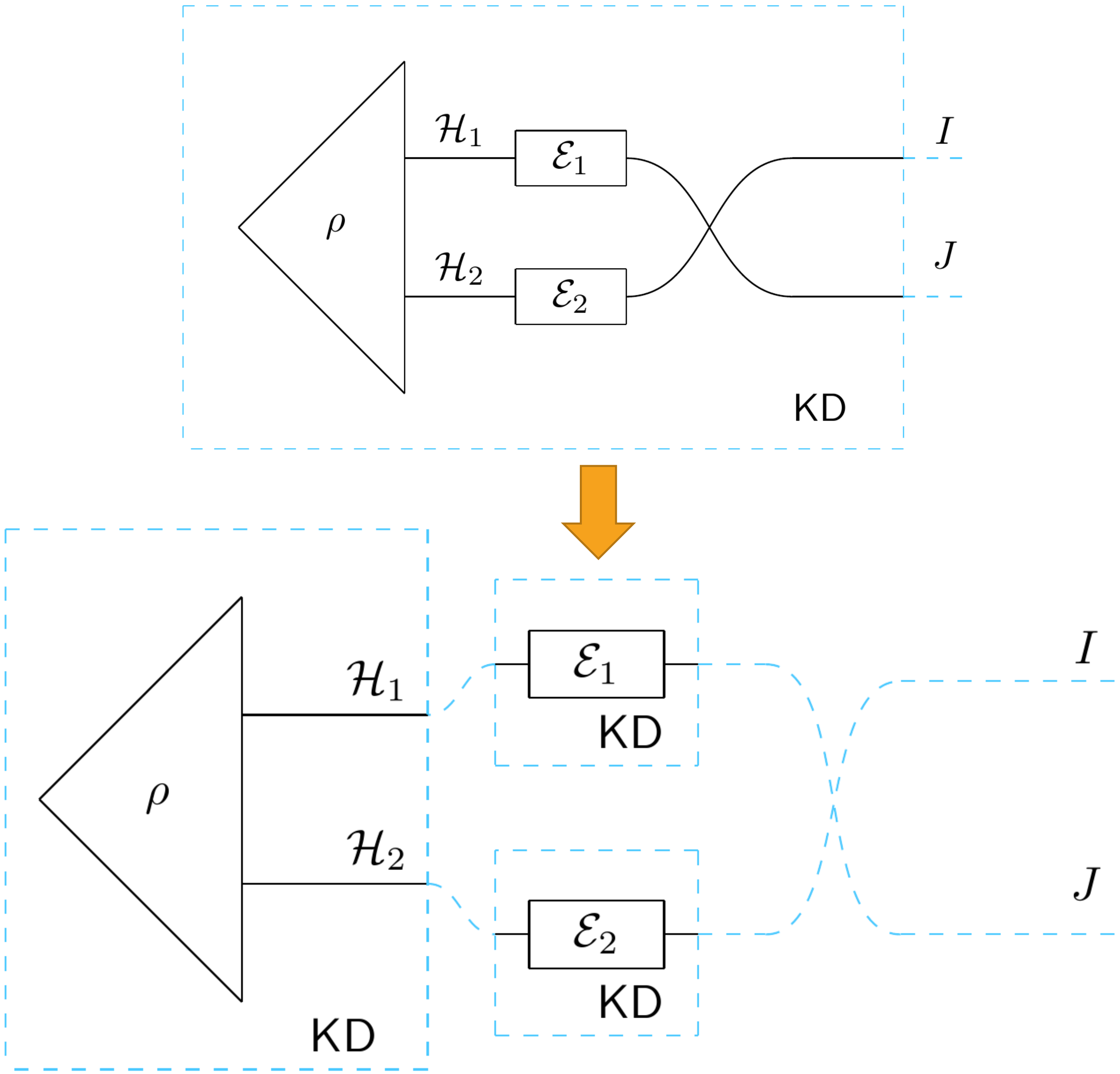
$$\langle f_j, g_k \rangle = \delta_{j,k} \quad g_k = S^{-1} f_k$$

Ole Christensen, *An introduction to frames and Riesz bases*, (2003)

DIAGRAM PRESERVATION

IMPORTANT PROPERTY

Diagram preservation
('functoriality')



BIPARTITE SYSTEMS

BIPARTITE SYSTEMS

$$F_{i,i'} := \langle a'_{i'}|a_i\rangle |a'_{i'}\rangle\langle a_i|$$

$$D_{i,i'} := \frac{|a_i\rangle\langle a'_{i'}|}{\langle a'_{i'}|a_i\rangle}$$

$$F_{i,i';j,j'} = \langle a'_{i'}, b'_{j'}|a_i, b_j\rangle |a'_{i'}, b'_{j'}\rangle\langle a_i, b_j|$$

$$F_{i,i';j,j'} = F_{i,i'} \otimes F_{j,j'}$$

$$D_{i,i';j,j'} = D_{i,i'} \otimes D_{j,j'}$$

$$\Gamma(k, k'; l, l'|i, i'; j, j', \mathcal{E}_1 \otimes \mathcal{E}_2) = \Gamma(k, k'|i, i', \mathcal{E}_1)\Gamma(l, l'|j, j', \mathcal{E}_2)$$

PROOFS

TOMOGRAPHY PROOF

Diagram preservation
('functoriality')

$$
\begin{aligned}
&\sum_{i,i'} \mu(i,i'|\rho) D_{i,i'} \\
=&\sum_{i,i'} \langle a_i|\,\rho\,|a'_{i'}\rangle\,\langle a'_{i'}|a_i\rangle\,\frac{|a_i\rangle\langle a'_{i'}|}{\langle a'_{i'}|a_i\rangle} \\
=&\sum_{i,i'} |a_i\rangle\langle a_i|\,\rho\,|a'_{i'}\rangle\langle a'_{i'}| \\
=&\rho
\end{aligned}
$$

TOMOGRAPHY PROOF

Diagram preservation ('functoriality')

$$\sum_{i,i'} \xi(E|i,i')F_{i,i'}$$

$$= \sum_{i,i'} \frac{\langle a'_{i'}|\, E \,|a_i\rangle}{\langle a'_{i'}|a_i\rangle} |a'_{i'}\rangle\!\langle a_i| \, \langle a'_{i'}|a_i\rangle$$

$$= \sum_{i,i'} |a'_{i'}\rangle\!\langle a'_{i'}|\, E \,|a_i\rangle\!\langle a_i|$$

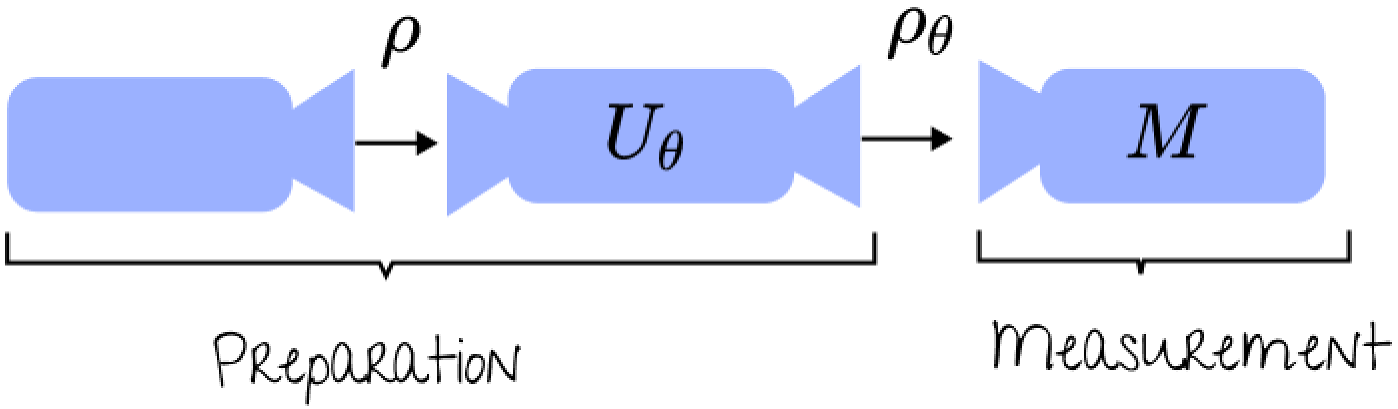
$$= E$$

TOMOGRAPHY PROOF

Diagram preservation
('functoriality')

$$
\begin{aligned}
&\sum_{j,j',i,i'} \Gamma(j,j'|i,i',\mathcal{E})\,\mathrm{Tr}[F_{i,i'}\;\cdot\;]D_{j,j'} \\
=&\sum_{j,j',i,i'} \mathrm{Tr}[F_{j,j'}\mathcal{E}(D_{i,i'})]\mu(i,i'|\;\cdot\;)D_{j,j'} \\
=&\sum_{j,j'} \mathrm{Tr}\left[F_{j,j'}\mathcal{E}\left(\sum_{i,i'}\mu(i,i'|\;\cdot\;)D_{i,i'}\right)\right]D_{j,j'} \\
=&\sum_{j,j'} \mathrm{Tr}[F_{j,j'}\mathcal{E}\left(\cdot\right)]D_{j,j'} \\
=&\sum_{j,j'} \mu\left(j,j'|\mathcal{E}(\cdot)\right)D_{j,j'} \\
=&\mathcal{E}(\cdot),
\end{aligned}
$$

POST-SELECTED METROLOGY



PREPARE-AND-MEASURE PARAMETER ESTIMATION

- Estimation of a single parameter

PREPARE-AND-MEASURE PARAMETER ESTIMATION

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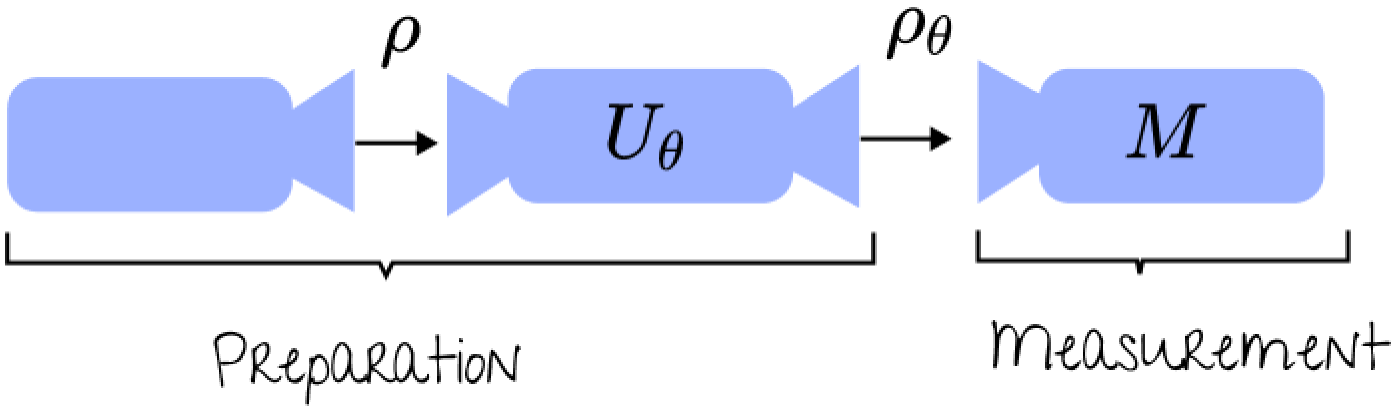
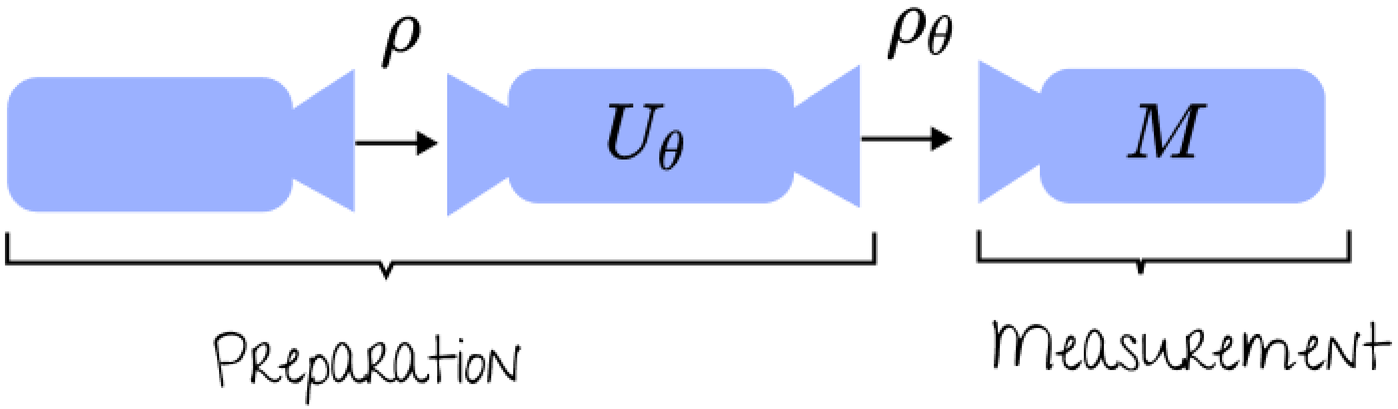


$$U_\theta = e^{i\theta A}$$

$$\mathcal{I}_Q(\theta|\rho_\theta) = \mathrm{Tr}(\rho_\theta \Lambda_{\rho_\theta}^2)$$

PREPARE-AND-MEASURE PARAMETER ESTIMATION



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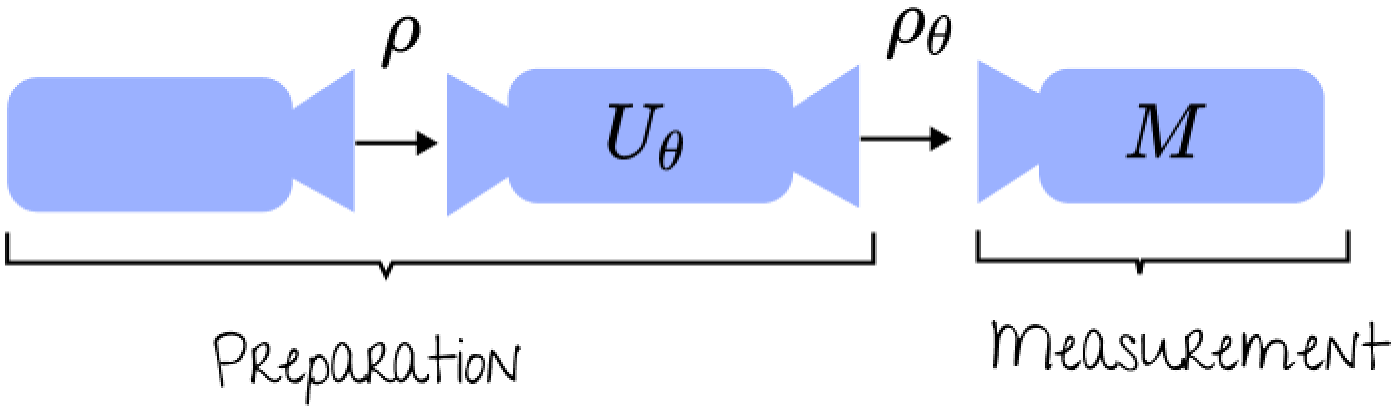
- Estimation of a single parameter
- The quantum Fisher information is obtained maximizing over all possible measurements



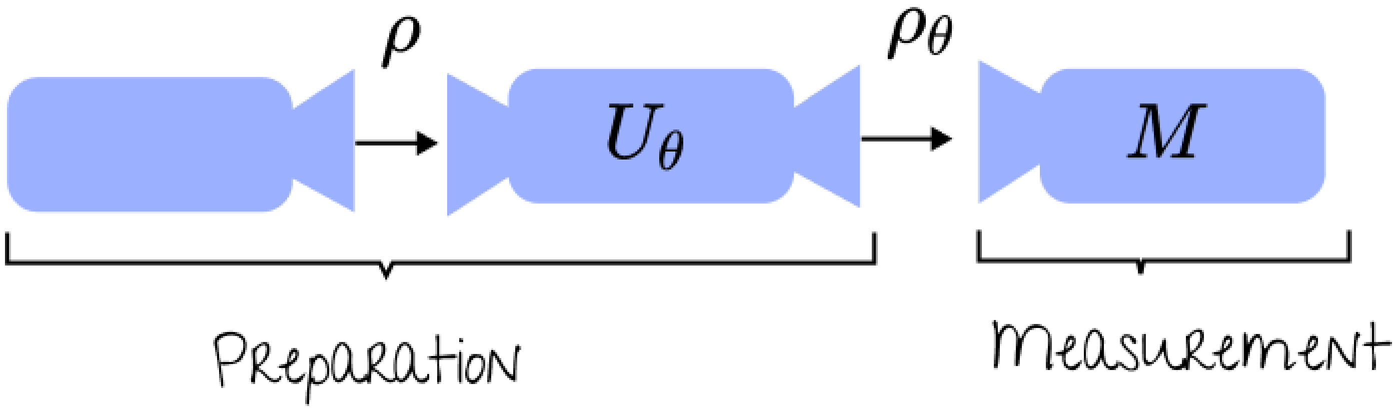
$$U_\theta = e^{i\theta A}$$

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{N_C \mathcal{I}(\theta)}$$

$$\mathcal{I}_Q(\theta|\rho_\theta) = \mathrm{Tr}(\rho_\theta \Lambda^2_{\rho_\theta})$$

PREPARE-AND-MEASURE PARAMETER ESTIMATION

- Estimation of a single parameter
- The quantum Fisher information is obtained maximizing over all possible measurements
- The bound is tight, i.e., there exists some measurement that extracts the largest possible information about the parameter
- We want the quantum Fisher information to be as high as possible to make the rhs small



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$$\Delta a := a_{max} - a_{min}$$

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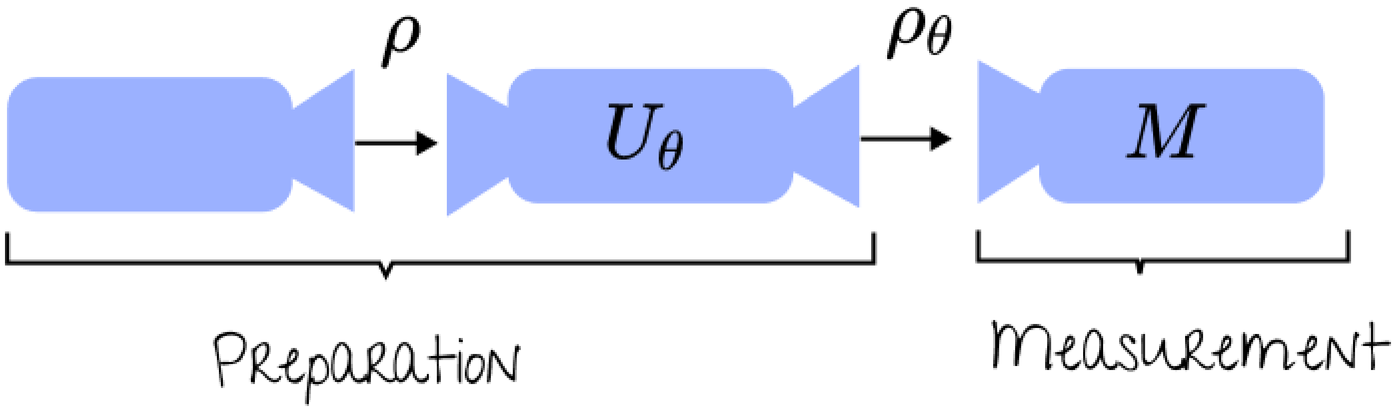

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$$\Delta a := a_{max} - a_{min}$$

$$\max_\rho \mathcal{I}_Q(\theta|\rho_\theta) = (\Delta a)^2$$

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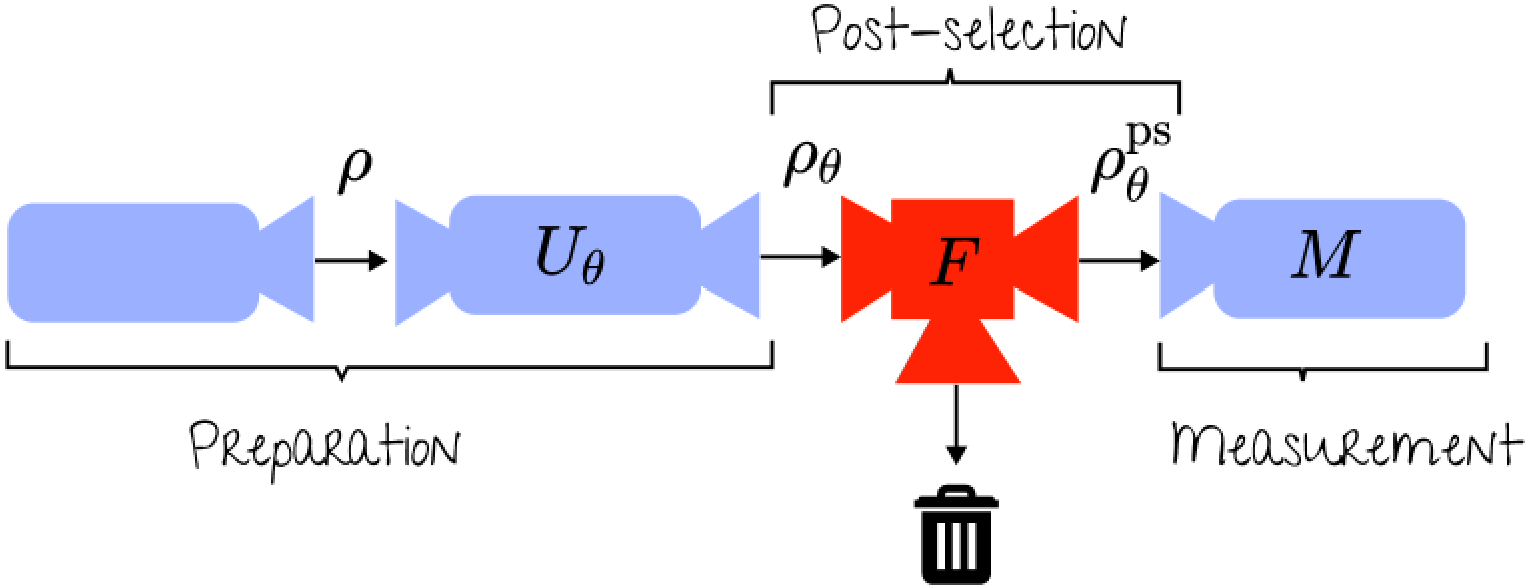

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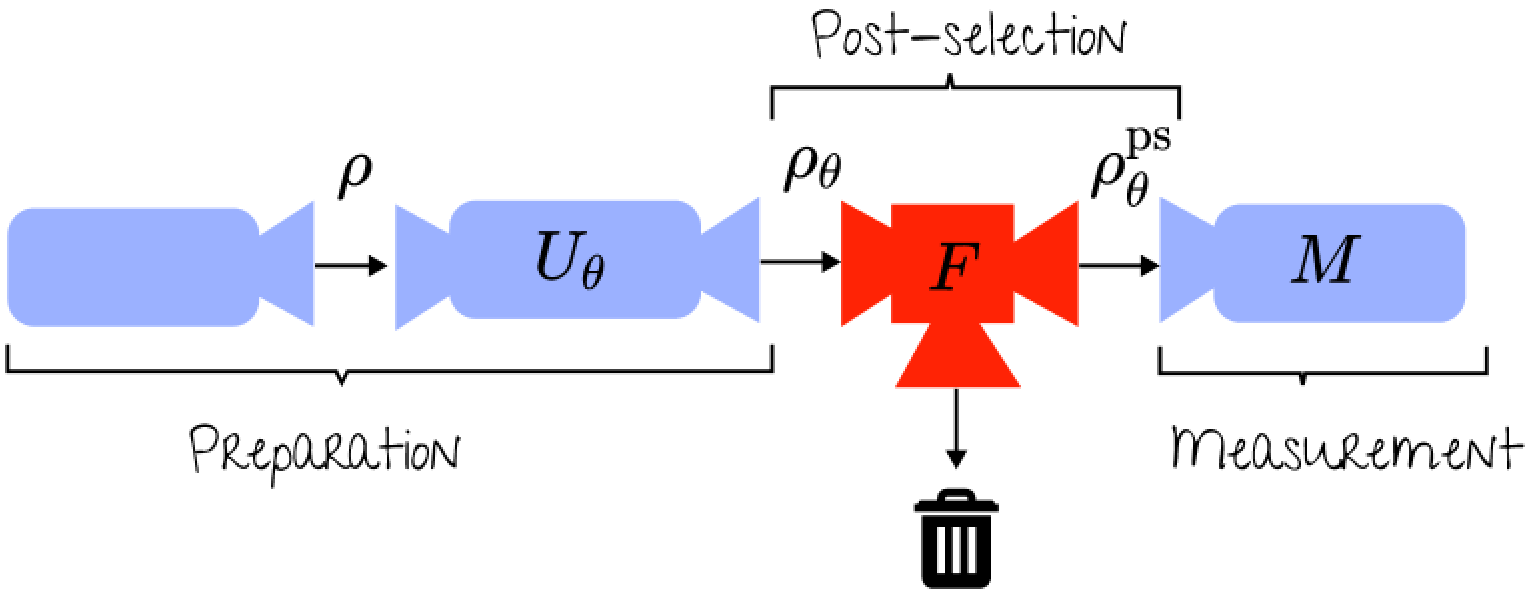
POST-SELECTION

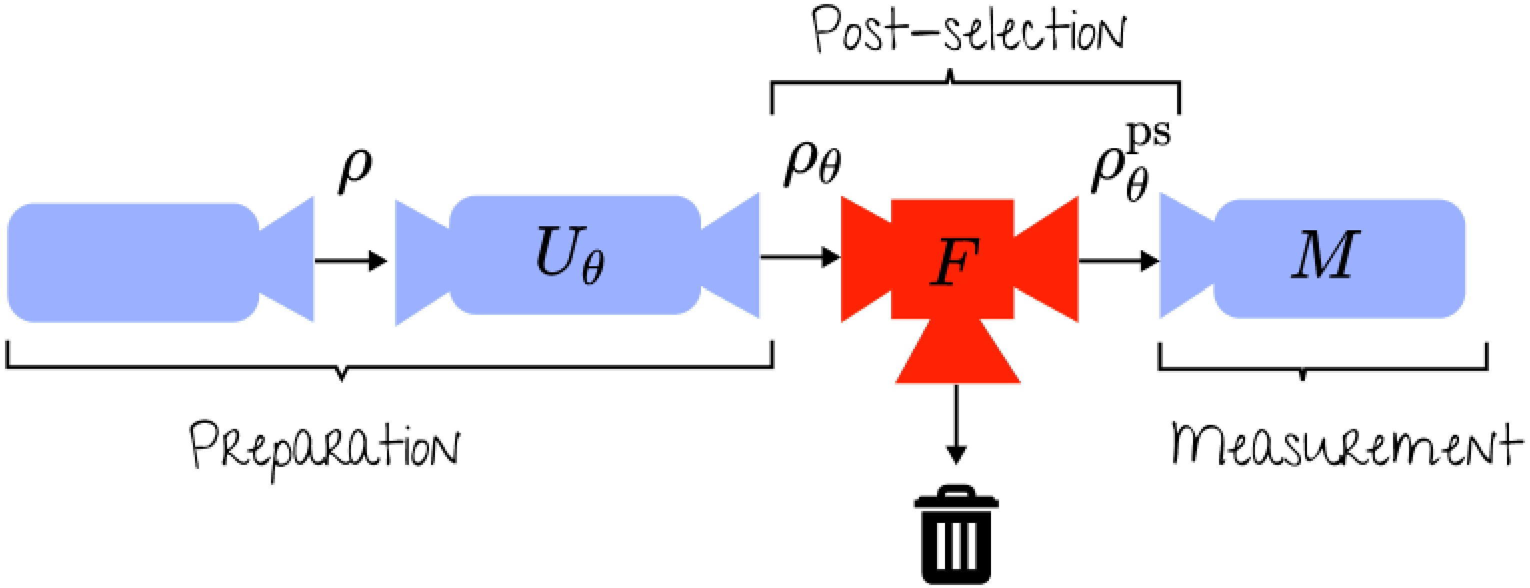


$$U_\theta = e^{i\theta A}$$

$$F = \sum_f |f\rangle\langle f|$$

$$p_\theta^{\mathrm{ps}} = \mathrm{Tr}(F\rho_\theta)$$

POST-SELECTION



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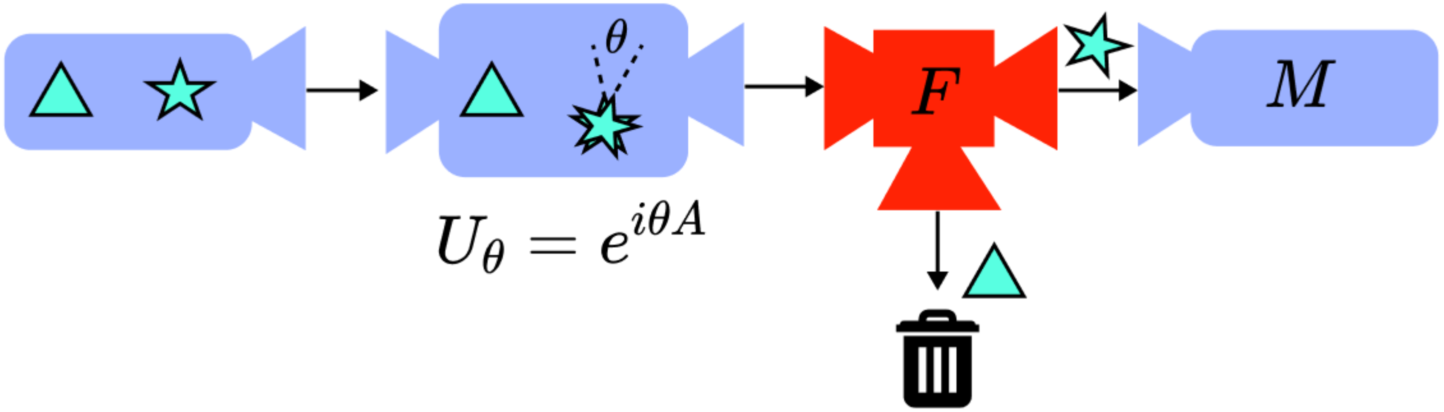

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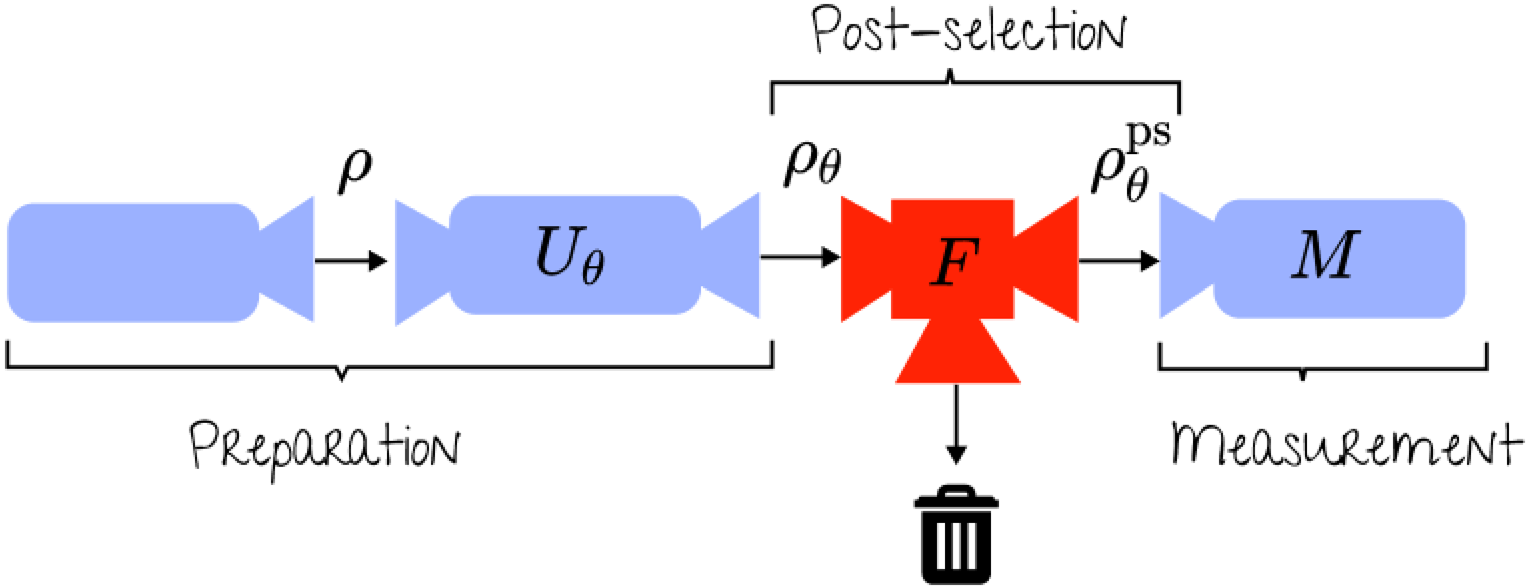

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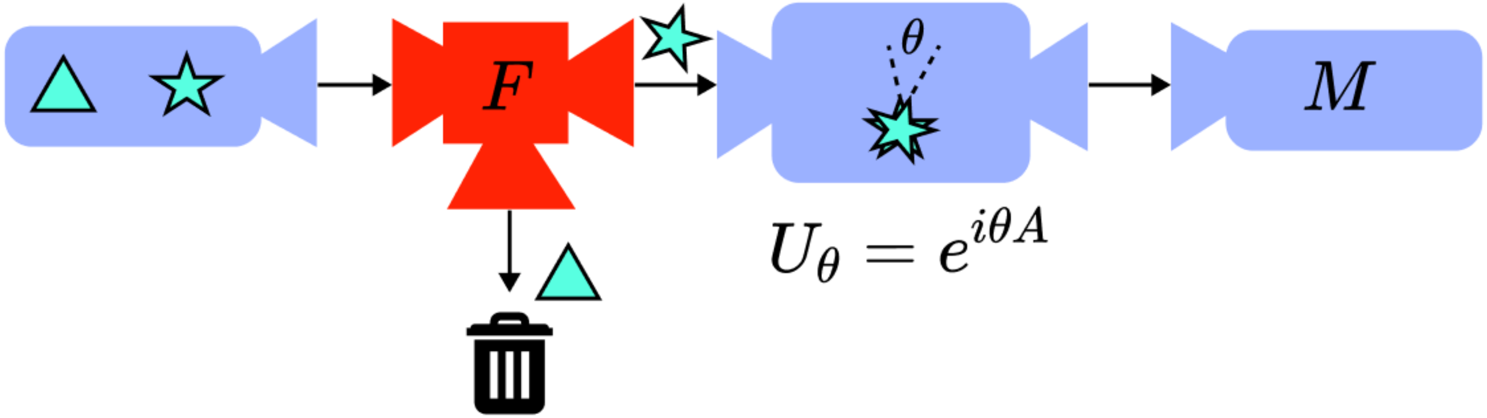

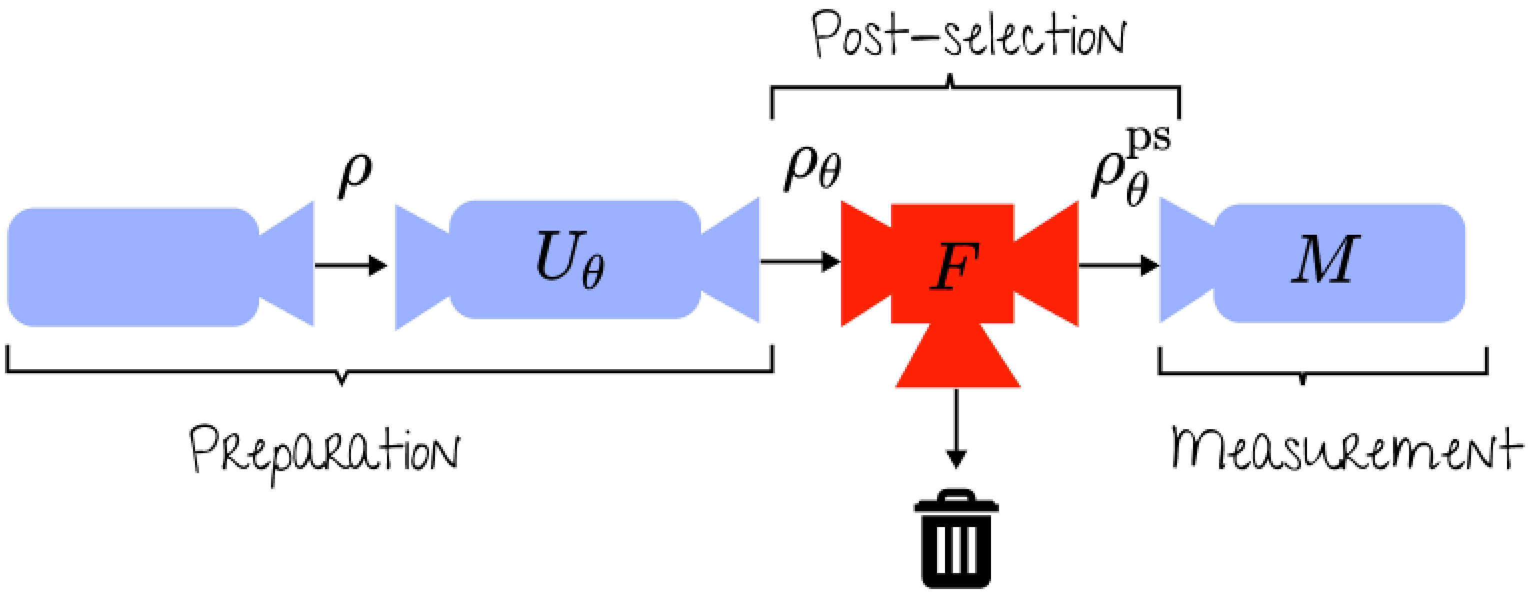

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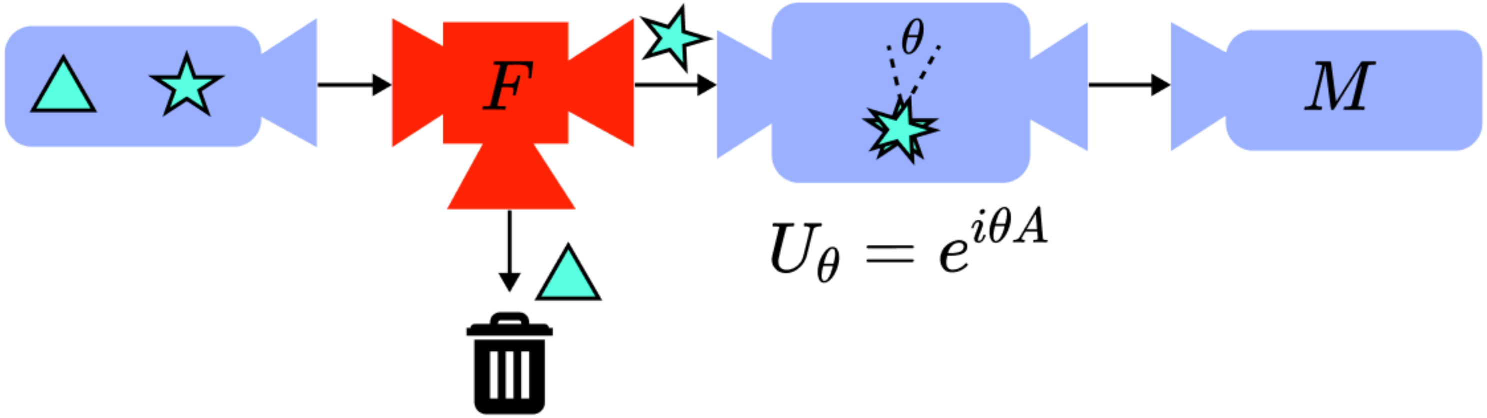
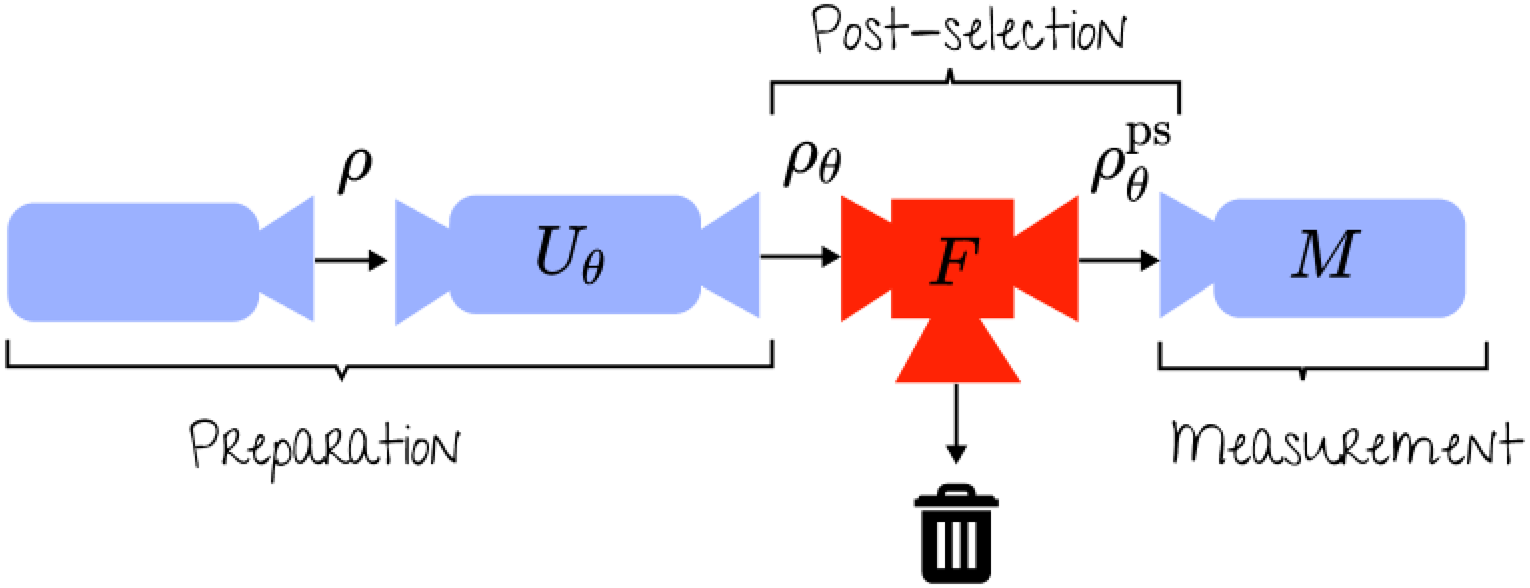


POST-SELECTION

$$\boxed{[F, A] = 0}$$



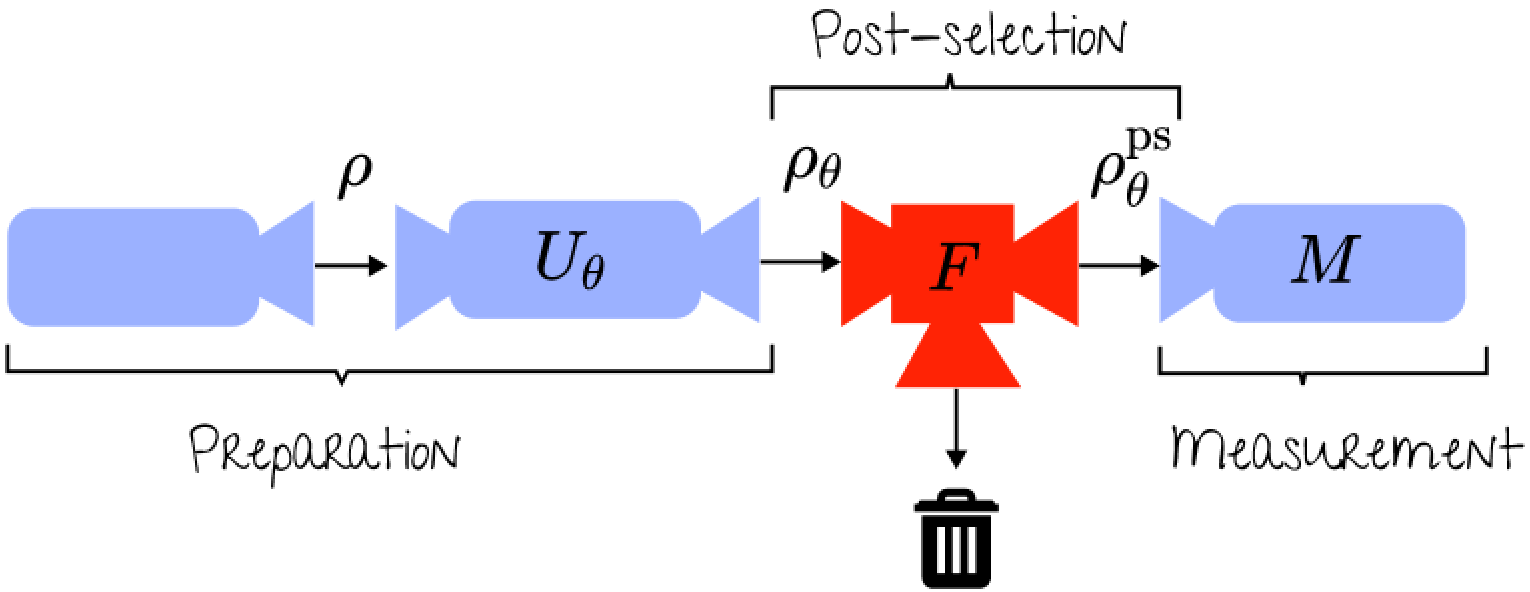
$$U_\theta = e^{i\theta A} \qquad F = \sum_f |f\rangle\langle f|$$

$$p_\theta^{\mathrm{ps}} = \mathrm{Tr}(F\rho_\theta)$$

$$\mathcal{I}_Q(\theta|\rho_\theta) \mapsto \mathcal{I}_Q(\theta|\rho_\theta^{\mathrm{ps}})$$

HIGHER FISHER INFORMATION

$$\boxed{[F, A] = 0}$$



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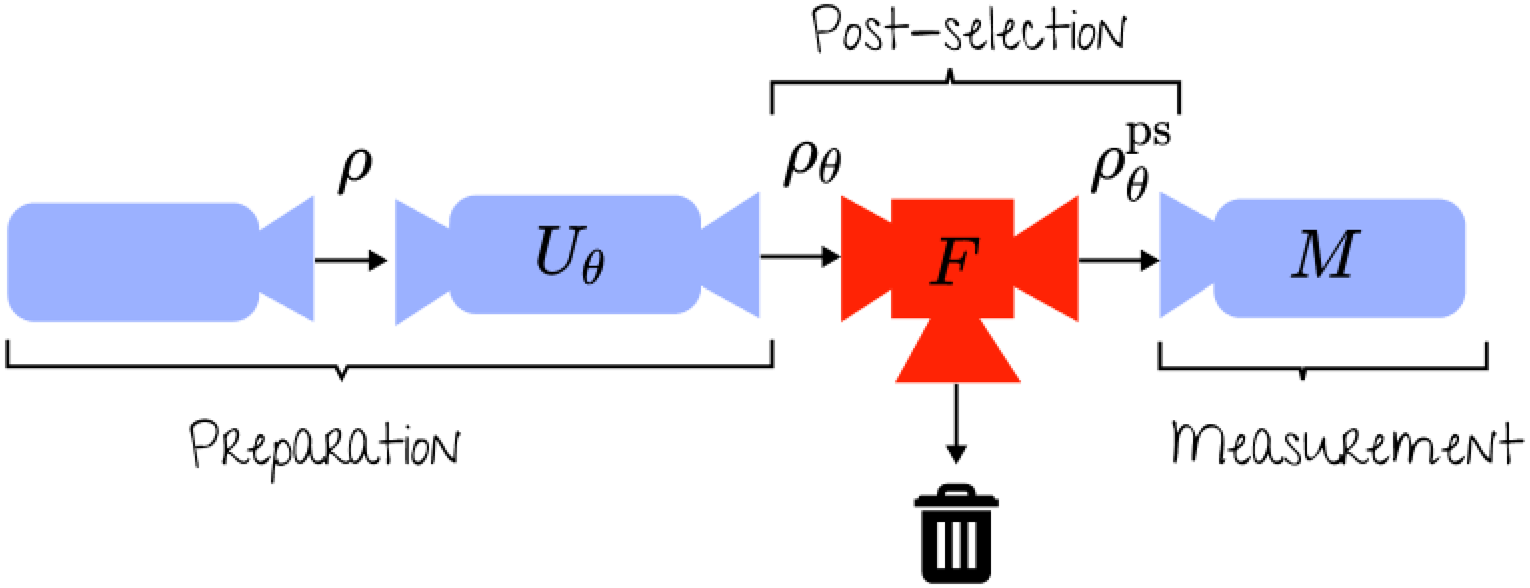
HIGHER FISHER INFORMATION

$$\boxed{[F, A] = 0}$$



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It is possible that $\mathcal{I}_Q(\theta|\rho_\theta^{\mathrm{ps}}) > \mathcal{I}_Q(\theta|\rho_\theta)$

HIGHER FISHER INFORMATION

$$\boxed{[F, A] = 0}$$

THEOREMS

ARTICLE

https://doi.org/10.1038/s41467-020-17559-w OPEN

Check for updates

Quantum advantage in postselected metrology

David R. M. Arvidsson-Shukur [1,2,3], Nicole Yunger Halpern [3,4,5], Hugo V. Lepage [1], Aleksander A. Lasek [1], Crispin H. W. Barnes [1] & Seth Lloyd [2,3]

$$\mathcal{I}_Q(\theta|\rho_\theta^{\mathrm{ps}}) \leq \max_{\rho} \mathcal{I}_Q(\theta|\rho_\theta)$$

Post-selected QFI cannot be higher than the optimal QFI without post-selection in a classical set up [F,A]=0

THEOREMS

- Post-selected QFI cannot be higher than the optimal QFI without post-selection in a classical set up [F,A]=0
- Whenever this happens it must be due to negativity!!

ARTICLE
https://doi.org/10.1038/s41467-020-17559-w OPEN

Quantum advantage in postselected metrology

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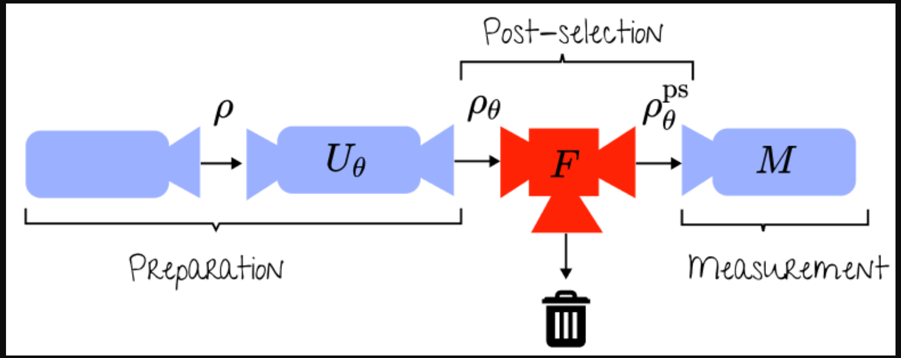
$$\mathcal{I}_Q(\theta|\rho_\theta^{\mathrm{ps}}) \leq \max_\rho \mathcal{I}_Q(\theta|\rho_\theta)$$

$$\mathcal{I}_Q(\theta|\rho_\theta^{\mathrm{ps}}) \stackrel{!}{>} \max_\rho \mathcal{I}_Q(\theta|\rho_\theta)$$

$$|\Psi_\theta\rangle = U_\theta|\Psi\rangle \qquad |\Psi_\theta^{\mathrm{ps}}\rangle = |\Psi_\theta\rangle/\sqrt{p_\theta^{\mathrm{ps}}}$$

$$|\dot{\Psi}_\theta\rangle \equiv \partial_\theta|\Psi_\theta\rangle$$

CONNECTION WITH KIRKWOOD-DIRAC QUASIPROBABILITY DISTRIBUTION



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$$\boxed{\mathcal{I}_Q(\theta|\Psi_\theta) = 4\langle\dot{\Psi}_\theta|\dot{\Psi}_\theta\rangle - 4|\langle\dot{\Psi}_\theta|\Psi_\theta\rangle|^2}$$

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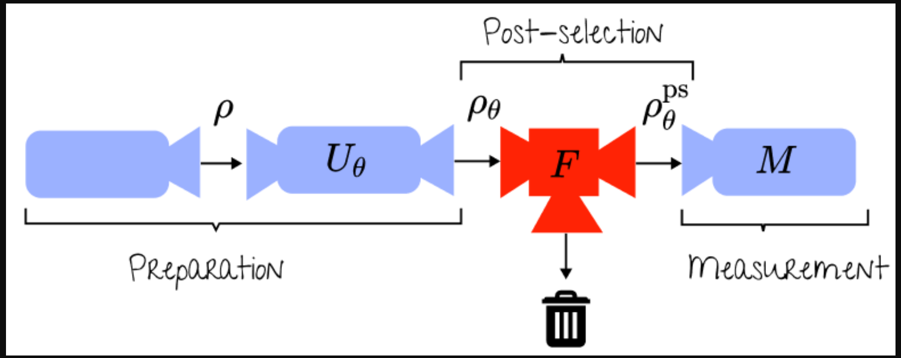

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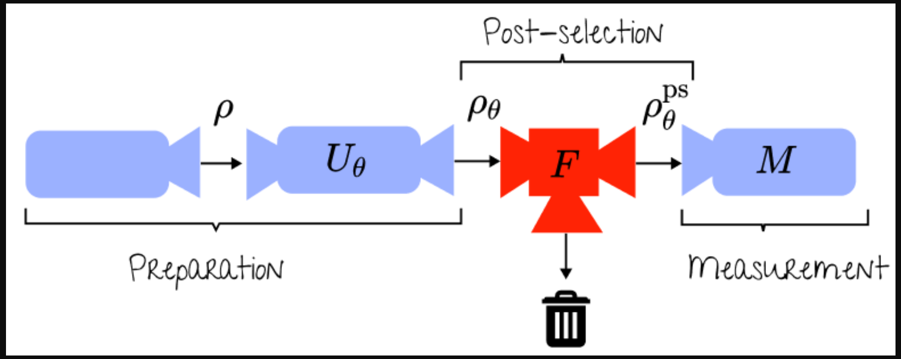

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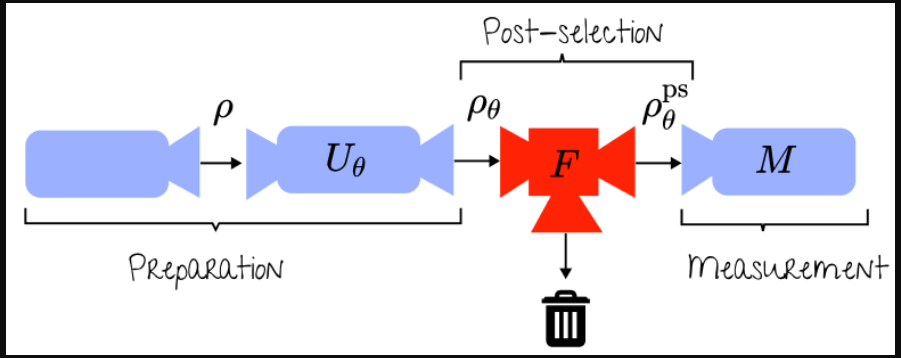

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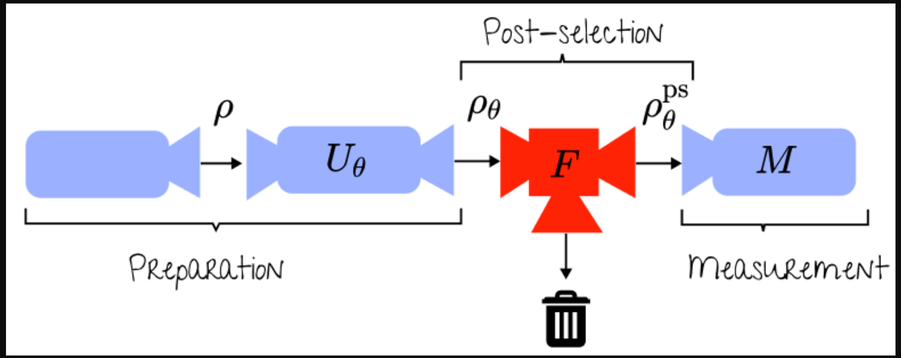

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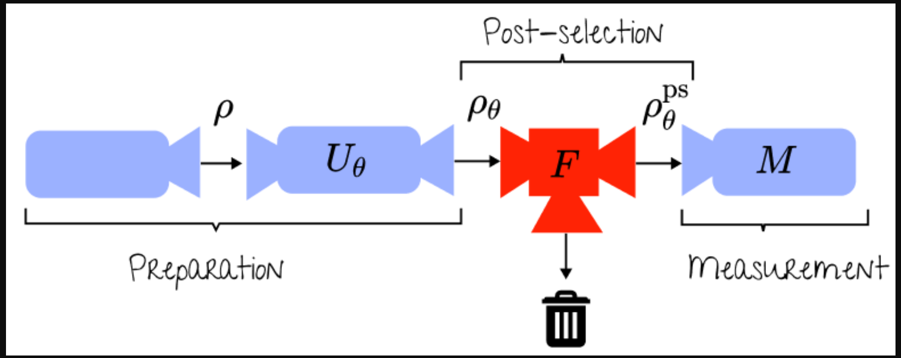
CONNECTION WITH KIRKWOOD-DIRAC QUASIPROBABILITY DISTRIBUTION



KIRKWOOD-DIRAC QUASIPROBABILITY DISTRIBUTION

$$\xi(\rho|a,b) = \mathrm{Tr}(|a\rangle\langle a|\rho|b\rangle\langle b|)$$

EXTENDED

$$\xi(\rho_\theta|a,a',b) = \mathrm{Tr}(|a\rangle\langle a|\rho_\theta|a'\rangle\langle a'||b\rangle\langle b|)$$

KIRKWOOD-DIRAC QUASIPROBABILITY DISTRIBUTION

EXTENDED

$$\xi(\rho|a,b) = \mathrm{Tr}(|a\rangle\langle a|\rho|b\rangle\langle b|)$$

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SCRAMBLING OF QUANTUM INFORMATION

$$F(t) := \mathrm{Tr}(W^\dagger(t)V^\dagger W(t)V\rho).$$

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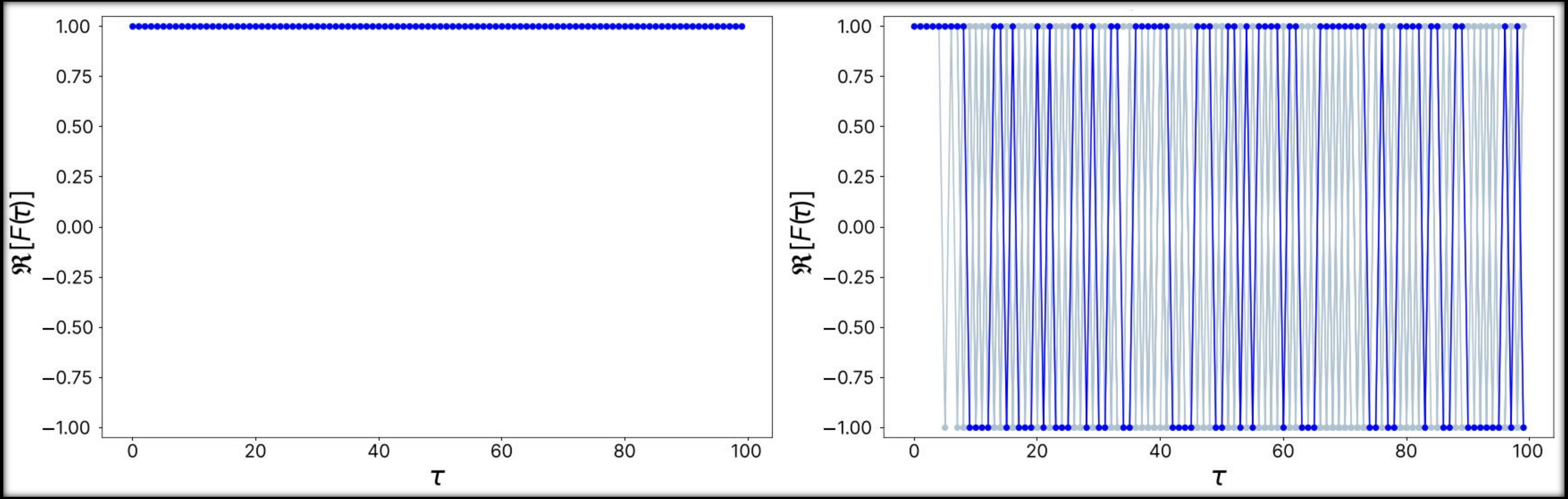

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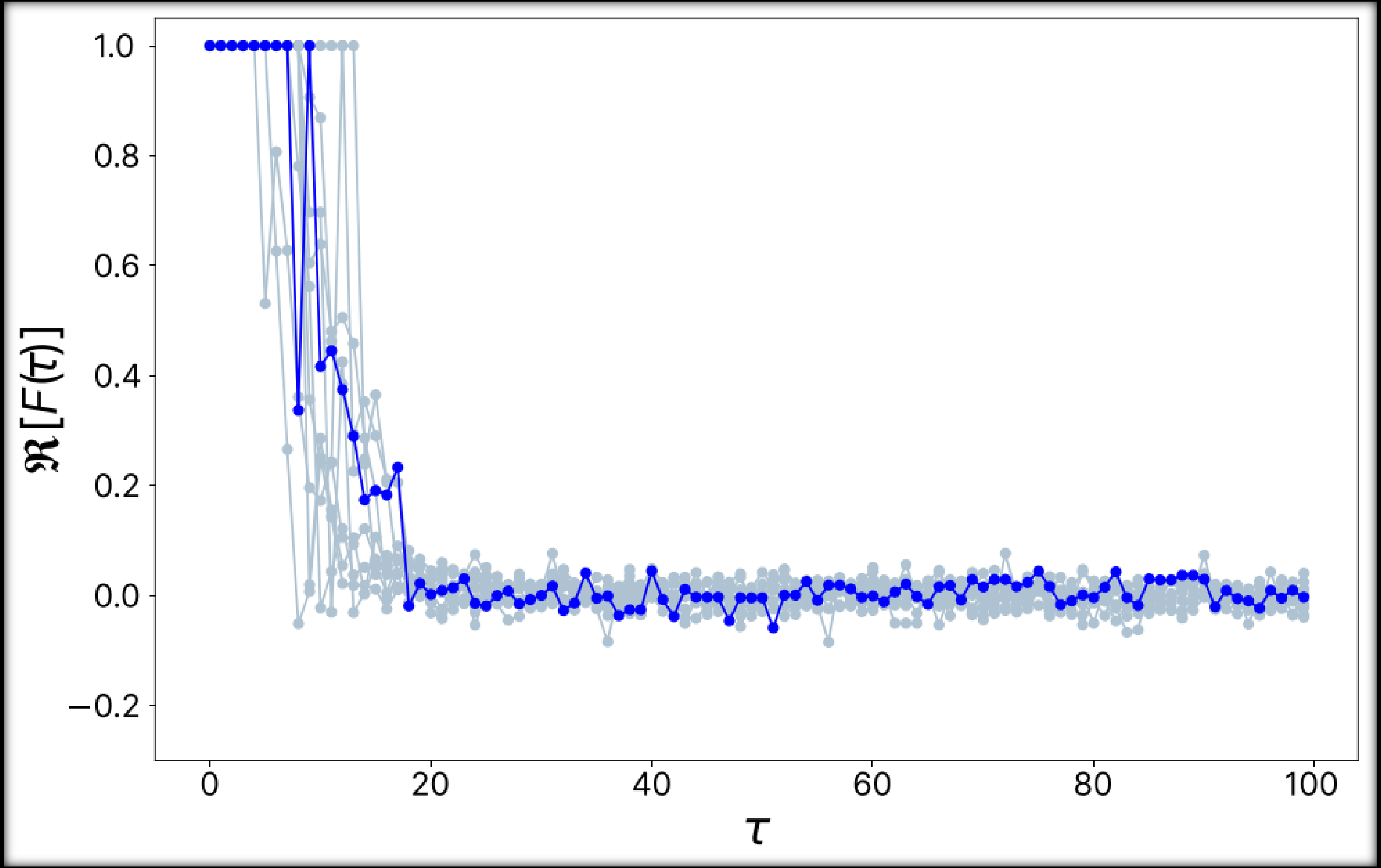
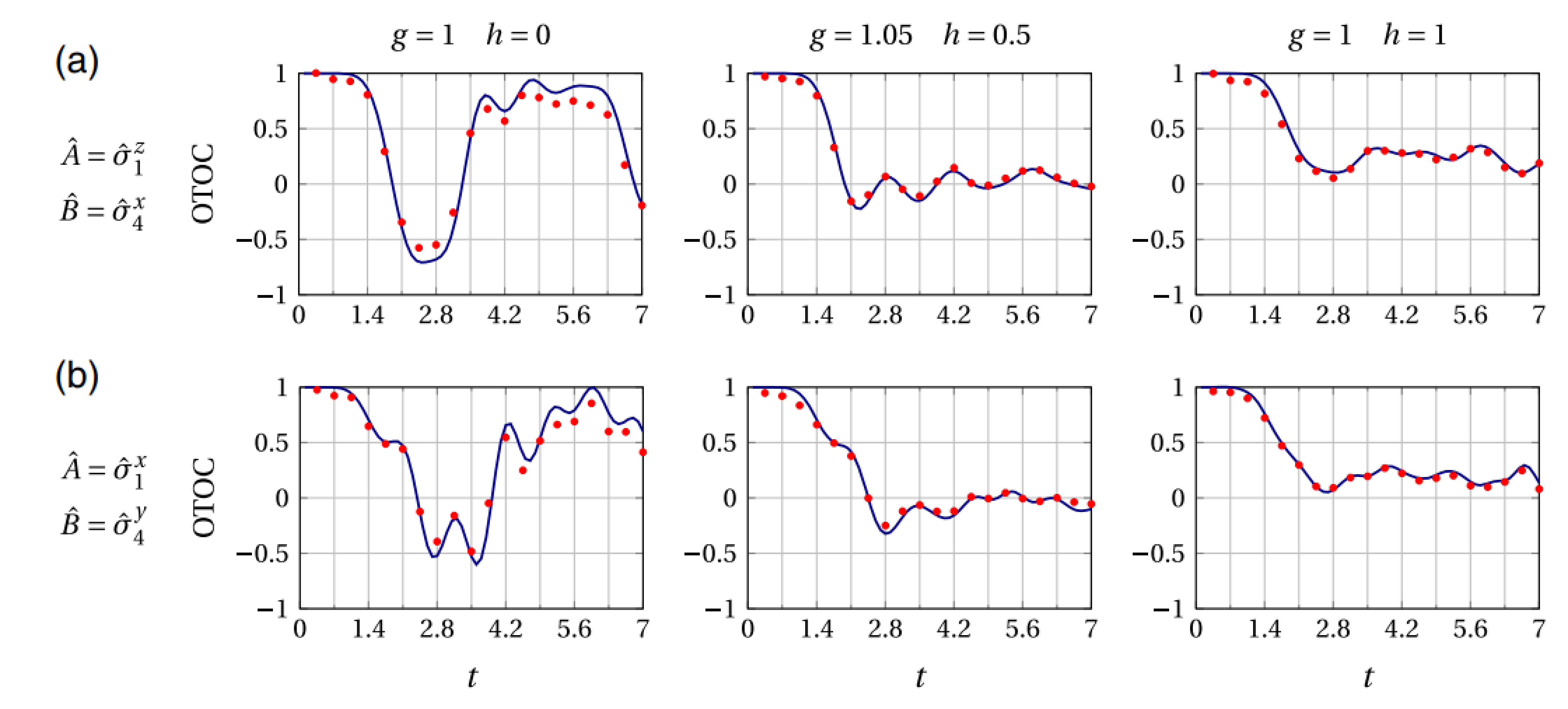


Jun Li *et al*, PRX 7, 031011 (2017)

$$F(t) := \mathrm{Tr}(W^\dagger(t) V^\dagger W(t) V \rho).$$



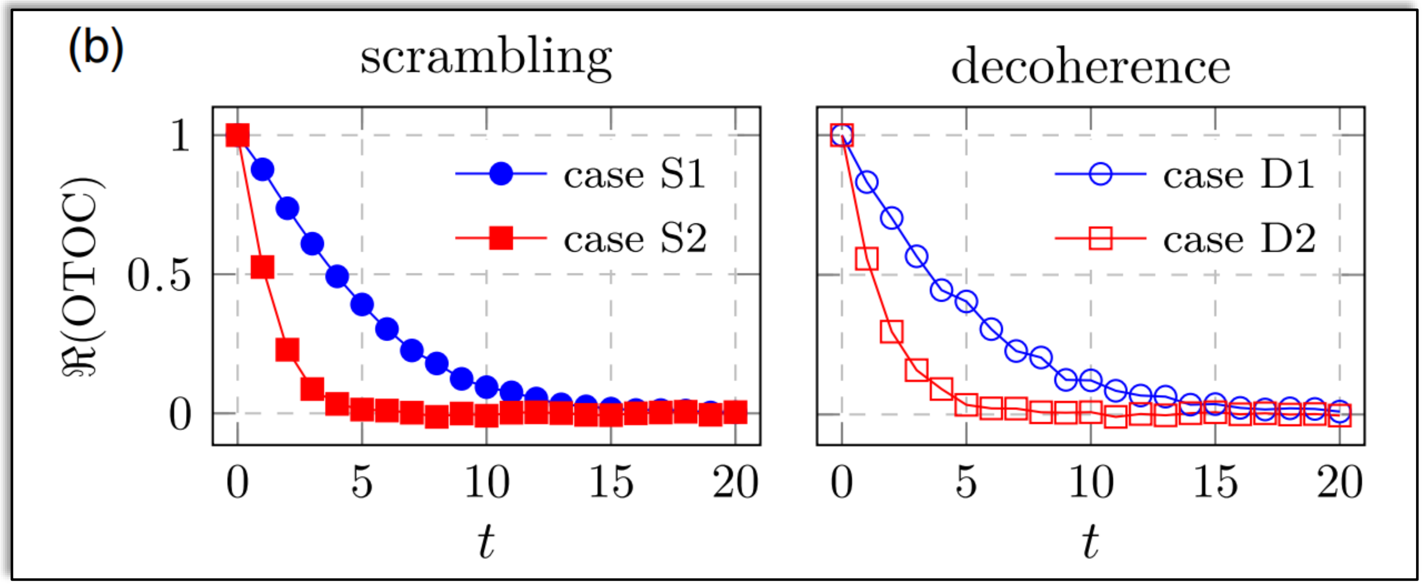
$$F(t) := \mathrm{Tr}(W^\dagger(t) V^\dagger W(t) V \rho).$$

$$F(t) := \sum_{(v_1,\lambda_{v_1}),(v_2,\lambda_{v_2}),(w_2,\lambda_{w_2}),(w_3,\lambda_{w_3})} v_1 w_2 v_2^* w_3^* \tilde{A}_\rho$$

$$\tilde{A}_\rho = \langle w_3, \lambda_{w_3} | U | v_2, \lambda_{v_2} \rangle \langle v_2, \lambda_{v_2} | U^\dagger | w_2, \lambda_{w_2} \rangle \langle w_2, \lambda_{w_2} | U | v_1, \lambda_{v_1} \rangle \langle v_1, \lambda_{v_1} | \rho U^\dagger | w_3, \lambda_{w_3} \rangle$$

ROBUSTNESS ISSUE

It is often hard to distinguish between scrambling and decoherence



PHYSICAL REVIEW LETTERS **129,** 050602 (2022)

Benchmarking Information Scrambling

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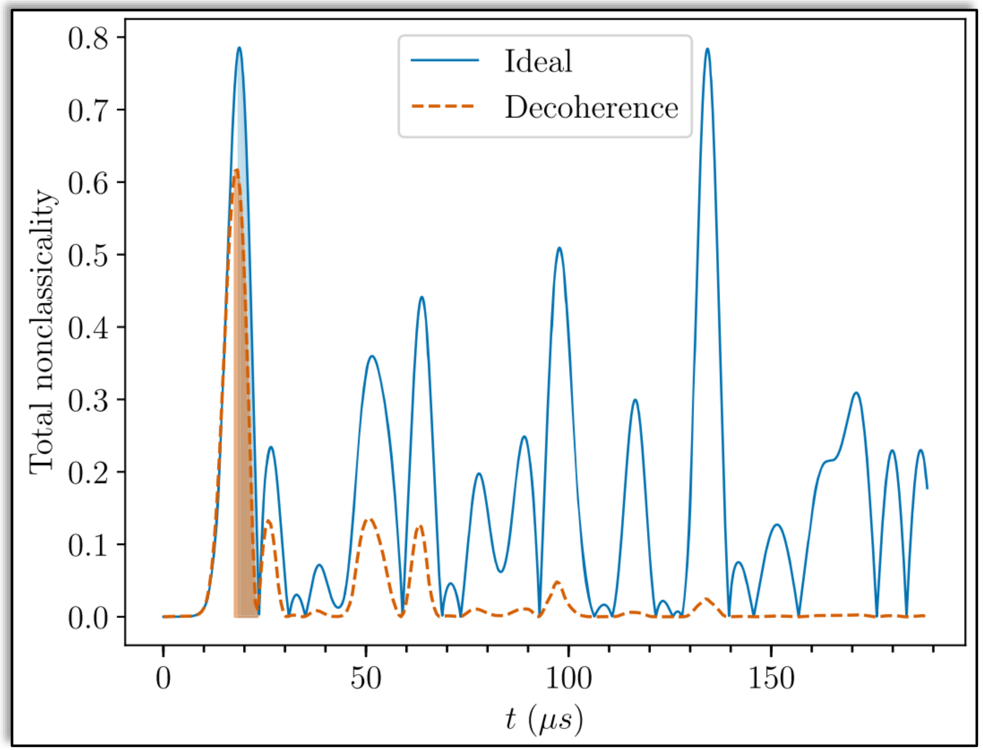
SCRAMBLING OF QUANTUM INFORMATION

$$\tilde{p}_t\left(v_1, w_2, v_2, w_3\right) := \mathrm{Tr}\left(\Pi^{W(t)}_{w_3}\Pi^{V}_{v_2}\Pi^{W(t)}_{w_2}\Pi^{V}_{v_1}\rho\right)$$

$$\tilde{N}(t) := \sum_{v_1, w_2, v_2, w_3} \left|\tilde{p}_t\left(v_1, w_2, v_2, w_3\right)\right| - 1$$



PHYSICAL REVIEW LETTERS **122**, 040404 (2019)

Out-of-Time-Ordered-Correlator Quasiprobabilities Robustly Witness Scrambling

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SCRAMBLING OF QUANTUM INFORMATION

$$\tilde{p}_t\left(v_1, w_2, v_2, w_3\right) := \mathrm{Tr}\left(\Pi^{W(t)}_{w_3}\Pi^{V}_{v_2}\Pi^{W(t)}_{w_2}\Pi^{V}_{v_1}\rho\right)$$

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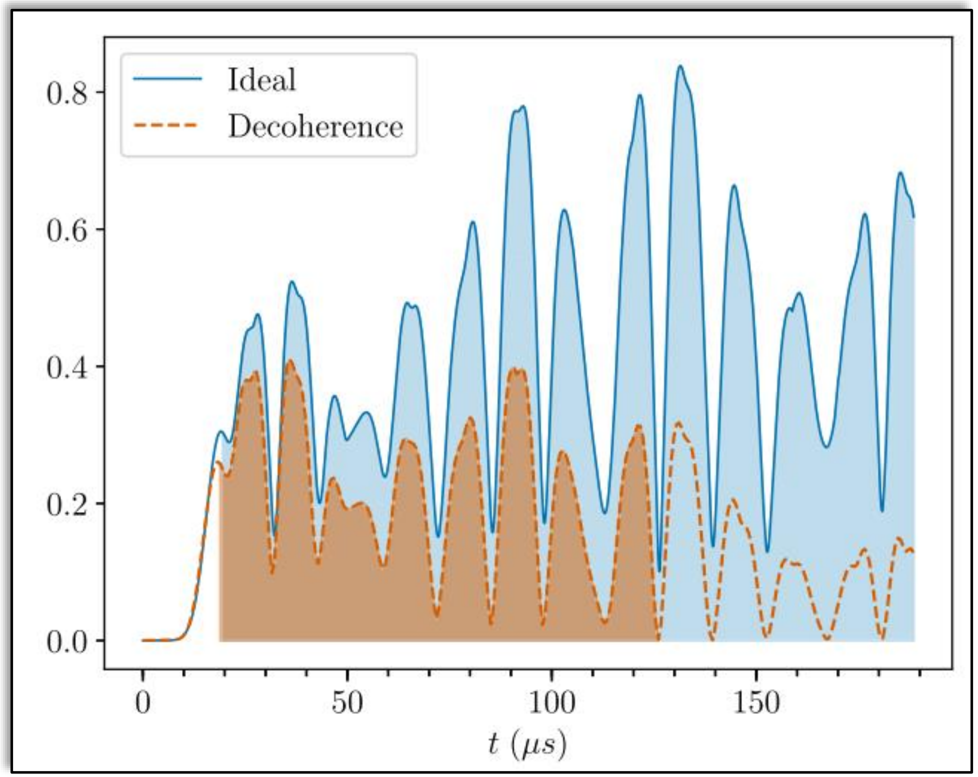

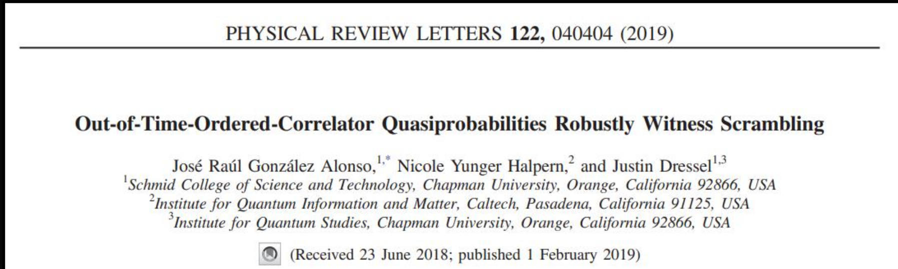
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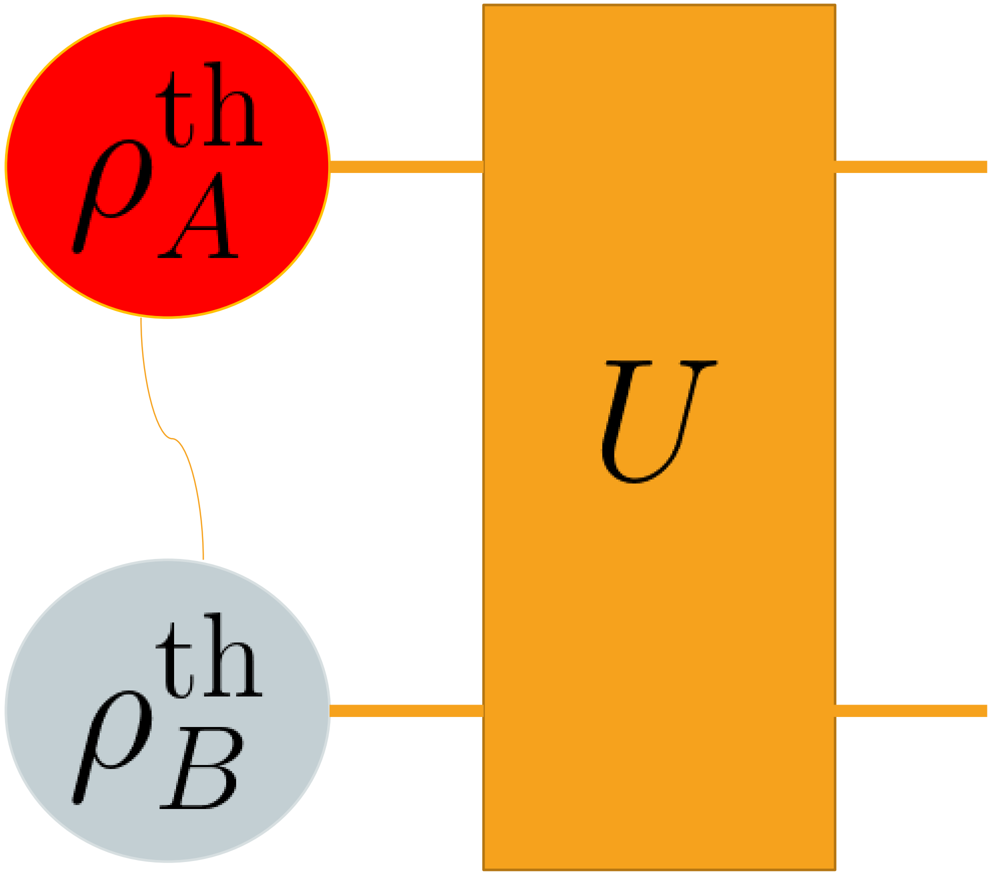
QUANTUM THERMODYNAMICS



QUANTUM THERMODYNAMICS

$$\langle \mathcal{Q}_A \rangle \equiv \langle U(t) H_A U^\dagger(t) - H_A \rangle$$
$$= \mathrm{Tr}\left\{ \rho (U(t) H_A U^\dagger(t) - H_A) \right\}$$

Average heat flow

$\langle \mathcal{Q}_A \rangle > 0$ The system A receives heat

$\langle \mathcal{Q}_A \rangle < 0$ The system B receives heat

QUANTUM THERMODYNAMICS

$$\mathcal{Q}_A = \sum_{i_A, i_B; f_A, f_B} p(i_A, i_B; f_A, f_B)(E_{i_A} - E_{f_A})$$

Not possible because of noncommutativity

The initial energy is not really well define (and measuring it leads to the TPM)

$$\mathcal{Q}_A = \sum_{i_A, i_B; f_A, f_B} q(i_A, i_B; f_A, f_B)(E_{i_A} - E_{f_A})$$

$$q(i_A, i_B; f_A, f_B) = \mathrm{Re}[\mu(\Pi^{\tau}_{f_A, f_B}, \Pi_{i_A, i_B} | \rho)]$$

QUANTUM THERMODYNAMICS

If we start with the energies E_{i_A}, E_{i_B}

$$\sum_{i_A, i_B} \mathrm{Re}[\mu(i_A, i_B; f_A, f_B|\rho)] = \mathrm{Tr}(\rho \Pi^{\tau}_{f_A, f_B})$$

$$E_{f_A}, E_{f_B}$$

$$\sum_{f_A, f_B} \mathrm{Re}[\mu(i_A, i_B; f_A, f_B|\rho)] = \mathrm{Tr}(\rho \Pi_{i_A, i_B})$$