A First Encounter with General Probabilistic Theories

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September 20, 2022

Outline I

Quantum Theory

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Motivations

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4. It teaches us relevant things about computation.
5. Good tools for theoretical investigations. Diagrams!

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2 The composition of two physical systems described by $\mathscr{H}_1$ and $\mathscr{H}_2$ is a new system described by $\mathscr{H}_1 \otimes \mathscr{H}_2$.
3 When no measurement is performed the change in states ψ is given by $|\psi(t)\rangle = U(t)|\psi\rangle$ for some strongly continuous one-parameter (semi)group $t \mapsto U(t)$.

States as density operators: Given some ensamble of states $\{(|\psi_i\rangle, p_i)\}$,

$$\rho := \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{1}$$

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6 When a measurement $M = \{E\}$ is performed on ρ and outcome E is obtained, there is a discontinuous change towards a new state $\rho' = \frac{\sqrt{E}\rho\sqrt{E}}{\mathrm{Tr}(\rho E)}$.

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Physics: Aren't they too mathematical? Wouldn't be nicer to have something like in special relativity?

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3. $\Omega := \{\rho \in \mathbb{H}_{d_H}(\mathbb{C})^+ : \mathrm{Tr}(\rho) = 1\}$. $\dim(\Omega) = d_H^2 - 1$.

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Closed: $\sigma_n \to \sigma$, $\langle v, \sigma v\rangle = \langle v, \lim_{n\to\infty} \sigma_n v\rangle = \lim_{n\to\infty}\langle v, \sigma_n v\rangle$.

Generating: Every element $\rho \in \mathbb{H}_{d_H}(\mathbb{C})$ is such that $\rho = \rho^+ - \rho^-$ with $\rho^+, \rho^- \in \mathbb{H}_{d_H}(\mathbb{C})^+$.

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Any matrix $\sigma \in \mathbb{H}_{d_H}(\mathbb{C})$ can then be embedded in vectors in $\mathbb{R}^{d_H^2}$.

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There exists a unique effect $u := \mathrm{Tr}(\mathbb{1}(\cdot))$that says with certainty if any normalized state is present since $u(\sigma) = \mathrm{Tr}(\sigma)$and defines the probability of preparing any state.
Measurements are sets of effects $M = \{e\}$ such that $\sum_{e \in M} e = u \iff \sum_{E \in M} E = \mathbb{1}$.

Transformations are reversible; for any unitary operator describing the evolution U there exists the inverse operator $U^{-1} = U^{\dagger}$.

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Definition

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4. $e_{AB}(\omega_A \circ (\cdot)), e_{AB}((\cdot) \circ \phi_B) \in E_{AB}, \forall \phi_B, \omega_A$ normalized. Similar construction for states.

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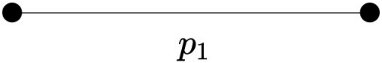
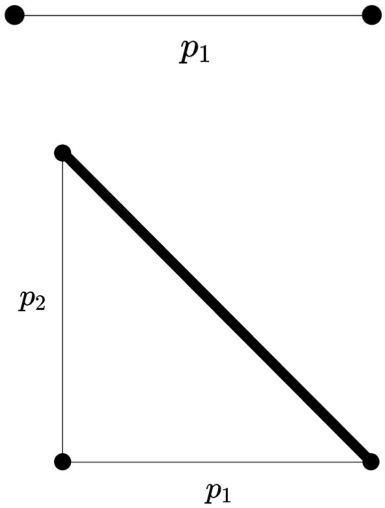
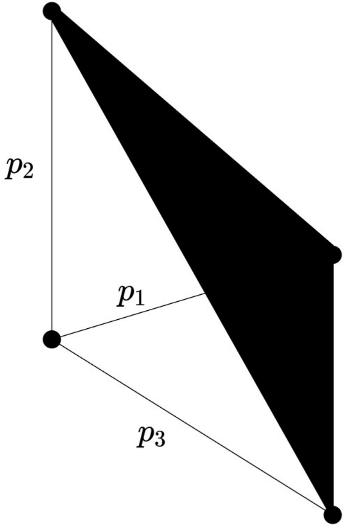

p_1

Simplex




p_2
p_1
p_3

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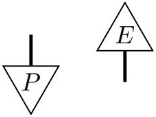
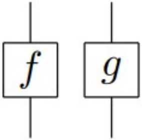
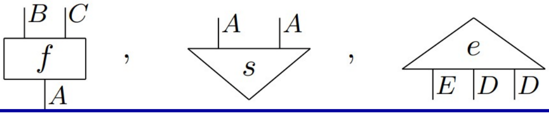
That is noncontextuality in GPT sense!

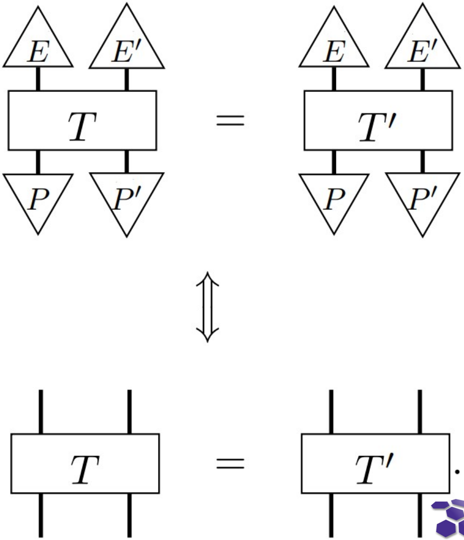
Nice feature of GPTs: Diagrammatic language!



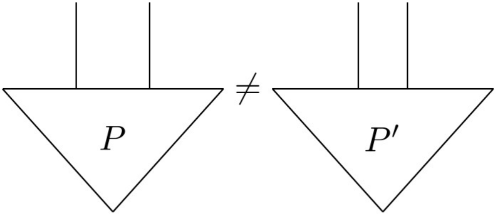
$f \otimes g \quad :=$ 



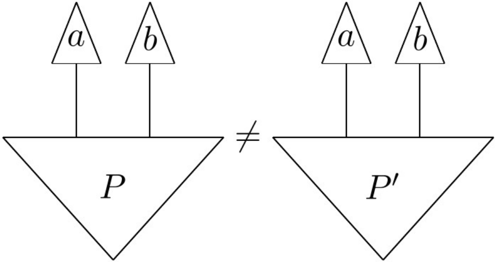
$\forall E, E', P, P'$



Local Discriminability



$\exists a, b$



Less Algebra and More Geometry:

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Therefore the real bits $d = 2$ are embeddable in $2(2+1)/2 = 3$ real vector space.
Here is a (choice) basis of the space $\frac{1}{2}\{\mathbb{1}, X, Z\}$.

$$\sigma = a_0\frac{\mathbb{1}}{2} + a_1\frac{1}{2}X + a_2\frac{1}{2}Z \sim \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

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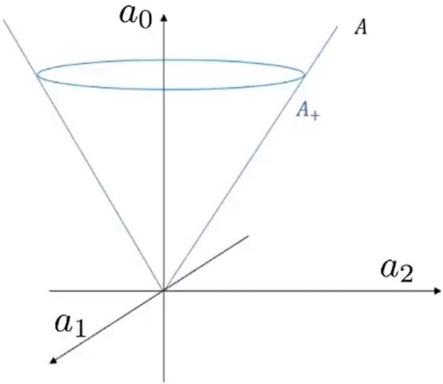

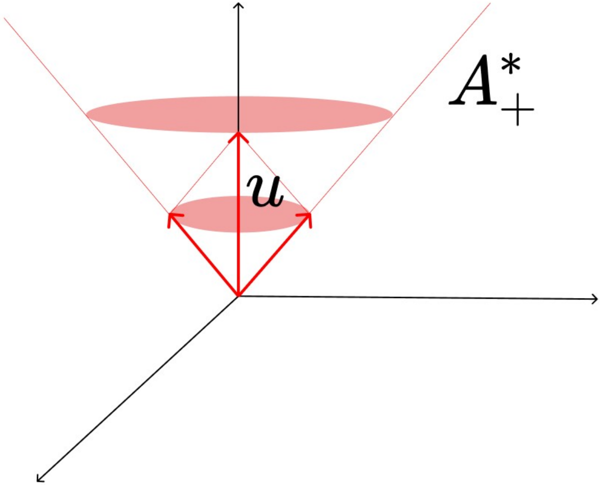
A^*_+
u

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Real QT: compositions

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Dim. of two q-rebits: $\dim(\mathbb{H}_2 \otimes \mathbb{H}_2) = \dim(\mathbb{H}_2)\dim(\mathbb{H}_2) = 9.$

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Conclusion: Real QT is weird

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Some separable states in QT become entangled in $\mathbb{R}QT$!

Real QT: can be falsified

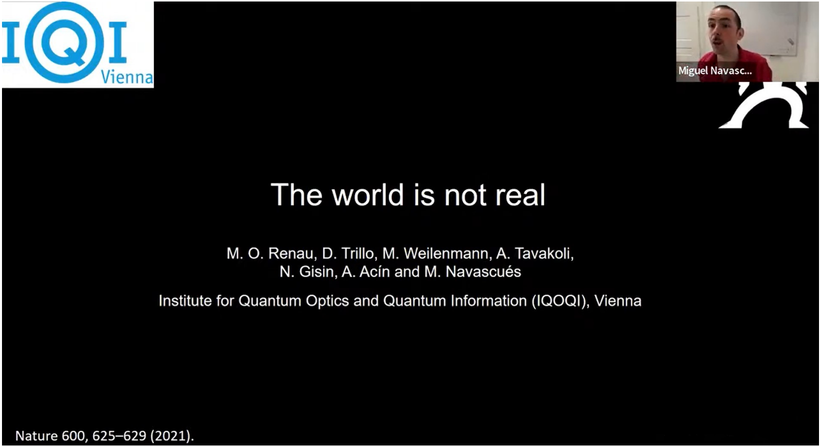
Wed 12th May 2021 at 14.00

Angelos Bampounis (INL / U Minho)

Journal club: Quantum physics needs complex numbers

Abstract Slides

Real QT: (almost) disproved



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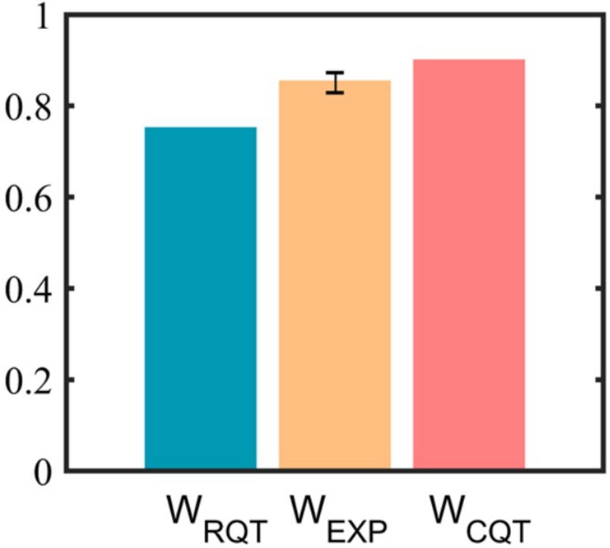


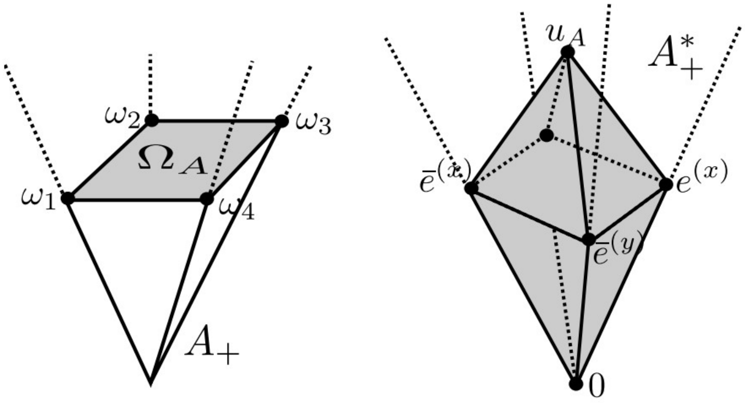
1. Local Tomography: Local operations suffice to learn about states.
2. No-restriction: Every mathematically possible effect is also physically possible. $E_A = A^*_+$
3. Strong self-duality: There exists an inner-product making effect and state space identical.

Wed 9th December 2020 at 14.00

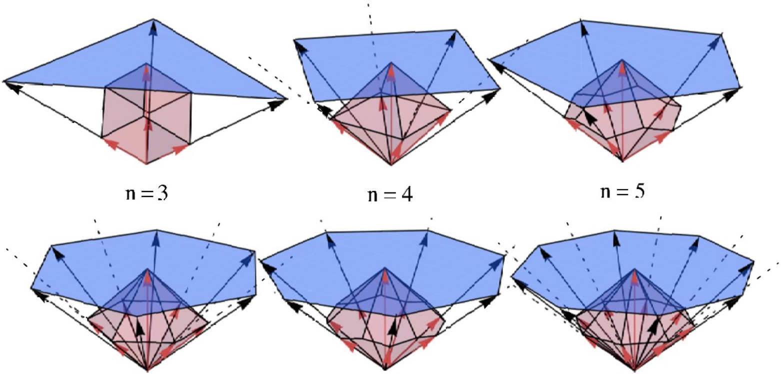
Michael Oliveira (U Minho)

From EPR to Bell, with a description on the correlation polytopes

Abstract **Slides**


ω_2
ω_3
Ω_A
ω_1
ω_4
A_+
u_A
A_+^*
$\bar{e}^{(x)}$
$e^{(x)}$
$\bar{e}^{(y)}$
0

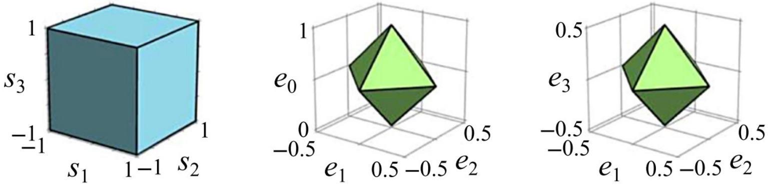
Non strongly self-dual theory: polygonal GPTs

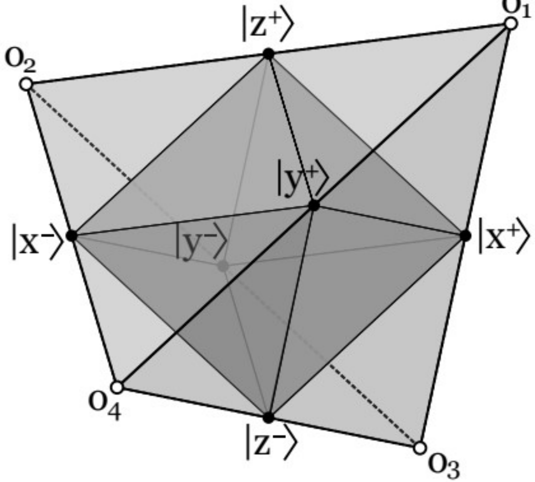


n = 3 n = 4 n = 5

n = 6 n = 7 n = 8

Non strongly self-dual theory: 'boxworld'




|z⁺⟩
O1
O2
|y⁺⟩
|x⁻⟩
|y⁻⟩
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O4
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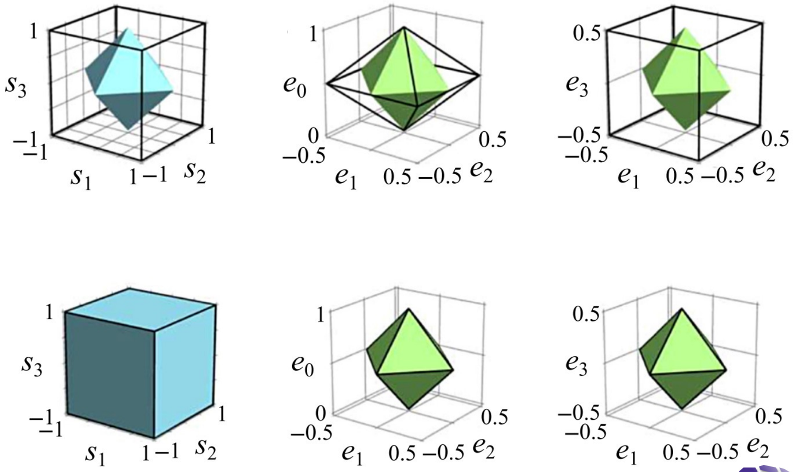
O3


1
s_3
−1
−1
s_1
1 −1
s_2
1
1
e_0
0
−0.5
e_1
0.5 −0.5
e_2
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e_3
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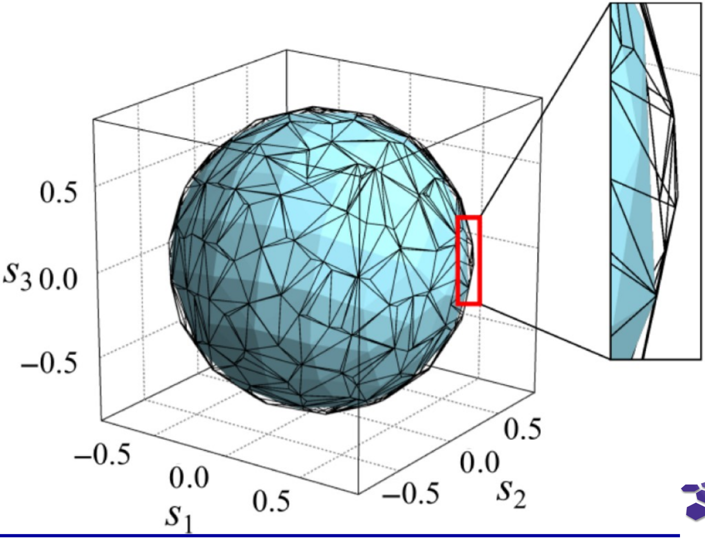
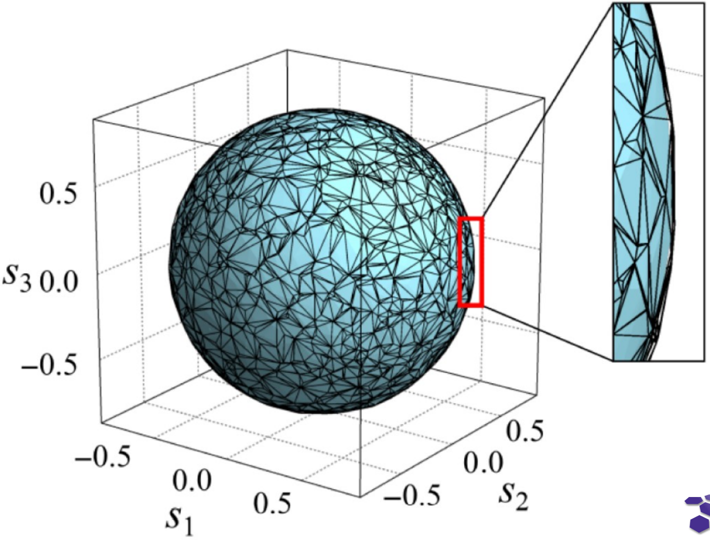
GPTs are experimentally accessible





0.5
s_3 0.0
–0.5
–0.5
0.0
0.5
s_1
–0.5
0.0
0.5
s_2

Causality Principle:

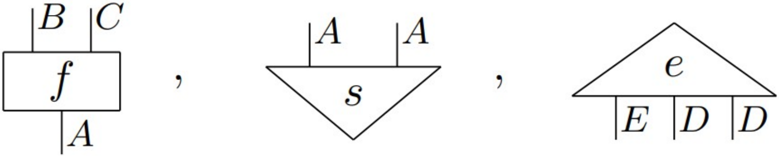
Causality Principle: The probability of preparing states is independent of future measurements.

Causality Principle: The probability of preparing states is independent of future measurements.

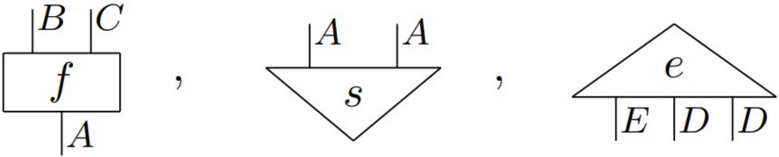
Theorem (Pavia group)

A GPT is causal if, and only if there exists one and only one deterministic effect u.

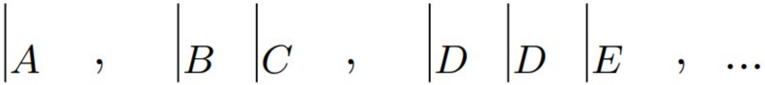
Causal Tomographically Local GPTs have a diagrammatic language.



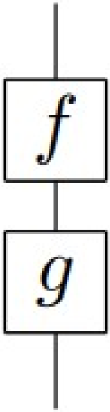
Causal Tomographically Local GPTs have a diagrammatic language.



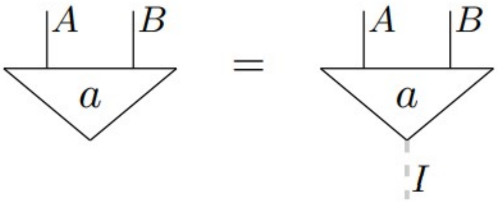
Mathematically, the structure is of a strict symmetric monoidal category.

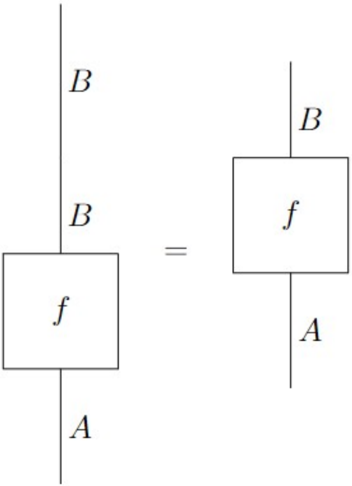

A ,
B C ,
D D E ,
...

$$f \circ g \quad :=$$

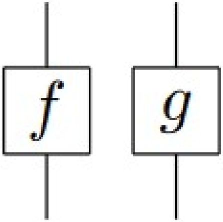


$$\mathbb{1}_A \quad := $$


A
B
a
=
A
B
a
I

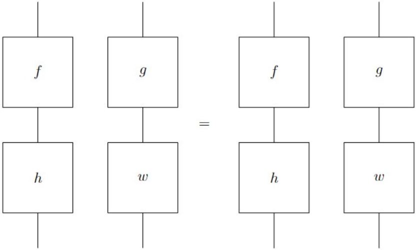

B
B
f
A
=
B
f
A

$$f \otimes g \quad :=$$



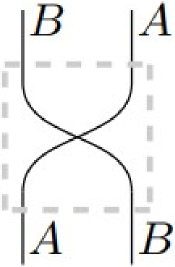
Diagrams

$$(f \otimes_1 g) \circ (h \otimes_1 w) =$$

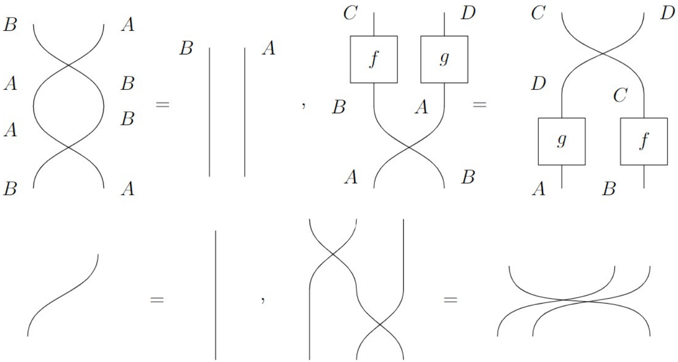


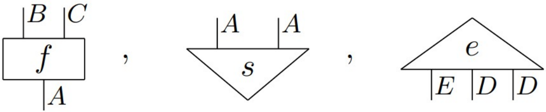
$$= (f \circ h) \otimes_1 (g \circ w)$$

$$\mathrm{SWAP}_{AB} \quad :=$$



Diagrams




B
C
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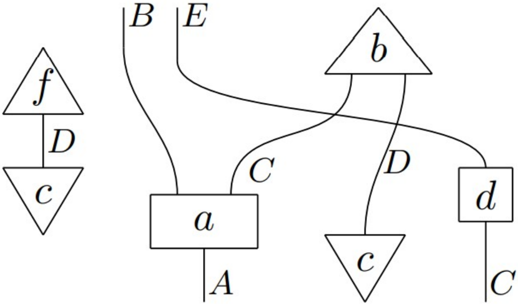
D


B
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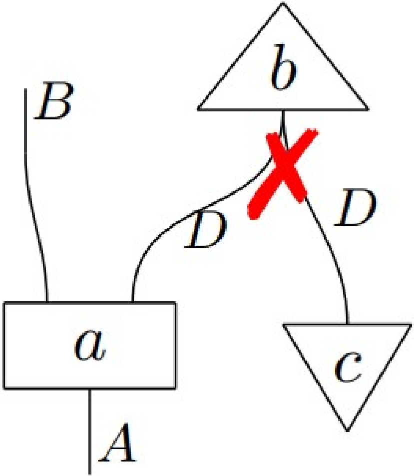
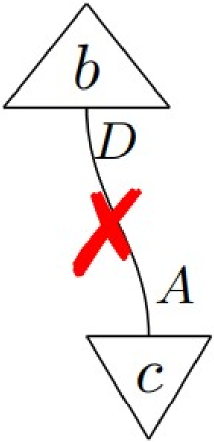
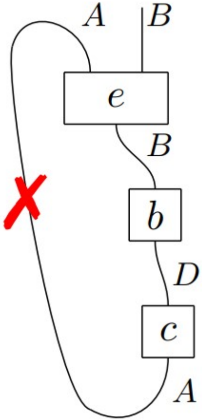
a
A
c
C


B
b
D
D
a
c
A

Forbidden



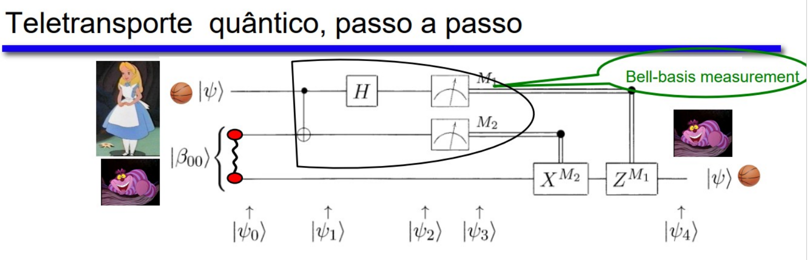

A
B
e
B
b
D
c
A

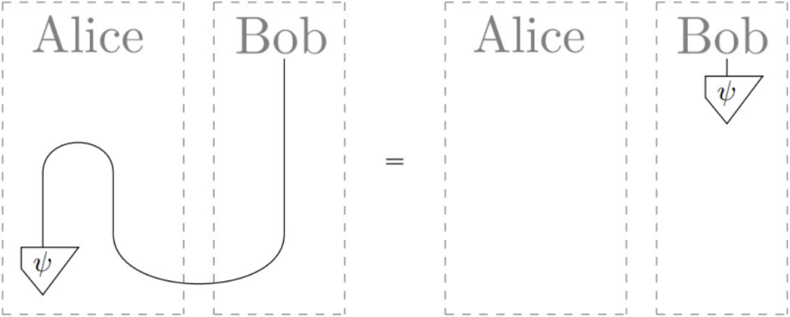
Thu 23rd July 2020 at 11.15

Ernesto F. Galvão (INL / UFF)

Quantum teleportation

Abstract Slides


Teletransporte quântico, passo a passo
Bell-basis measurement
$|\psi\rangle$
$|\beta_{00}\rangle$
H
M_1
M_2
X^{M_2}
Z^{M_1}
$|\psi\rangle$
$|\psi_0\rangle$
$|\psi_1\rangle$
$|\psi_2\rangle$
$|\psi_3\rangle$
$|\psi_4\rangle$


Alice
Bob
ψ
=
Alice
Bob
ψ

Appendix

Theorem

Let $(H, \|\cdot\|)$ be a Hilbert space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For any bounded $\phi : H \to \mathbb{K}$ ($\phi \in H^*$) there is a unique vector $v_\phi \in H$ such that $\phi = \phi_{v_\phi}$ where $\phi_v := \langle v, \cdot \rangle$. For any bounded $\phi : H \to \mathbb{K}$, $\|v_\phi\| = \|\phi\|_\infty$. For all bounded $\phi, \phi' \in H^*$ and $\alpha \in \mathbb{K}$ it is true that $v_{\phi+\phi'} = v_\phi + v_{\phi'}$ and $v_{\alpha\phi} = \overline{\alpha} v_\phi$. In words, the mapping $\phi \mapsto v_\phi$ is antilinear.

The proof goes as follows; for uniqueness, suppose that there are two vectors v_ϕ^1, v_ϕ^2 such that $\phi = \phi_{v_\phi^1} = \phi_{v_\phi^2}$. Then $\phi_{v_\phi^1} = \langle v_\phi^1, \cdot \rangle = \langle v_\phi^2, \cdot \rangle$ which happens iff $\langle v_\phi^1, v \rangle = \langle v_\phi^2, v \rangle$ for all $v \in H$. Let $v = v_\phi^1 - v_\phi^2$ we get $\langle v_\phi^1, v_\phi^1 - v_\phi^2 \rangle - \langle v_\phi^2, v_\phi^1 - v_\phi^2 \rangle = 0$ which implies that $v_\phi^1 - v_\phi^2 = 0$, hence uniqueness.

From the sesquilinearity of the inner product we have that $(\phi + \phi')(v) = \langle v_\phi + v_{\phi'}, v\rangle$ for all v therefore $v_{\phi+\phi'} = v_\phi + v_{\phi'}$ and $\phi(\alpha v) = \langle v_\phi, \alpha v\rangle = \overline{\alpha}\langle v_\phi, v\rangle$ hence $v_{\alpha\phi} = \overline{\alpha} v_\phi$.

Lastly, we show the relationship between the two norms. By deinition we have

$\|\phi\|_\infty = \sup_{v\in H, \|v\|=1} |\phi(v)| = \sup_{v\in H, \|v\|=1} |\langle v_\phi, v\rangle| \leq \|v_\phi\|$

where in the last equality we use the Cauchy-Schwarz inequality. Therefore $\|\phi\|_\infty \leq \|v_\phi\|$. But we also have that,

$$\|v_\phi\| = \frac{|\langle v_\phi, v_\phi\rangle|}{\|v_\phi\|} = \left\langle v_\phi, \frac{v_\phi}{\|v_\phi\|}\right\rangle \leq \sup_{v\in H, \|v\|=1} |\langle v_\phi, v\rangle| = \|\phi\|_\infty$$

This result holds in general for any C^*-algebra $\mathfrak{U}$ and the positive elements thereof $\mathfrak{U}^+$. Therefore, one way to prove would be to use the theorem that for any polynomial p, the spectrum of operators A of the algebra satisfy that $\mathrm{sp}(p(A)) = p(\mathrm{sp}(A))$. We only need to choose the correct polynomial $p(x) := tx$ and the result follows for any real t.

Another way of showing this without using known results is as follows; considering only $t \geq 0$ and $t \in \mathbb{R}$. Let $t \neq 0$ at first. $z \in t\mathrm{sp}(A) \iff \exists \lambda_z \in \mathbb{C}$ with $z = t\lambda_z$ such that $A - \mathbb{1}\lambda_z$ does not have inverse. This is true if, and only if $tA - \mathbb{1}t\lambda_z = tA - \mathbb{1}z$ does not has inverse, hence $t\mathrm{sp}(A) = \mathrm{sp}(tA)$ for $t \neq 0$. The case $t = 0$ follows from $\mathrm{sp}(tA) = \{0\} = t\mathrm{sp}(A)$.

We can state this for the more generic case of C^*-algebras, which have sets of matrices/hermitian operators as examples.

Lemma (generating)

Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{A}^{\mathbb{R}}$ the self-adjoint section. For any $A \in \mathscr{A}^{\mathbb{R}}$ there is A^+, A^- positive such that $A = A^+ - A^-$ and $\|A^+\|, \|A^-\| \leq \|A\|$.

Proof.

The case $A = 0$ is true trivially. Let $A \neq 0$ then $A = \frac{\|A\|}{4}\left(\frac{A}{\|A\|} + \mathbb{1}\right)^2 - \frac{\|A\|}{4}\left(\frac{A}{\|A\|} - \mathbb{1}\right)^2 = p_1(A) - p_2(A)$. Where p_1, p_2 are two polynomials positive for every A, defining new self-adjoint operators $p_1(A), p_2(A)$ that have smaller norm and that must have positive spectrum, therefore they are positive.

$$\begin{pmatrix} a_0 + a_2 - \alpha & a_1 \\ a_1 & a_0 - a_2 - \alpha \end{pmatrix} \implies (a_0+a_2-\alpha)(a_0-a_2-\alpha)-a_1^2 = 0$$

$$a_0^2 - a_0a_2 - \alpha a_0 + a_2a_0 - a_2^2 - a_2\alpha - a_0\alpha + a_2\alpha + \alpha^2 = a_1^2$$

$$\alpha(\alpha - 2a_0) = a_1^2 + a_2^2 - a_0^2 \implies a_1^2 + a_2^2 - a_0^2 \leq 0$$

The last implication is because for any positive real matrix σ we have $\mathrm{Tr}(\sigma) = a_0$, since the trace is the sum of eigenvalues, any particular eigenvalue must be smaller then a_0, hence $\alpha - 2a_0$ is always negative. The conclusion is then that for α to be nonnegative the rhs of the equation must be nonnegative.

Proof of Pavia Lemma

Fist we show that if the probability of preparing a state is independent then the deterministic effect is unique. The probability of some state ω being prepared is $p(\omega|M) = \sum_{e \in M} e(\omega)$ in a given measurement M. Since the effects normalizing u_1, u_2 satisfy $u_1 = \sum_{e \in M} e$ and $u_2 = \sum_{e' \in M'} e'$ for two measurement procedures we have that $u_1(\omega) = p(\omega|M) = p(\omega|M') = u_2(\omega)$, because we assumed that the probability of being prepared was independent of future measurements to be applied, hence $p(\omega|M) = p(\omega) = p(\omega|M')$.

Second we show that if the deterministic effect is unique we get that the probability of preparing the state must be independent of future measurements. Now this direction is basically trivial from the above consideration. Whenever the deterministic effect is unique, given any two measurement procedures $M = \{e\}, M' = \{e'\}$ we have that $u = \sum_{e \in M} e = \sum_{e' \in M'} e'$. Therefore the probability of preparing the state ω will be equal in both cases since $p(\omega|M) = \sum_{e \in M} e(\omega) = u(\omega) = \sum_{e' \in M'} e'(\omega) = p(\omega|M')$.

Definition (Category)

Let C_0 and C_1 be two classes. The class C_0 is named the class of objects in the category and the class C_1 is named the class of arrows (or morphisms) in the category. There are two functions $\mathrm{dom} : \mathsf{C}_1 \to \mathsf{C}_0$ and $\mathrm{cod} : \mathsf{C}_1 \to \mathsf{C}_0$ such that if $f \in \mathsf{C}_1$ we have that $\mathrm{dom}(f), \mathrm{cod}(f) \in \mathsf{C}_0$ and we write $\mathrm{dom}(f) \xrightarrow{f} \mathrm{cod}(f)$. Defining $H := \{(g, f) \in \mathsf{C}_1 \times \mathsf{C}_1 \mid \mathrm{dom}(g) = \mathrm{cod}(f)\}$ there is a function $\circ : H \to \mathsf{C}_1$ and we write $g \circ f = \circ(g, f)$. There is a function $1 : \mathsf{C}_0 \to \mathsf{C}_1$. Then a category is a 6-tuple $\mathsf{C} = (\mathsf{C}_0, \mathsf{C}_1, \mathrm{dom}, \mathrm{cod}, \circ, 1)$ such that these structures satisfy the following demands:

(a) If g and f are arrows, then $g \circ f$ is also an arrow. Therefore the composition operation $\circ$ is closed within the category. This means that, as in the definition of the 6-tuple we have that $\mathrm{cod}(g) = \mathrm{dom}(f)$ implies that $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$.

(b) The map $\circ$ is associative.

(c) We have that $\mathrm{dom}(1(A)) = \mathrm{cod}(1(A))$ for every $A \in \mathsf{C}_0$ and $\forall f, g \in \mathsf{C}_1$ with $A \xrightarrow{f} \mathrm{cod}(f)$ and $\mathrm{dom}(g) \xrightarrow{g} A$ for any A it is true that 1_A does not changes the arrows, $f \circ 1_A = f$ and $1_A \circ g = g$.

(d) **Lemma:** The map 1_A is unique for every $A \in \mathsf{C}_0$.

Definition

A strict monoidal category C is a category such that there are two maps $\otimes_0 : \mathsf{C}_0 \times \mathsf{C}_0 \to \mathsf{C}_0$ and $\otimes_1 : \mathsf{C}_1 \times \mathsf{C}_1 \to \mathsf{C}_1$, and a unit object $I \in \mathsf{C}_0$ such that the following holds,

(a) $\forall A, B, C \in \mathsf{C}_0$ we have that $(A \otimes_0 B) \otimes_0 C = A \otimes_0 (B \otimes_0 C)$.

(b) $\forall A \in \mathsf{C}_0$ we have that $A \otimes_0 I = I \otimes_0 A = A$.

(c) There is an interaction rule between $\circ$ and $\otimes_1$. They satisfy the so-called interchange law, so $\forall f_1, f_2, g_1, g_2 \in \mathsf{C}_1$ we have that $(g_1 \otimes_1 g_2) \circ (f_1 \otimes_1 f_2) = (g_1 \circ f_1) \otimes_1 (g_2 \circ f_2)$

Theorem: Every monoidal category is equivalent to a strict monoidal category. In particular, every monoidal category of sets has a strict monoidal structure, so there is no need to change the category structure.

A symmetric monoidal category (SMC) is a monoidal category with a swap morphism $\sigma_{A,B} : A \otimes B \to B \otimes A$ that is defined for any object satisfying:

(a) Swapping twice is the identity, $\sigma_{A,B} \circ \sigma_{B,A} = 1_{A \otimes B}$.

(b) Parallel composition followed by swapping is equal to swapping in the respectively type-objects and then composing in parallel with reversed order, $\sigma_{A,B} \circ (g \otimes f) = (f \otimes g) \circ \sigma_{C,D}$.

(c) Swapping the identity object with another object is the identity arrow, $\sigma_{A,I} = 1_A$.

(d) This is somewhat a coherence relation between sapping and the identity arrows, $(1_B \otimes \sigma_{A,C}) \circ (\sigma_{A,B} \otimes 1_C) = \sigma_{A,B \otimes C}$.