Efficient gate teleportation in higher dimensions

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November 23, 2022

1. Introduction

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Magic state distillation is more efficient in the higher-dimensional *qudit* setting.

Natural questions

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Question 2: Which Clifford hierarchy gates can be *efficiently* implemented?

2. Background & overview

Stabiliser formalism: Pauli gates

The Pauli gates X, Z are defined for any prime dimension d:

$$Z\left|z\right\rangle = \omega^{z}\left|z\right\rangle \qquad\qquad X\left|z\right\rangle = \left|z+1\right\rangle.$$

where $\omega = e^{2\pi i/d}$ and $z \in \mathbb{Z}_d$.

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where $\omega = e^{2\pi i/d}$ and $z \in \mathbb{Z}_d$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & \omega^{d-1} \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

These matrices satisfy $ZX = \omega XZ$.

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Gates U in the Clifford hierarchy as they can be fault-tolerantly implemented via gate teleportation (Gottesman-Chuang, 1999) to achieve universality using a magic state.

Semi-Clifford gates

Semi-Clifford gates are special Clifford hierarchy gates that can be fault-tolerantly implemented via *one-dit* gate teleportation using a magic state of half the size (Zhou-Leung-Chuang, 2000).

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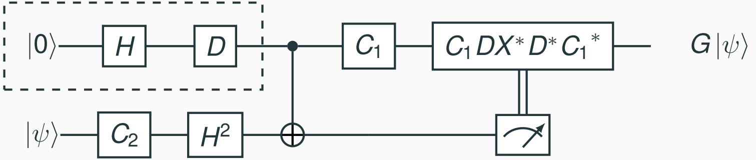
The *semi-Clifford gates* are ‘diagonal up to Clifford’:

$$\mathcal{SC}_k = \mathcal{C}_2 \mathcal{D}_k \mathcal{C}_2.$$

One-dit teleportation of $\mathcal{SC}_3$ gates

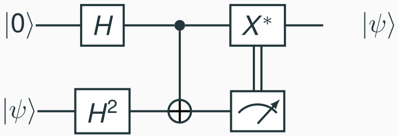
Suppose $G \in \mathcal{SC}_3$, i.e. $G = C_1 D C_2$ for $C_1, C_2 \in \mathcal{C}_2$, $D \in \mathcal{D}_3$

MAGIC STATE: $|M\rangle = D\,|+\rangle$



All gates and measurements are stabiliser operations. Preparing the magic state can be done fault-tolerantly.

One-dit teleportation

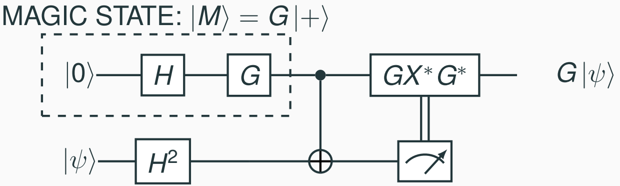


$H^2 |z\rangle = |-z\rangle$

$CX |z_1\rangle |z_2\rangle = |z_1\rangle |z_1 + z_2\rangle$

Arithmetic operations are modulo p.

One-dit teleportation of $\mathcal{C}_3$



$GX^*G^* \in \mathcal{C}_2$

This works for any $G \in \mathcal{C}_3$ that commutes with the CX gate, e.g. diagonal G.

A slight modification works for $G \in \mathcal{SC}_3$.

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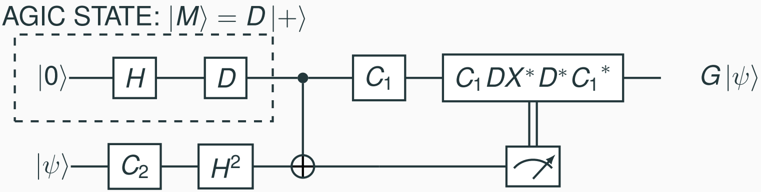

Apply $C_1 D$ at the end of the circuit and teleport $C_2\,|\psi\rangle$.

Straightforwardly parallelised for *n*-qudit gates.

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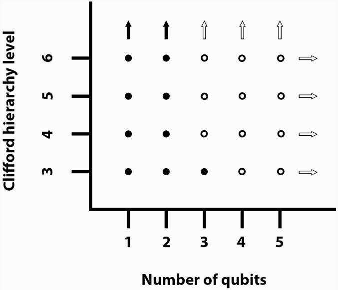
For $n > 2, k > 3$ or $n > 3$, *not all* k-th level gates are semi-Clifford. (Beigi-Shor and Gottesman-Mochon, 2009)



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$$D[{\omega_m}^{\phi}] = \sum_{\hat{z} \in \mathbb{Z}_d^n} {\omega_m}^{\phi(\hat{z})} \left|\hat{z}\right\rangle \left\langle\hat{z}\right|$$

where ω_m is the primitive d^m-th root of unity and $\phi : \mathbb{Z}_d^n \to \mathbb{Z}_{d^m}$ is a polynomial.

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- A generalisation of the efficient gate teleportation protocol for qubit semi-Clifford gates to the qudit case.
- An algorithm for recognising and diagonalising semi-Clifford gates.
- A proof that all third-level gates of one qudit or two qutrits admit efficient gate teleportation.

Methods

Canonical commutation relations

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Roughly: any two representations of the CCRs are unitarily equivalent.

Discrete Stone-von Neumann theorem (one qudit)

Definition
*An ordered pair of unitaries (U, V) is a **conjugate pair** if*

1. $U^d = \mathbb{I}$ *and* $V^d = \mathbb{I}$*,*
2. $UV = \omega VU$*.*

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There is a unitary G such that $U = GZG^*$ and $V = GXG^*$ given by:

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where $|u_0\rangle = G\,|0\rangle$ is an eigenvector of U with eigenvalue 1.

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G is a $(k+1)$-th level gate $\iff$ U and V generate k-th level gates.

An elementary proof: I

Lemma

Suppose $U, V \in GL_d(\mathbb{C})$ *satisfy* $UV = \omega VU$. *Then the matrices* $U^a V^b$ *are traceless for* $(a, b) \in \mathbb{Z}_d^2 \setminus (0, 0)$.

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Proof.
Suppose first that $b \neq 0$.

$$\mathrm{Tr}(U^a V^b) = \mathrm{Tr}(U^{a-1} U V^b) = \omega^b \mathrm{Tr}(U^{a-1} V^b U) = \omega^b \mathrm{Tr}(U^a V^b)$$

Since, $\omega^b \neq 1$, the above expression vanishes. It similarly vanishes if $a \neq 0$.

$\square$

An elementary proof: II

Lemma

Suppose $U, V \in M_d(\mathbb{C})$ *form a conjugate pair. Then the matrices* $\{U^i V^j \mid i, j \in \mathbb{Z}_d\}$ *are orthogonal in* $M_d(\mathbb{C})$ *with the Hilbert-Schmidt inner product* $\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}(A^* B)$ *and hence form a basis of* $M_d(\mathbb{C})$*.*

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Proof.
$\langle U^i V^j, U^k V^l\rangle_{\mathrm{HS}} = \mathrm{Tr}(V^{-j} U^{-i} U^k V^l)$ vanishes unless $i \equiv k$ and $j \equiv l \pmod{d}$.

Since U, V are unitary, their products are nonzero.

An orthogonal set of nonzero matrices is linearly independent.

$\square$

An elementary proof: III

Lemma

Suppose (U, V) and $(\tilde{U}, \tilde{V})$ are two conjugate pairs. There is a unitary Q, unique up to phase, such that $\phi_Q(M) = QMQ^*$ maps U to $\tilde{U}$ and V to $\tilde{V}$.

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Proof.
Define $\phi(U) = \tilde{U}$ and $\phi(V) = \tilde{V}$; this extends to a unique $*$-automorphism on $M_d(\mathbb{C})$.

The $*$-automorphisms of simple matrix algebras are in correspondence with unitaries up to phase (Skolem-Noether, 1927).

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Proof.
Apply Q to both sides of the equation $|z\rangle = X^z|0\rangle$.

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Key idea: Study k-th level gates via their conjugate tuples of $(k-1)$-th level gates.

Results

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5. Generate the level $k + 1$ gates using the Stone-von Neumann theorem
6. Go to step 2

A short proof of the CGK classification of $\mathcal{D}_k$

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where ω_m is the primitive d^m-th root of unity and $\phi : \mathbb{Z}_d \to \mathbb{Z}_{d^m}$ is a polynomial.

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Recognising and diagonalising semi-Clifford gates

Definition
A Lagrangian semibasis is a linearly independent set of n vectors $\{(\hat{p}_i, \hat{q}_i)\}_{i\in[n]} \subseteq \mathbb{Z}_d^{2n}$ satisfying $[(\hat{p}_i, \hat{q}_i), (\hat{p}_j, \hat{q}_j)] = 0$.

$\iff$ $Z^{\hat{p}_i} X^{\hat{q}_i}$ generate a maximal abelian subgroup of the Pauli group.

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Theorem
Suppose $G \in \mathcal{C}_k^n$ *and denote its conjugate tuple by* $U_i = GZ_iG^*, V_i = GX_iG^*$.

G is semi-Clifford if and only if there exists a Lagrangian semibasis $\{(\hat{p}_i, \hat{q}_i)\}_{i\in[n]} \subseteq \mathbb{Z}_d^{2n}$ *such that, for each* $i \in [n]$, $U^{\hat{p}_i} V^{\hat{q}_i}$ *is a Pauli gate.*

Applications of the semi-Clifford criterion

Theorem
Every third-level gate of one qudit (of any prime dimension) is semi-Clifford: $\mathcal{SC}_3^1 = \mathcal{C}_3^1$.

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Discussion

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Every k^{th}-level gate of one or two qudits (of any prime dimension) is semi-Clifford.

- A complete analytic classification of the Clifford hierarchy and semi-Clifford gates
- Usefulness for optimising circuit and gate synthesis?