On Quantum Bayesian Networks

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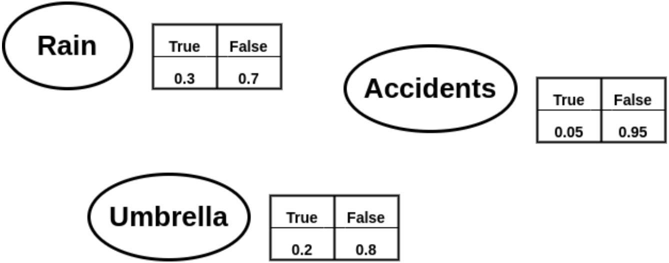
Outline

Section 1

Bayesian networks for a decision-making process

Working with Uncertainty

A variety of domains handle objects and events that are uncertain (presents probabilistic behavior):



They are most of the time-related meaning that the probability of occurring is not independent:

$$P(Rain, Accidents) \neq P(Rain) * P(Accidents) \quad (1)$$

Probability Distribution Table

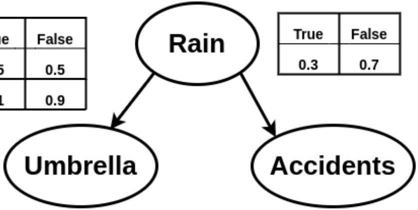
The complete information required to describe the occurrence of mutual events is given by the probability of each combination:

Rain	Umbrella	Accidents	P(Rain,...)
True	True	True	0.08
True	True	False	0.12
True	False	True	0.03
True	False	False	0.025
False	True	True	0.01
...	...	...	...
...	...	...	...

This representation is very expensive in computational resources as it requires an exponential amount of space in relation to the number of variables.

Bayesian networks

Bayesian networks are a compact representation of the joint probability distribution table.



P(Umbrella| Rain)	True	False
Rain = True	0.5	0.5
Rain = False	0.1	0.9

True	False
0.3	0.7

P(Accidents| Rain)	True	False
Rain = True	0.15	0.85
Rain = False	0.05	0.95

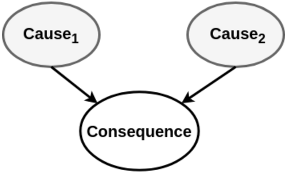
Any probability from the distribution table can be obtained by:

$$P(x_1, ..., x_n) = \prod_{i=1}^{n} P(x_i | Parents(X_i)) \tag{2}$$

Note: They are inspired by the Bayes updating rule.

Uses of Bayesian networks

In a variety of domains, constructing the network from the data can expose causal relations between the elements of study:



... while in other domains their capacity to infer conditional probabilities is explored:

$$P(Var|Evidence_1, Evidence_2) = \frac{P(Var, Evidence_1, Evidence_2)}{P(Evidence_1, Evidence_2)} \quad (3)$$

Decision Making

In a decision-making process, conditional probabilities are used to compute the utility of an action:

$$EU(a|e) = \sum_{r} P(Result = r|a, e) * U(r) \tag{4}$$

By the maximum expected utility principle, a rational entity will choose the action with the greatest expected utility with respect to its own set of beliefs:

$$action = argmax_a EU(a|e) \tag{5}$$

Bayesian networks for decision making

Decision process

$$\text{Bayesian network} \xrightarrow{\text{Rejection Sampling}} Conditional\ Probabilities \xrightarrow{\text{Utility function}}$$

$$Expected\ utilities \xrightarrow{\text{Maximum expected utility}} Best\ action$$

However, inference on a Bayesian network is an NP-Hard problem, which means that the number of iterations required to compute a value grows exponentially with the size of the problem.

Section 2

Quantum decision-making

Quantum inference

A quantum algorithm for inference on Bayesian networks was introduced by Low *et al* [2], leading to a quadratic speed up for some networks.

$$\text{Bayesian Networks} \xrightarrow{\text{Encoding}} \textit{Quantum state} \xrightarrow{\text{Amplitude Amplification}}$$
$$\textit{Updated quantum state} \xrightarrow{\text{Measurement}} \textit{Sample}$$

$$\equiv$$

Rejection Sampling

This comes from the fact that this kind of inference can be reduced to a search problem, to which the Amplitude Amplification algorithm can be applied.

Quantum decision process: Method 1

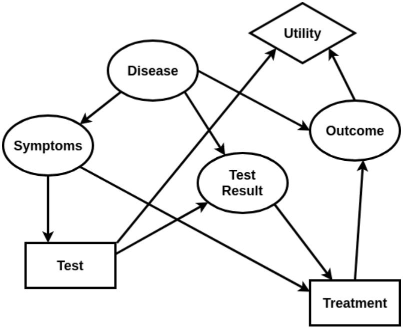
With a quantum computer, decision problems could be improved:

$$EU(a|e) = \sum_{R} \underbrace{P(Result = r|a, e)}_{\textbf{Quantum}} * \underbrace{U(r)}_{\textbf{Classical}} \tag{6}$$

The inference process which is a sub-process of the decision process suffers a quadratic speedup.

Quantum decision process: Method 2

With the use of a decision network, the inference process computes the expected utilities directly:

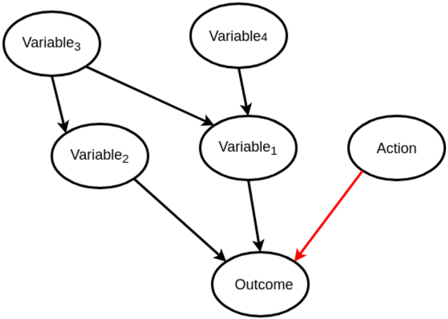


Resulting in a quadratic speedup for the whole decision process:

$$EU(a|e) = \underbrace{\sum_{R} P(Result = r|a, e) * U(r)}_{\textbf{Quantum}} \tag{7}$$

Quantum decision process: Method 3

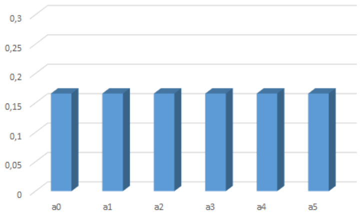
The structure of the data can be used to create a different algorithm for the decision-making process.



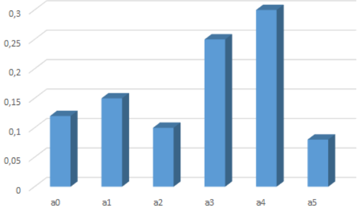
Related variables are entangled in the quantum state. Therefore, a transformation on one of them has an impact on the other.

Quantum decision process: Method 3

The utility function can be applied to the Outcome variable, which has a very interesting result on the Action variable.



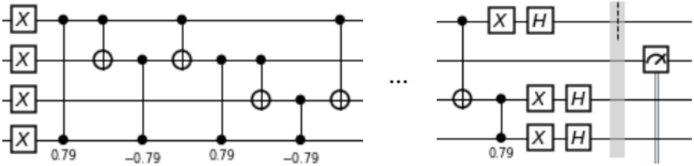
⇒



This process exchanges the order of the action variable's probabilities according to their expected utility!

Implementation of Method 3

A simple decision-making process was implemented on IBM quantum simulator as a proof-of-concept of **Method 3**:



The results were in correspondence with the theoretical predictions:

States	Theoretically expected probability	Percentage of Samples
$Action_0$	0,58	0.544
$Action_1$	0,42	0.456

Section 3

Complexity estimation

Complexity estimation

The complexity of each process is defined by the following 2 terms:

- Number of samples (S)
- Number of iterations by sample (I_s)

Total Number of Iterations

$$I_t = S * I_s$$

Remark: The number of samples defines the precision of the value.

Number of Samples

The relation between the precision and the number of samples is defined by the error terms of a multinomial distribution (**Method 1 and 3**):

$$N = \frac{A * \pi_i(1 - \pi_i)}{\delta^2}, (i = 1, 2, ..., k) \tag{8}$$

The Wald interval is considered for the case of a binomial distribution (**Method 2**):

$$N = z^2 \cdot \frac{\pi_i \left(1 - \pi_i\right)}{\delta^2} \tag{9}$$

Number of iterations per sample

The number of iterations in the Amplitude Amplification algorithm is related to the probability of the goal state:

$$I_s = \sqrt{\frac{1}{P(state)}} \tag{10}$$

For **Methods 1 and 2** this value is

$$I_s = \sqrt{\frac{N_a}{P(e)}} \tag{11}$$

while, for **Method 3**,

$$I_s = \sqrt{\frac{N_r}{P(e)}} \tag{12}$$

Total Number of Iterations

Method 1:

$$n * 2^m * \sqrt{\frac{N_a}{P(e)}} * \frac{A * \pi_i(1-\pi_i)}{\delta_c^2} \tag{13}$$

Method 2:

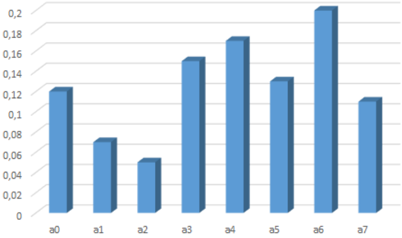
$$n * 2^{m'} * \sqrt{\frac{N_a}{P(e)}} * \frac{z^2 * \pi_i\,(1-\pi_i)}{\delta_b^2} \tag{14}$$

Method 3:

$$n * 2^{m'} * \sqrt{\frac{N_r}{P(e)}} * \frac{A * \pi_i(1-\pi_i)}{\delta_a^2} \tag{15}$$

Complexity comparison

In reinforcement learning, the action is selected in correspondence to a probability distribution:



The distribution required is exactly the one constructed by **Method 3**, meaning that with this method only one sample solves the decision problem.

Section 4

Picturing a quantum Bayesian algorithm

String diagrams

String diagrams are graphical representations of terms in a monoidal (dagger compact) category, which provides a semantics for quantum programs [1].

This abstraction offers a framework to verify proprieties of quantum algorithms.

Grover's Algorithm in string diagrams

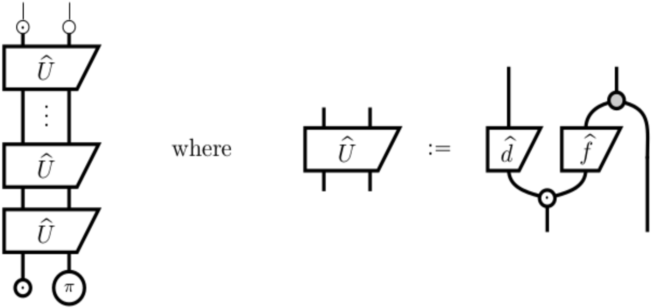


Figure: Grover's algorithm in string diagrams.

A verification example

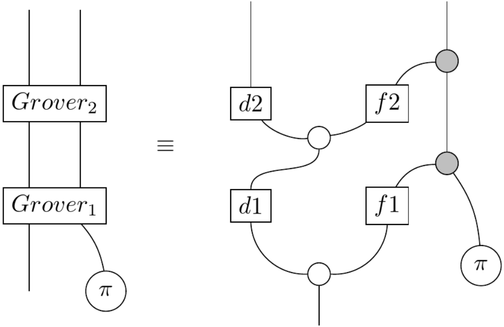
In our algorithm, the utility function is applied after a search for the correct values of the evidence variables. However, the utility function is applied within a search.

Question

Is the composition of 2 quantum searches equivalent to a quantum search with the union of the restrictions?

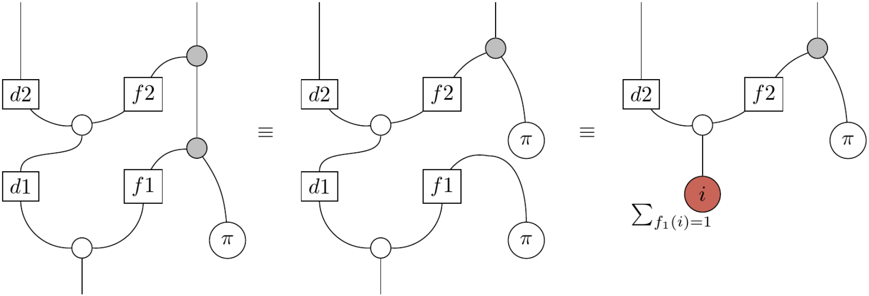
Proof

Both searches can be written as follows, where each one can have only one iteration.



Proof

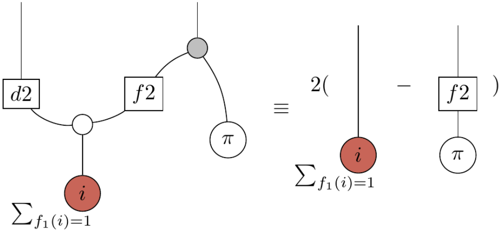
Equivalence-preserving transformations can be applied:



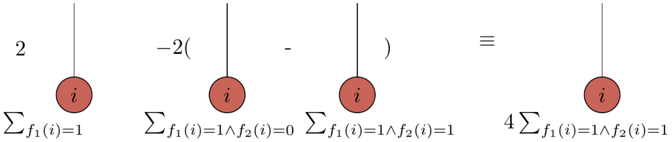
Until the resulting state is given to the second search process.

Proof

Finally, process d_2 can be described as below:

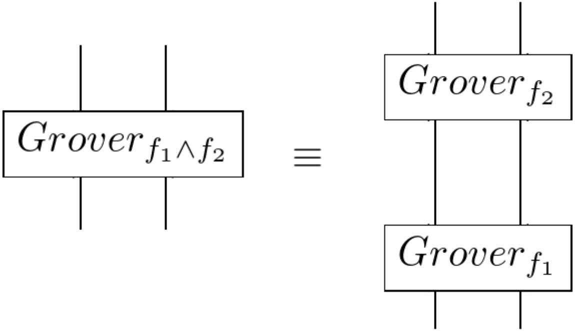


Resulting in:



Proven equivalence

The result shows that the programmer has a choice between the two implementations:



Section 5

Conclusions

Conclusions

- Decision-making processes can be implemented on a quantum device, with a speedup.
- The computational complexity of an algorithm helps the programmer to select the best solution.
- String diagrams revealed themselves as a very useful tool to analyze quantum processes.

References I

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Picturing Quantum Processes: {A} First Course in Quantum Theory and Diagrammatic Reasoning.
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