Neither Contextuality nor Nonlocality Admits Catalysts

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Catalysts

Entanglement theory famously allows for catalysts: there are quantum states $|\psi_1\rangle, |\psi_2\rangle$ and $|\phi\rangle$ such that

- there's no LOCC-protocol $|\psi_1\rangle \to |\psi_2\rangle$
- but there is a LOCC-protocol $|\psi_1\rangle \otimes |\phi\rangle \to |\psi_2\rangle \otimes |\phi\rangle$

More generally, in an arbitrary resource theory we'd speak of catalysis whenever there's three resources d, e, f such that

- there's no free transformation $e \to f$
- but there is a free transformation $d \otimes e \to d \otimes f$

If the resource theory is catalysis-free, this never happens. Writing $e \rightsquigarrow f$ for the existence of a free transformation, this is equivalent to saying that $d \otimes e \rightsquigarrow d \otimes f$ implies $e \rightsquigarrow f$ for any d, e, f.

Punchline

Let

- d, e, f range over various correlations (contextual or not)
- $d \otimes e$ denote having d and e independently side-by-side
- $d \rightsquigarrow e$ denote the existence of a transformation $d \to e$ in the resource theory of contextuality

Theorem

If $d \otimes e \rightsquigarrow d \otimes f$, then $e \rightsquigarrow f$. Ditto for the resource theory of non-locality.

Overview

- As the resource theory of contextuality we use that of

  '*A comonadic view of simulation and quantum resources*'
  Abramsky, Barbosa, MK., Mansfield, LiCS 2019.

  giving a formalization of the the wirings and prior-to-input-classical communication paradigm studied in physics.

- The resource theory of non-locality: the n-partite version of the above

- Proof idea: if you can catalyze once you can catalyze arbitrarily many times. For big enough n this implies that one needs only a compatible (and hence non-contextual) part of d.[1]

[1] or rather, a compatible subset of $\mathrm{MP}(d)$

Formalising empirical data

A **measurement scenario** $S = \langle X_S, \Sigma_S, O_S \rangle$:

- X_S – a finite set of measurements
- Σ_S – a simplicial complex on X_S
  faces are called the **measurement contexts**
- $O_S = (O_x)_{x \in X_S}$ – for each $x \in X_S$ a non-empty outcome set O_x. Joint outcomes over $U \subseteq X_S$ denoted by $\mathcal{E}_S(U)$.

A	B	(0, 0)	(0, 1)	(1, 0)	(1, 1)
a_0	b_0	1/2	0	0	1/2
a_0	b_1	1/2	0	0	1/2
a_1	b_0	1/2	0	0	1/2
a_1	b_1	0	1/2	1/2	0

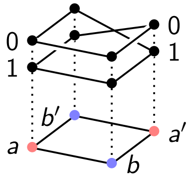
$X = \{a_0, a_1, b_0, b_1\}, \; O_x = \{0, 1\}$

$\Sigma = \downarrow \{ \{a_0, b_0\}, \{a_0, b_1\}, \{a_1, b_0\}, \{a_1, b_1\} \}.$

An **empirical model** $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on S:

- each $e_\sigma \in \text{Prob}\,(\mathcal{E}_S(\sigma))$ is a probability distribution over joint outcomes for σ.
- *generalised no-signalling* holds: for any $\sigma, \tau \in \Sigma_S$, if $\tau \subseteq \sigma$,

$$e_\sigma|_\tau = e_\tau$$



Contextuality

An empirical model $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on a measurement scenario (X, Σ, O) is **non-contextual** if there is a distribution d on $\prod_{x \in X} O_x$ such that, for all $\sigma \in \Sigma$:

$$d|_\sigma \;=\; e_\sigma .$$

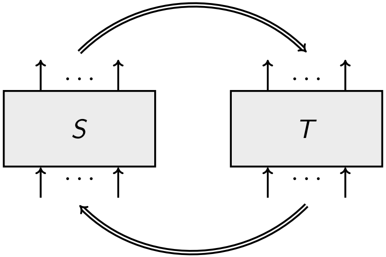
That is, we can **glue** all the local information together into a global consistent description from which the local information can be recovered.

If no such global distribution exists, the empirical model is **contextual**.

Contextuality: family of data that is **locally consistent** but **globally inconsistent**.

The import of Bell's and Kochen–Spekker's theorems is that there are behaviours arising from quantum mechanics that are contextual.

Maps between scenarios



A deterministic map $S \to T$ proceeds as follows:

- map inputs of T (measurements) to inputs of S
- run S
- map outputs of S (measurement outcomes) to outputs of T

Same but formally

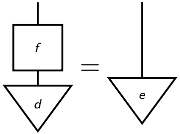
A deterministic map $(\pi, h) : S \to T$ is given by:

- A simplicial function $\pi : (X_T, \Sigma_T) \to (X_S, \Sigma_S)$.
- For each $x \in X_T$, a function $h_x : O_{\pi(x)} \to O_x$.

Simpliciality of π means that contexts in Σ_T are mapped to contexts in Σ_S.

Simulations

Given $d : T$, $e : S$, a deterministic simulation $d \to e$ is a deterministic map $f : S \to T$ that transforms d to e.



For instance, the *PR*-box can be simulated from a liar's paradox on a triangle, by collapsing one edge to a point.
But what if we want to

(i) let a measurement of T to depend on a measurement protocol of S?

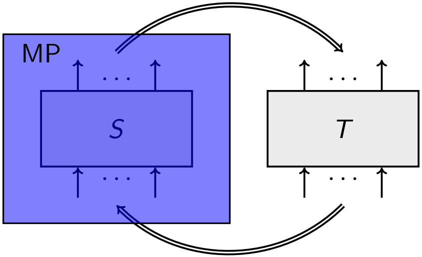
(ii) use classical randomness?

The MP construction

Given a scenario $S = \langle X_S, \Sigma_S, O_S \rangle$ we build a new scenario MP(S), where:

- measurements are the (deterministic) measurement protocols on S. A measurement protocol on S is either empty or consists of a measurement in $x \in X_S$ and of a function from outcomes of x to measurement protocols on $S|_{\mathsf{lk}_x}$
- outcomes are the joint outcomes observed during a run of the protocol
- measurement protocols are compatible if they can be combined consistently

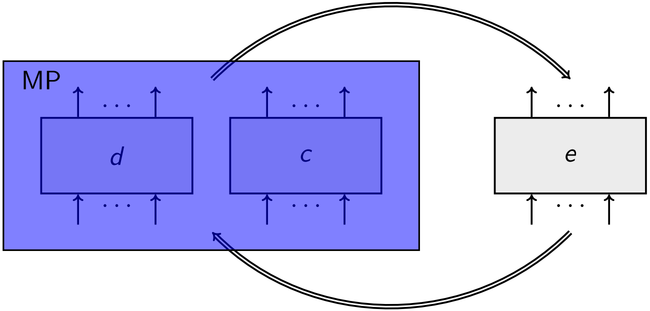
Adaptive procedure



An adaptive map $S \to T$ proceeds as follows:

- map measurements of T to **measurement protocols** over S, i.e., measurements of $\mathrm{MP}(S)$
- run S
- map outcomes of $\mathrm{MP}(S)$ to outputs of T

Adaptive procedure with classical randomness



Requirement: c is noncontextual.

General simulations

Given empirical models e and d, a **simulation** of e by d is a deterministic simulation

$$\mathsf{MP}(d \otimes c) \to e$$

for some noncontextual model c.

The use of the noncontextual model c is to allow for classical randomness in the simulation.

We denote the existence of a simulation of e by d as $d \rightsquigarrow e$, read "d simulates e".

Why bother?

The convertibility relation $\rightsquigarrow$ results in a resource theory of contextuality with nice properties:

- Expressive enough to capture less formally defined transformations in the literature (in the single-shot exact case)
- Added precision can help with new results
- Contextual fraction is a monotone
- Contextuality is equivalent to insimulability from a trivial model. Variants for logical and strong contextuality.

No-catalysis

Theorem (No-catalysis)

If $d \otimes e \rightsquigarrow d \otimes f$ then $e \rightsquigarrow f$

Proof.

First step — reduce to the deterministic case:
If $d \otimes e \rightsquigarrow d \otimes f$, then there is a deterministic simulation $\mathrm{MP}(d \otimes e \otimes c) \to d \otimes f$ for some non-contextual c. Setting $g := e \otimes c$ we thus have a deterministic map $\mathrm{MP}(d \otimes g) \to d \otimes f$.
Second step—if you can catalyze once, you can do so many times: we can get deterministic simulations $\mathrm{MP}(d \otimes (g^{\otimes n})) \to d \otimes (f^{\otimes n})$ for any n so that the i-th copy of f uses d and the i-th copy of g, but otherwise the copies of f are simulated similarly.

Concluding the proof

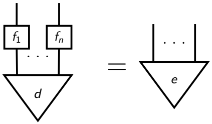
Cont.

Final step—things needed from d are compatible: as the underlying map is simplicial, questions asked from d when simulating different copies of f are always compatible. Considering big enough n, this means that the set of all possible behaviours in d needed for the simulation forms a compatible subset of MP(d). Join all of these to a single measurement protocol over d and simulate 1st copy of f by first measuring this single measurement of MP(d) and then proceeding to the 1st copy of g. This results in a deterministic simulation $\text{MP}(d' \otimes g) \to f$, where d' has a single measurement (representing the whole MP over d) and is thus noncontextual.

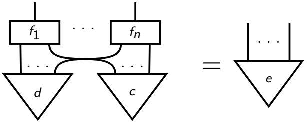
□

Non-locality

We think of the resource theory of non-locality as an n-partite version of that of contextuality: an object is a model $e : \bigotimes_{i=1} S_i$ over an n-partite scenario, and a simulation $d \to e$ is an n-tuple of adaptive maps that, taken together, transform d to e.
In the deterministic case:



in the randomness-assisted case



where c is local. This captures the LOSR-paradigm.

No catalysts for non-locality and beyond

For a minor variant of the previous proof, an n-partite simulation $d \otimes e \to d \otimes f$ produces an n-partite simulation $e \to f$, proving the theorem for non-locality.

In fact, the proof shows more: if $\mathcal{X}$ is a class of models that (i) contains all non-contextual models and (ii) is closed under $\otimes$, then one can define $\mathcal{X}$-assisted simulations $d \to e$ as deterministic simulations $\mathsf{MP}(d \otimes x) \to e$ where $x \in \mathcal{X}$. Write $d \rightsquigarrow_{\mathcal{X}} e$ for the existence of such a simulation.

Theorem

For any such $\mathcal{X}$, $d \otimes e \rightsquigarrow_{\mathcal{X}} d \otimes f$ if and only if $e \rightsquigarrow_{\mathcal{X}} f$

Thus we can't use a PR box as a catalyst, even if we can freely use quantum correlations.

Questions...

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MK, "Neither Contextuality nor Nonlocality Admits Catalysts" (2021), Phys. Rev. Lett. 127, 160402
arXiv:2102.07637