Relationship between Covariance of Wigner functions and Transformation Noncontextuality

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arXiv:2004.06318


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Covariance of Wigner functions

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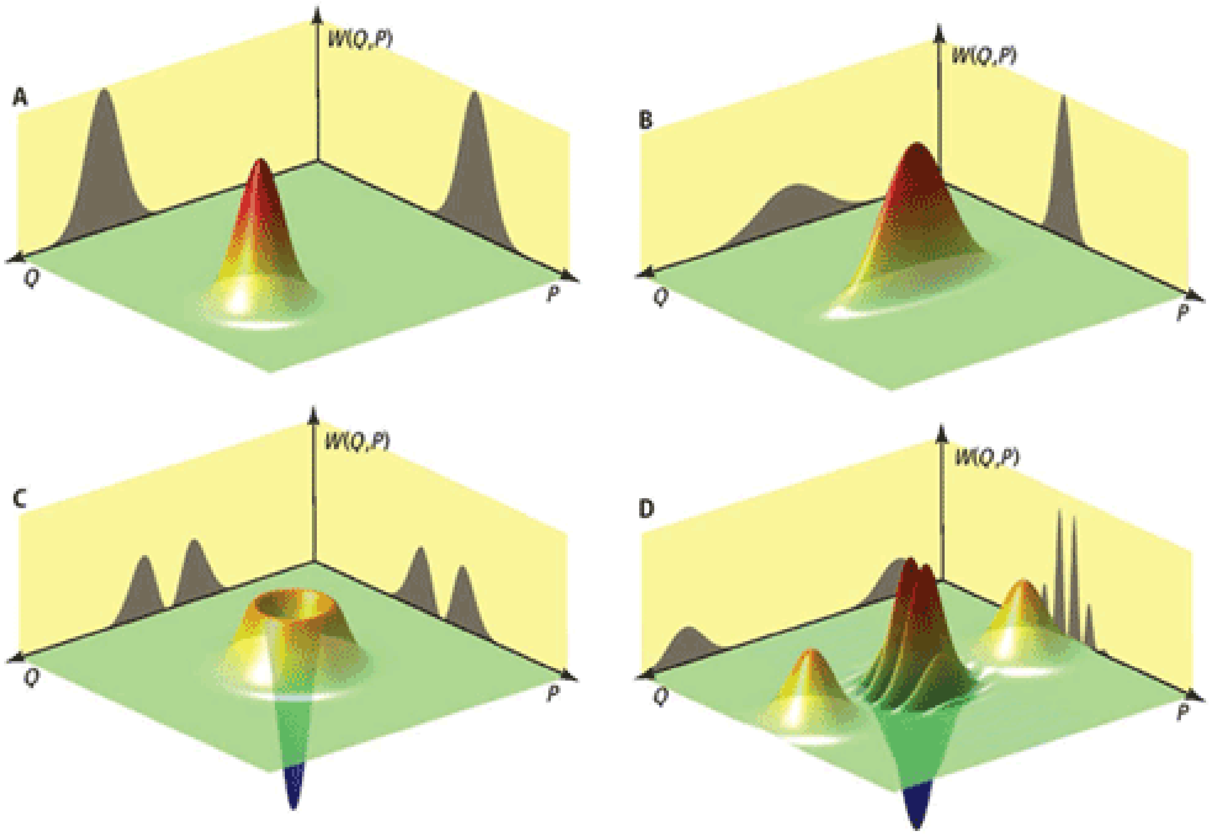
Transformation Noncontextuality

Wigner Functions

Continuous case



Discrete case

(X_2, P_2)	0	0	0	0
	0	0	0	0
	$1/4$	$1/4$	0	0
	$1/4$	$1/4$	0	0
	(X_1, P_1)			

Qubit's state:

$$|\psi\rangle = |00\rangle \, .$$

Discrete case

(X_2, P_2)

$-1/8$	$1/8$	$1/8$	$1/8$
$1/8$	$-1/8$	$1/8$	$1/8$
$1/8$	$1/8$	$-1/8$	$1/8$
$1/8$	$1/8$	$1/8$	$-1/8$

(X_1, P_1)

Qubit's state:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Definition

Phase space point

Phase space point operator

$$W_\rho(\lambda) = \mathrm{Tr}[A(\lambda)\rho]$$

Quantum state

Phase space point operator

$$A^{\gamma}(\lambda) = \frac{1}{N_\Lambda} \sum_{\lambda' \in \Lambda} \chi([\lambda, \lambda'])T^{\gamma}(\lambda')$$

Weyl operator

$$T^{\gamma}(\lambda) = w^{\gamma(\lambda)} Z(\lambda_Z) X(\lambda_X)$$

$$\lambda = (\lambda_Z, \lambda_X) \in \mathbb{Z}_d^{2n}$$

R. Raussendorf et al, PRA 95, 052334 (2017).

Gross' Wigner functions (*Odd prime qudits stabilizer*) :

J. Math. Phys. 47 (12): 122107(2006)

$$\chi(a) = e^{\frac{2\pi i}{d}a},\ a \in \mathbb{Z}_d, \quad w^{\gamma(\lambda)} = \chi(-2^{-1}\gamma(\lambda)), \quad \gamma(\lambda) = \lambda_X \cdot \lambda_Z.$$

Delfosse et al.'s Wigner functions (*CSS rebits*) :

PRX 5, 021003 (2015)

$$\chi(a) = (-1)^a,\ a \in \mathbb{Z}_d, \quad w^{\gamma(\lambda)} = 1,\ \gamma(\lambda) = 0.$$

Gibbons-Wootters' Wigner functions (*Single qubit stabilizer*) :

PRA 70, 062101 (2004)

$$\chi(a) = 1,\ a \in \mathbb{Z}_d,\ w^{\gamma(\lambda)} = i^{\gamma(\lambda)},\ \gamma(\lambda) = \lambda_Z \cdot \lambda_X \bmod 4.$$

Phase space point operator

$$\rho = \sum_{\lambda} W_{\rho}(\lambda) A(\lambda)$$

Covariance
of Wigner Functions

We consider closed subtheories of Quantum Theory $(\mathcal{S}, \mathcal{T}, \mathcal{M})$.

Specified by a group of unitaries.

Definition

$$W_{U\rho U^\dagger}(\lambda) = W_\rho(S\lambda + a) \quad \forall\, \rho \in \mathcal{S},\ \forall\, \lambda \in \Lambda.$$

$\forall\, U \in \mathcal{T}$

Simplectic matrix

Translation term

Definition

$$UA(\lambda)U^\dagger = A(S\lambda + a) \quad \forall\, \lambda \in \Lambda.$$

$\forall\, U \in \mathcal{T}$

Simplectic matrix

Translation term

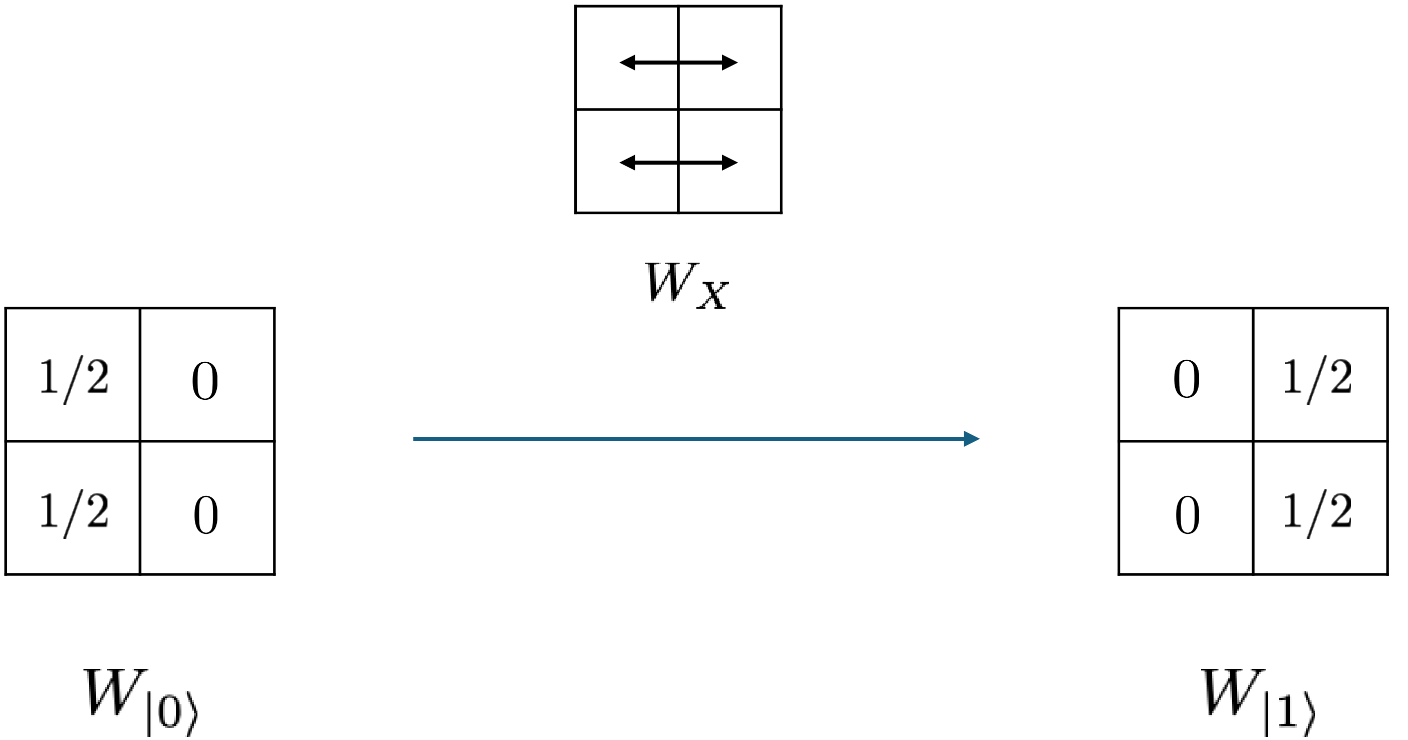
Example

$1/2$	0
$1/2$	0

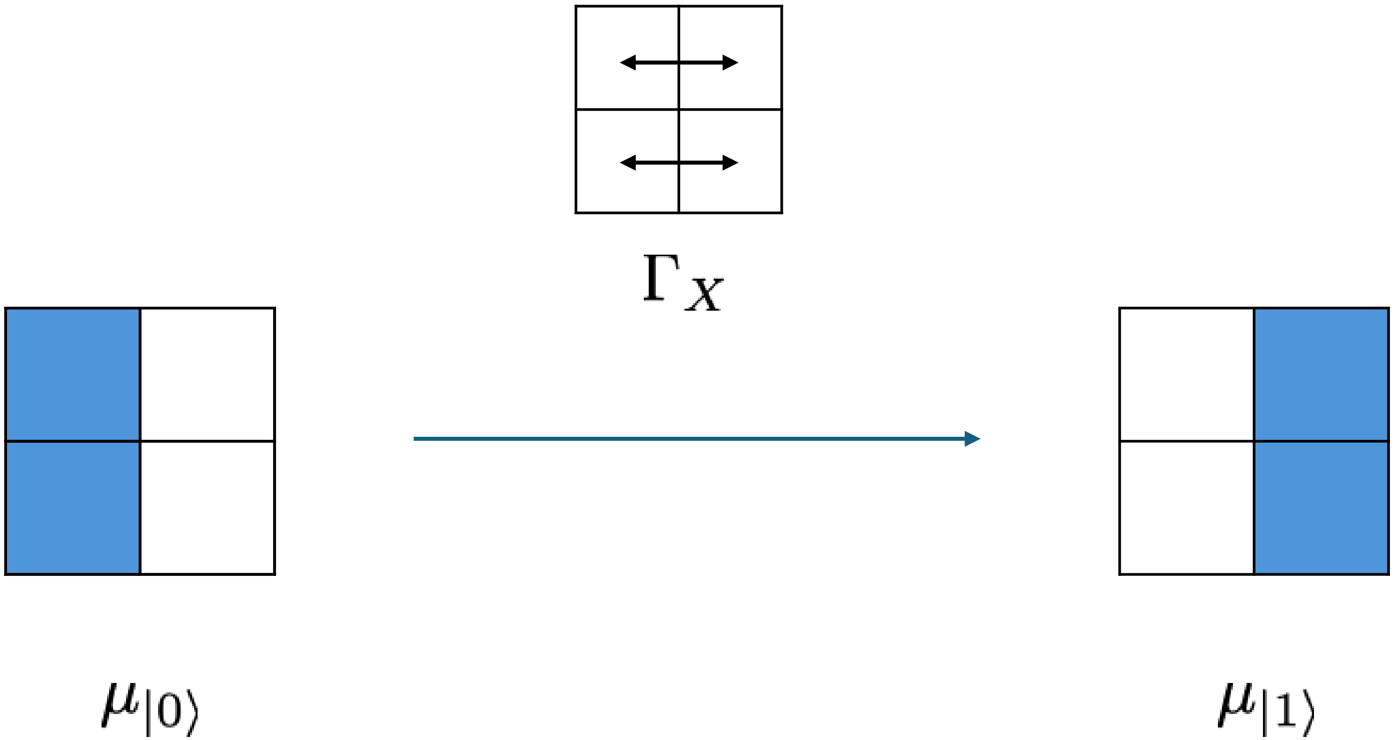
$W_{|0\rangle}$

$\xrightarrow{\text{X}}$

Example



Example

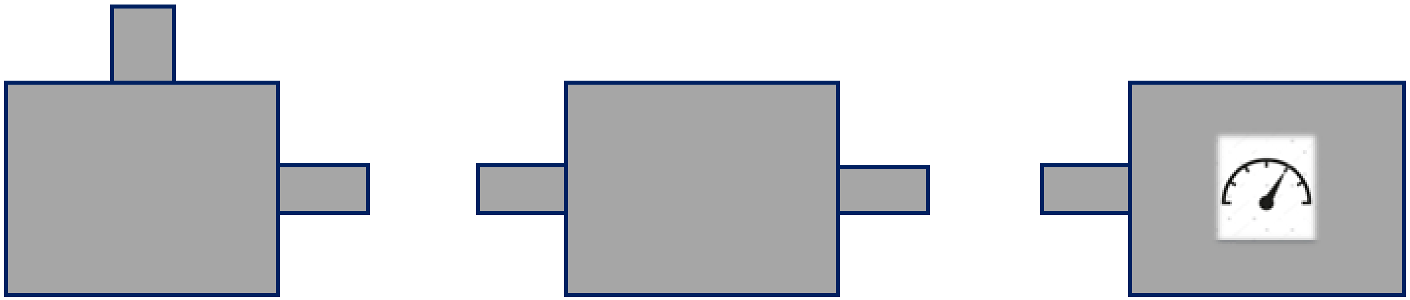


L. Catani and D. E. Browne, *New J. Phys.* 19 073035 (2017).

Transformation Noncontextuality

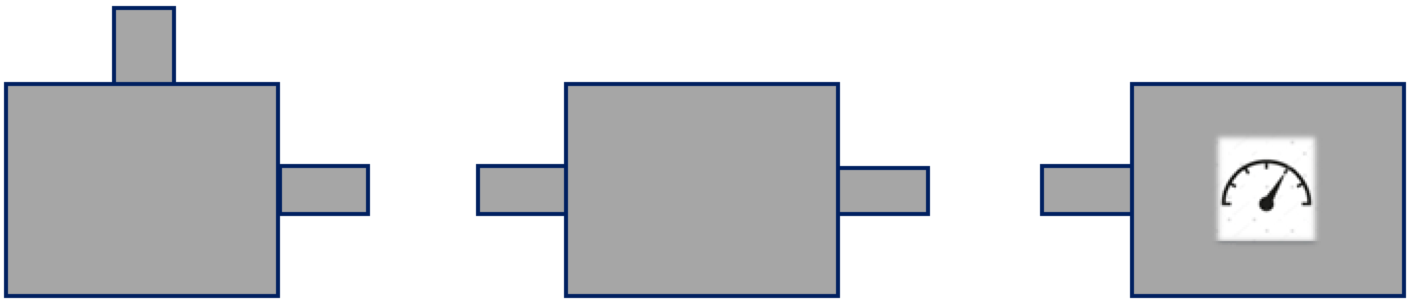
Operational Approach to Physical Theories

A *physical theory* is just a tool to predict the statistics of outcomes from experimental procedures.



$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{p(k|M,T,P)}$$

Operational Quantum Theory



ρ

(Density Operator)

$\mathcal{E}$

(CPTP Map)

$\{E_k\}$

(POVM)

$$p(k|M,T,P) = Tr[\varepsilon(\rho)E_k]$$ (Born Rule)

Ontological Models Framework

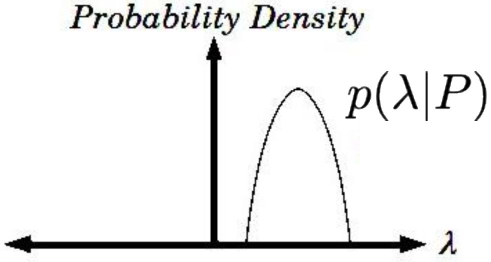
N. Harrigan, R.W. Spekkens, Found. of Phys. **40**, 2, 155-157 (2010)

Ontological Model

Each system $\longrightarrow$ ontic state space Λ with ontic states $\lambda \in \Lambda$.

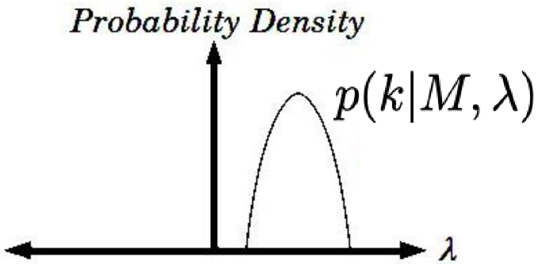
Experimental procedures:

$P \longrightarrow p(\lambda|P)$



$T \longrightarrow p(\lambda'|\lambda, T)$

$M, k \longrightarrow p(k|M, \lambda)$



Ontological Model

Statistics (classical probability theory):

$$p(k|P,T,M) = \int d\lambda d\lambda' \, p(k|M,\lambda')p(\lambda'|\lambda,T) \, p(\lambda|P) \ .$$

Noncontextual Ontological Model

Two transformations T, T' are *operationally equivalent,* $T \simeq T'$, if

$$p(k|P,T,M) = p(k|P,T',M) \qquad \forall\ P, M.$$

In a transformation noncontextual ontological model,

$$T \simeq T' \implies p(\lambda'|\lambda,T) = p(\lambda'|\lambda,T') \quad \forall\ T,T',\lambda,\lambda'.$$

R.W. Spekkens, Phys. Rev. A **71**, 052108 (2005).

Recap

Given a subtheory $(\mathcal{S}, \mathcal{T}, \mathcal{M})$,

Covariance:

$$U A(\lambda) U^\dagger = A(S\lambda + a) \quad \forall\ \lambda, U.$$

Transformation Noncontextuality:

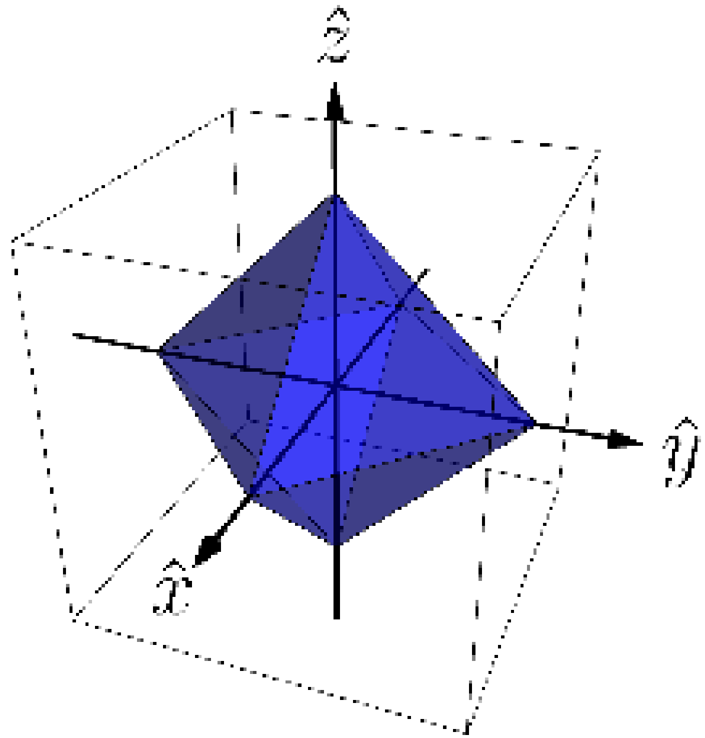
$$T \simeq T' \implies p(\lambda'|\lambda, T) = p(\lambda'|\lambda, T') \quad \forall\ T, T', \lambda, \lambda'.$$

Key Example:
The Single Qubit Stabilizer Theory

States: eigenstates of $\hat{X}, \hat{Y}, \hat{Z}$.

Observables: $\hat{X}, \hat{Y}, \hat{Z}$.

Transformations: $\langle H, P \rangle$.



$\hat{z}$
$\hat{y}$
$\hat{x}$

Two kinds of Wigner functions:

$$W_+(\lambda) = \mathrm{Tr}[\rho A_+(\lambda)]$$

$$A_+(0,0) = \mathbb{I} + \hat{X} + \hat{Y} + \hat{Z}$$

$$A_+(0,1) = \mathbb{I} + \hat{X} - \hat{Y} - \hat{Z}$$

$$A_+(1,0) = \mathbb{I} - \hat{X} + \hat{Y} - \hat{Z}$$

$$A_+(1,1) = \mathbb{I} - \hat{X} - \hat{Y} + \hat{Z}$$

$$W_-(\lambda) = \mathrm{Tr}[\rho A_-(\lambda)]$$

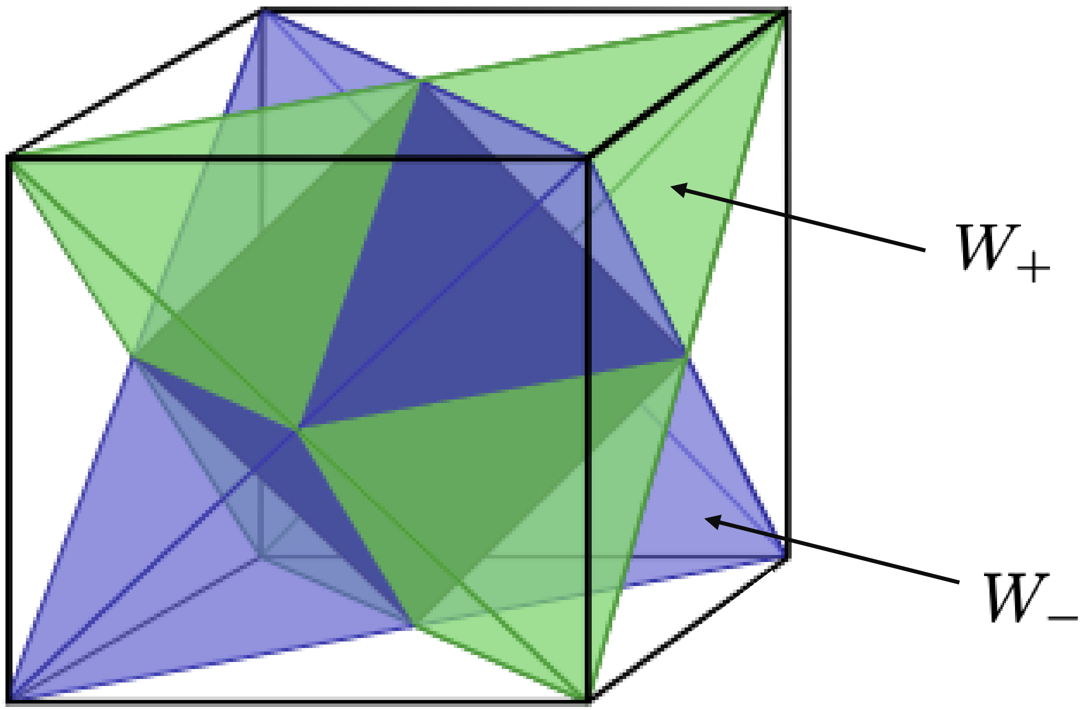
$$A_-(0,0) = \mathbb{I} + \hat{X} + \hat{Y} - \hat{Z}$$

$$A_-(0,1) = \mathbb{I} + \hat{X} - \hat{Y} + \hat{Z}$$

$$A_-(1,0) = \mathbb{I} - \hat{X} + \hat{Y} + \hat{Z}$$

$$A_-(1,1) = \mathbb{I} - \hat{X} - \hat{Y} - \hat{Z}$$

C. Cormick, E. Galvão, D. Gottesman, J. Paz, A. Pittenger, PRA 73, 012301(2006).



W_+
W_-

The single qubit stabilizer theory is **not** covariantly represented.

The Hadamard gate

$$\boxed{\begin{aligned} H\hat{X}H &= \hat{Z} \\ H\hat{Z}H &= \hat{X} \\ H\hat{Y}H &= -\hat{Y} \end{aligned}}$$

$$\Downarrow$$

$$HA_+(\lambda)H = A_-(\lambda')$$

The single qubit stabilizer theory is **not** noncontextual.

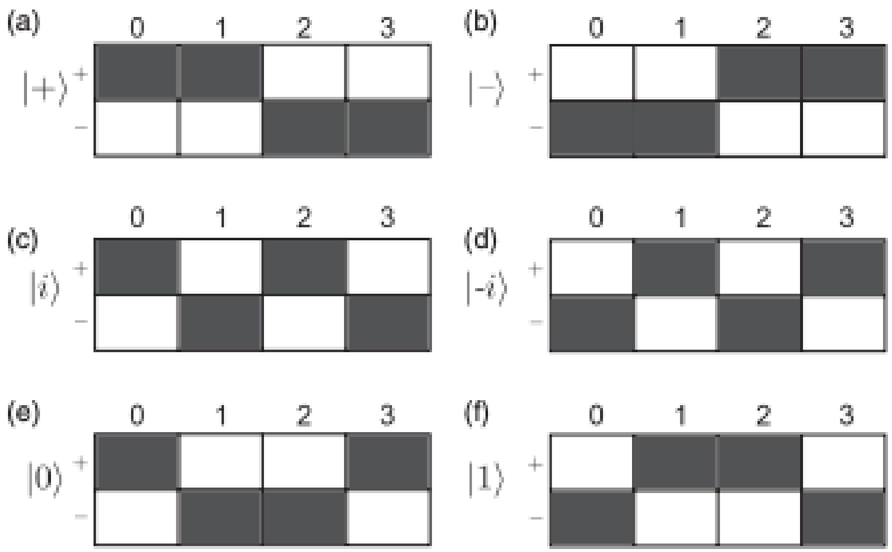
The completely depolarizing channel

$$T(\rho) = \frac{1}{4}\sum_U U\rho U^\dagger,$$

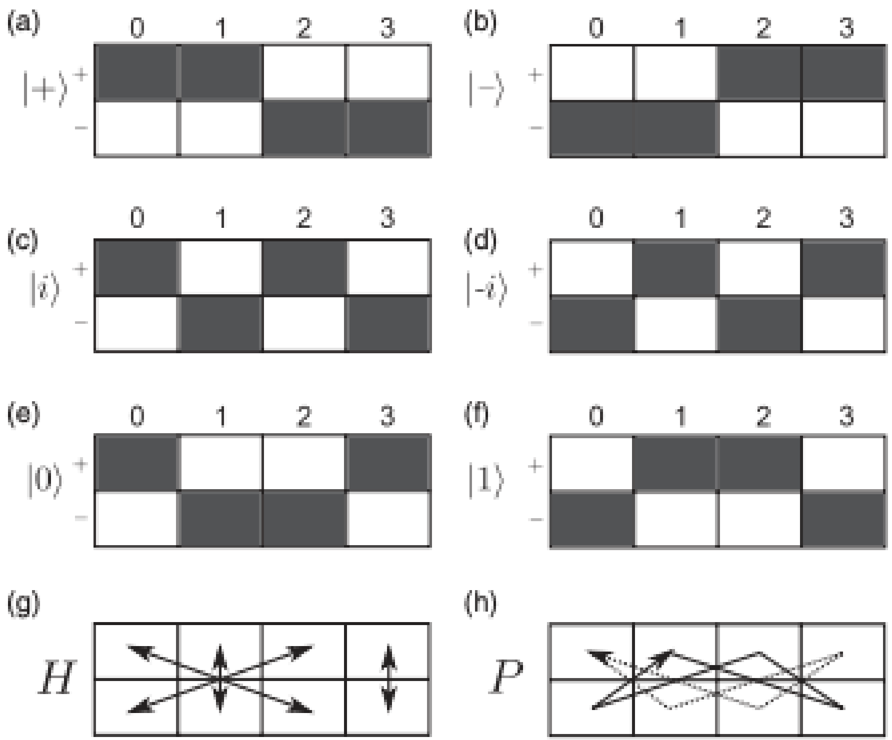
$$T'(\rho) = \frac{1}{4}\sum_U (HU)\rho(HU)^\dagger, \qquad U \in \{\mathbb{I}, X, Y, Z\}.$$

T and *T'* are operationally equivalent, but ontologically distinct in any ontological model of the single qubit stabilizer theory.

P. Lillystone, J. Wallman, J. Emerson, Phys. Rev. Lett. 122, 140405 (2019).



J. Wallman, S. Bartlett, Phys. Rev. A 85, 062121 (2012).



J. Wallman, S. Bartlett, Phys. Rev. A 85, 062121 (2012).

The single qubit stabilizer theory is transformation contextual for the same reason it breaks covariance.

How to connect these two notions?

Positive Quasiprobability Representation

$$\Updownarrow$$

Noncontextual Ontological Model

R. Spekkens, Phys. Rev. Lett. 101, 020401(2008).

Covariance

$\Downarrow$

Transformation NC

$(\mathcal{S}, \mathcal{T}, \mathcal{M})$

$$\varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger$$

$$W_\varepsilon(\lambda'|\lambda) = \mathrm{Tr}[\varepsilon(A(\lambda))A(\lambda')]$$ (Definition)

$$= \mathrm{Tr}[\sum_k A(S_k\lambda + a_k)A(\lambda')]$$ (Covariance)

$$\propto \sum_k \delta_{S_k\lambda+a_k,\lambda'} \geq 0.$$ (Linearity)
(Orthonormality)

Covariance

⇧

Transformation NC

$$W_\varepsilon(\lambda|\lambda') = \mathrm{Tr}[\varepsilon(A(\lambda))A(\lambda')] \geq 0 \quad \forall\varepsilon.$$ (Positivity)

$$\Downarrow$$

$$W_U(\lambda|\lambda') \geq 0$$

Let us define $B \equiv UA(\lambda)U^\dagger$.

$$B = \sum_{\lambda'} W_U(\lambda|\lambda')A(\lambda')$$ (*A* basis of Hermitian operators)

$$\mathrm{Tr}[B] = \sum_{\lambda'} W_U(\lambda|\lambda') = 1$$ (Cyclicity + linearity + *A* has unit trace)

$$\mathrm{Tr}[B^2] = \sum_{\lambda'} W_U^2(\lambda|\lambda') = 1$$ (Cyclicity + Orthonormality of *A*)

(Positivity)

$$W_U(\lambda|\lambda') \geq 0$$

$$\sum_{\lambda'} W_U(\lambda|\lambda') = 1$$ (*A* has unit trace)

$$\sum_{\lambda'} W_U^2(\lambda|\lambda') = 1$$ (Orthonormality of *A*)

$$W_U(\lambda|\lambda') \geq 0 \qquad \sum_{\lambda'} W_U(\lambda|\lambda') = 1 \qquad \sum_{\lambda'} W_U^2(\lambda|\lambda') = 1$$

$$W_U(\lambda|\lambda') = \begin{cases} 0 & \forall\ \lambda' \neq \tilde{\lambda} \\ 1 & \text{for } \lambda' = \tilde{\lambda} \end{cases}$$

$$\Downarrow$$

$$\underline{B} = \sum_{\lambda'} W_U(\lambda|\lambda') A(\lambda') \underline{= A(\tilde{\lambda})}$$

Positivity implies that phase point operators are mapped to phase point operators.

But are these maps symplectic transformations?

Covariance of Wigner functions

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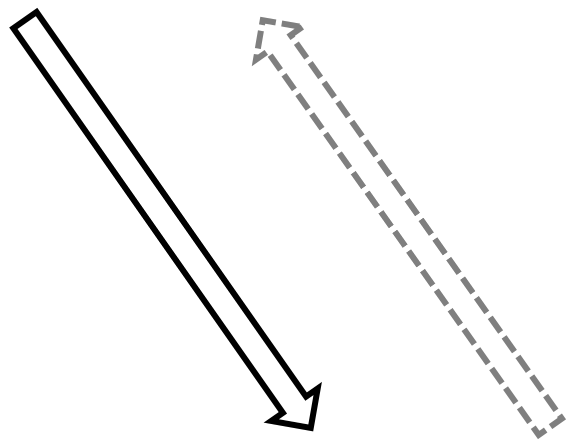
Transformation Noncontextuality

Covariance of Wigner functions



Transformation Noncontextuality

Extra Slides

Follow ups

- Possible strategy to prove the conjecture.
- Relationship with Positivity Preservation.
- Extend to all NC ontological models.
- Extend to all channels

Properties of WF

1) It is a quasi-probability distribution : $W_\rho : \mathbb{Z}_d^{2n} \to \mathbb{R}, \ \sum_\lambda W_\rho(\lambda) = 1$

2) Covariance with Paulis : $W_{T(u)\rho T^\dagger(u)}(\lambda) = W_\rho(\lambda + u).$

3) Marginals : $\sum_{\lambda_Z} W_\rho(\lambda) = Pr_\rho(\lambda_X)$

4) $Tr(\rho\sigma) = N \sum_\lambda W_\rho(\lambda) W_\sigma(\lambda)$

5) $\rho = \sum_\lambda W_\rho(\lambda) A(\lambda)$

Some properties of $A(\lambda)$:

1. Hermitianity : $A(\lambda) = A(\lambda)^\dagger.$
2. Unit trace : $Tr[A(\lambda)] = 1.$
3. $Tr[A(\lambda)A(\lambda')] = \delta_{\lambda,\lambda'}.$
4. $\sum_\lambda A(\lambda) = \mathbb{I}.$

Definition for CPTP maps

CPTP map $\mathcal{E}$ *= completely positive trace preserving map.*
Any CPTP map can always be written in a Kraus decomposition

$$\varepsilon(\rho) = \sum_k p_k E_k \rho E_k^\dagger, \quad where \sum_k E_k^\dagger E_k = \mathbb{I}.$$

Given a CPTP map $\mathcal{E}$ *such that* $\rho' = \varepsilon(\rho)$*, it is covariant if its decompositions into Kraus operators* $\{E_k\}$ *are covariant,*
i.e. $E_k A(\lambda) E_k^\dagger = A(T_k\lambda + a_k), \quad \forall\lambda \in \mathbb{Z}_d^{2n}, \ \forall k.$

N.B. This is different from $\varepsilon(A(\lambda)) = A(T\lambda + a), \quad \forall\lambda \in \mathbb{Z}_d^{2n}.$

Recap

Notions of classicality (transformations) so far:

- Positivity preservation

Given ε *such that* $\rho' = \varepsilon(\rho)$ *and* W_ρ *is positive, then* $W_{\rho'}$ *is positive* $\forall \rho \in S$.

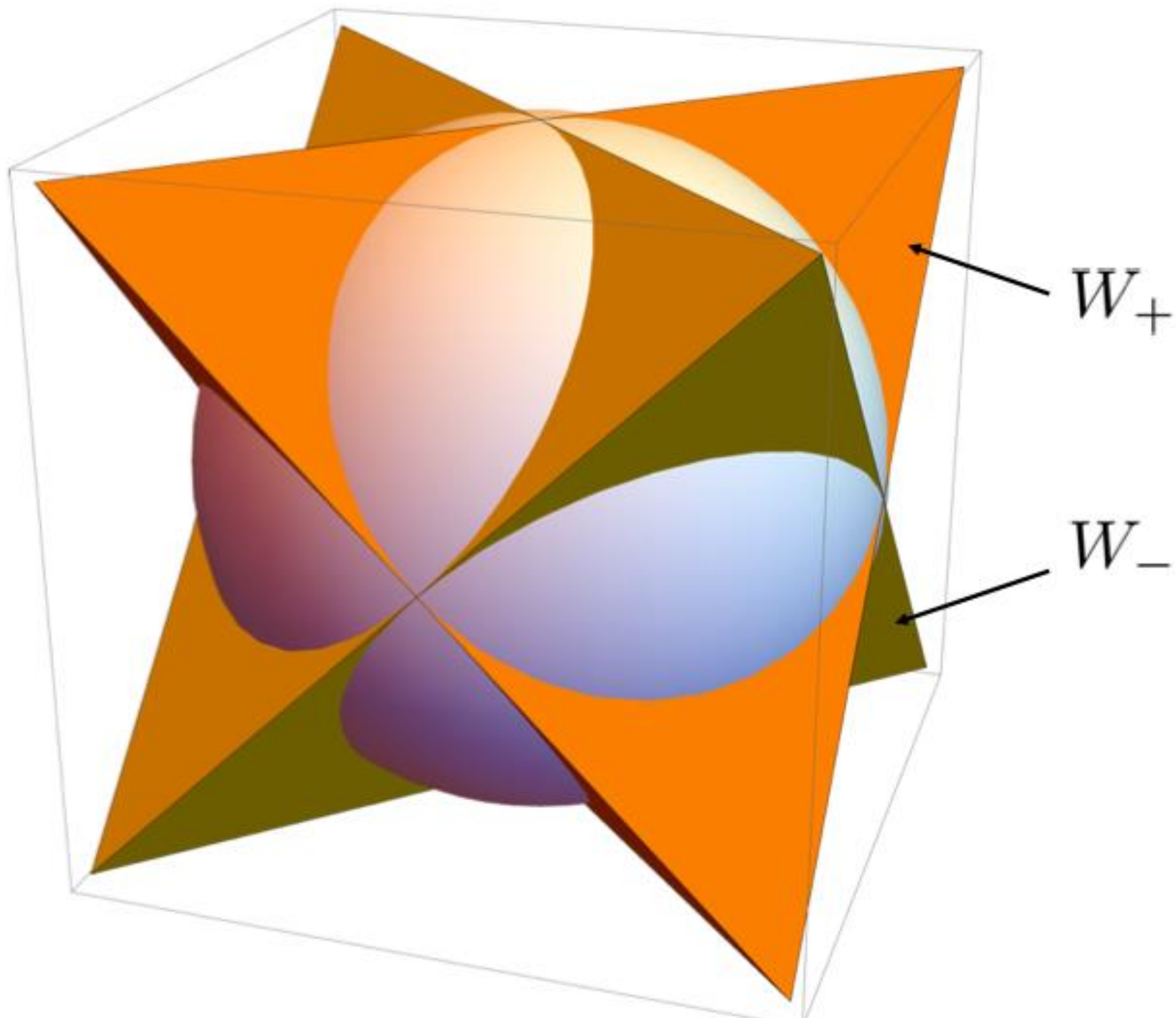
- Covariance

Given ε *such that* $\rho' = \varepsilon(\rho)$, *it has Kraus decompositions*

$$\varepsilon(\rho) = \sum_k p_k E_k \rho E_k^\dagger, \text{ s.t. } E_k A(\lambda) E_k^\dagger = A(T_k \lambda + a_k), \ \forall \lambda \in \mathbb{Z}_d^{2n}, \ \forall k.$$

- Transformation non-contextuality

$$\Gamma_T(\lambda'|\lambda) = \Gamma_{T'}(\lambda'|\lambda) \ \ \forall T, T' \in \mathcal{E}.$$


W_+
W_-

- Positivity preservation does not imply covariance.

- Positivity preservation does not imply transformation non-contextuality.

- Transformation non-contextuality and covariance seem to be broken by the same reason in the example of single qubit stabilizer (e.g. the Hadamard gate).
  What is their relation?